# SOME EXTENSIONS OF THE DROZ-FARNY LINE THEOREM

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Abstract. We provide some generalizations of the Droz-Farny theorem and give synthetics proofs for them.

## 1. Introduction

Let $ABC$ be a triangle with the orthocenter $H$. Let $\ell_1$ and $\ell_2$ be two perpendicular lines through $H$. Let $A_1$, $B_1$, and $C_1$ be the points where $\ell_1$ intersects $BC$, $AC$, and $AB$, respectively. Similarly, let $A_2$, $B_2$, and $C_2$ be the points where $\ell_2$ intersects $BC$, $AC$, and $AB$, respectively. In 1889 Arnold Droz-Farny stated that the midpoints of the three segments $A_1A_2$, $B_1B_2$, and $C_1C_2$ are collinear. He proposed this theorem without a proof and attracted the attention of quite a few authors. Both trigonometric and analytic approaches were given by Darij Grinberg, Floor van Lamoen [1, 5]. Milorad Stevanovic [4] offered a vector proof. Also, Darij Grinberg proposed a proof using inversion and another proof making use of angle chasing. Recently, Minh Ha Nguyen and The Vinh Luong present a synthetic proof using the notion of cross ratio and equal ratio in quadrilaterals [2]. After that in [3] Cyril Letrouit gives a clever extension using directly similar triangles.

In our paper we give an extension of the theorem by Droz-Farny and a synthetic proof taking up the idea of using equal ratio in quadrilaterals.

Fix a triangle $ABC$. Let $P$ be a point on its circumcircle. Let $P_a$, $P_b$ and $P_c$ be the reflections of $P$ in the lines $BC$, $AC$, and $AB$, respectively. Recall that, this points are collinear. The line $P_aP_bP_c$ is called *the Steiner line* of the point $P$ with respect to the triangle $ABC$. A significant property associated with a Steiner line is that it contains the orthocenter $H$ of the triangle $ABC$.

**Theorem 1.1.** *Let $ABC$ be a triangle and let $P$ be a point on its circumcircle. Let $\ell_{St}$ be the Steiner line of $P$ with respect to $ABC$ and let $Q$ be any point on $\ell_{St}$. Denote by $\ell_1$ and $\ell_2$ the bisectors of the angle formed by lines $\ell_{St}$ and $PQ$. Let $A_1$, $B_1$, $C_2$ and $A_2$, $B_2$, $C_2$ be points of intersection of lines $\ell_1$ and $\ell_2$ with sides of the triangle $ABC$. Then the midpoints of the three segments $A_1A_2$, $B_1B_2$, and $C_1C_2$ are collinear.*

In the case $Q = H$, we get the Droz-Farny theorem.

## 2. The proof of the Main Theorem

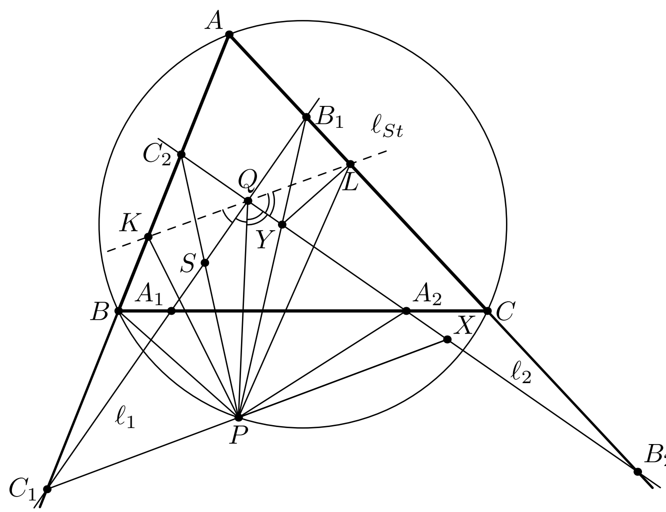
For ease of reference, recall our notations. Let $H$ be the orthocenter of the triangle $ABC$, and $P$ be a point on its circumcircle. Let $\ell_{St}$ be the Steiner line of $P$ with respect to triangle $ABC$, and $Q$ be a point on $\ell_{St}$. Bisectors $\ell_1$ and $\ell_2$ of the angle formed by lines $\ell_{St}$ and $PQ$ meet $BC$, $CA$, $AB$ at $A_1$, $B_1$, $C_1$ and $A_2$, $B_2$, $C_2$, respectively. We need to prove that the midpoints of $A_1A_2$, $B_1B_2$, $C_1C_2$ are collinear.

We give a proof of this extended theorem with the aid of a well-known lemma concerning equal ratio in quadrilaterals and the following lemma

**Lemma 2.1.** *Let $P$ be an arbitrary point on the circumcircle of the triangle $ABC$ with the orthocenter $H$. Let $\ell_{St}$ be the Steiner line of $P$ with respect to the triangle $ABC$. Bisectors $\ell_1$ and $\ell_2$ of the angle formed by lines $\ell_{St}$ and $PQ$ meet $BC$, $CA$, $AB$ at $A_1$, $B_1$, $C_1$, $A_2$, $B_2$, $C_2$, respectively. Then*

(1) $\dfrac{QB_1}{QC_1} = \dfrac{A_2C}{A_2B}$;

(2) $\dfrac{BC_2}{BC_1} = \dfrac{CB_2}{CB_1}$.




*Proof.* Let us prove part (1). Without loss of generality we assume that $P$ belongs the arc $BC$ not containing $A$, the points $A_1$ and $A_2$ is between $B$ and $C$ while $Q$ lies between $B_1$, $C_1$ as illustrated in the figure. We need to prove that $\frac{QB_1}{QC_1} = \frac{A_2C}{A_2B}$.

Indeed, let $\ell_{St}$ intersects $AB$, $AC$ at $K$, $L$ and $PC_1$ meets $QA_2$ at $X$. In the triangle $PQK$, $QC_1$ is the internal angle bisector and $KC_1$ is the external angle bisector. It follows that $PC_1$ is the external angle bisector. In the triangle $PQK$, $PX = PC_1$ is the external angle bisector and $QX$ is the external angle bisector, which implies that $KX$ is the internal angle bisector. Hence, if $KX$ meets $QC_1$ at $S$ then $S$ is the incenter of the triangle $PQK$. Therefore, $PS$ is the internal angle bisector passing through $C_2$ which is the intersection of two external angle bisectors of the triangle $PQK$, thereby, we also obtain that $S$ is the orthocenter of the triangle $XC_2C_1$.

Let $PB_1$ meets $QA_2$ at $Y$. In the triangle $PQL$ we have that $QB_2$ is the internal angle bisector, and $LB_1$, $QB_1$ are external angle bisectors, which means that $QB_1$ is the internal bisector. Therefore, $LY$ is the internal angle bisector of the triangle $PQL$, which implies the perpendicularity of $LY$ and $AC$. In the triangle $PQL$, $QB_2$ is the internal angle bisector and $LB_2$ is the external angle bisector, which yields the fact that $PB_2$ is an external angle bisector. Moreover, $PB_2$ and $PY$ are perpendicular. Consequently, quadrilaterals $YQB_1L$ and $PQC_2C_1$ are concyclic, which gives rise to the equality

$$\angle PB_1C = \angle YQL = \angle YQP = \angle PC_1B.$$

Furthermore, $P$ lies on the circumcircle of the triangle $ABC$, which implies that $\angle PBC_1 = \angle PCB_1$. Hence, triangle $PBC_1$ is similar to triangle $PCB_1$, which also implies the similarity of $PBC$ and $PC_1B_1$. Simultaniously, it follows from the similarity of triangles $PBC$ and $PC_2B_2$ that $\angle PCA_2 = \angle PB_2A_2$. This also implies concyclicity of quadrilateral $PA_2CB_2$, from which we obtain

$$\angle PA_2B = \angle PB_2C = \angle LYB_1 = \angle LQB_1 = \angle PQC_1.$$

Now we can conclude that the triangle $PA_2B$ is similar to the triangle $PQC_1$, from which we derive the ratio $\frac{QB_1}{QC_1} = \frac{A_2C}{A_2B}$.

Now we prove part (2). From the first part, we derive $\angle PB_2C = \angle LYB_1 = \angle LQB_1 = \angle PQC_1 = \angle PC_2B$ and $\angle PCB_2 = \angle PBC_1$. From this we achieve that the triangle $PCB_2$ is similar to the triangle $PBC_2$. Also $\angle PB_2C = \angle PC_2B$ implies that the two right triangles $PC_2C_1$ and triangle $PB_2B_1$ are similar. From the similarity of such triangles, we deduce the ratio $\frac{BC_2}{BC_1} = \frac{CB_2}{CB_1}$. □

For the general case we need to use the concept of signed distances for our reasoning.

**Lemma 2.2.** *Two triples of collinear points are $A_1$, $B_1$, $C_1$ and $A_2$, $B_2$, $C_2$ satisfying $\frac{B_1A_1}{B_1C_1} = \frac{B_2A_2}{B_2C_2}$. Let $A_0$, $B_0$, $C_0$ be the points that divide the segments $A_1A_2$, $B_1B_2$, $C_1C_2$ each into the same ratio. Then we have that $A_0$, $B_0$, $C_0$ are collinear.*

The proof of this lemma is evident and now we shall use two lemmas to prove the extended theorem.

*Proof.* By Lemma 2.1, we have the following computations in which the distances are signed

$$\frac{A_1B_1}{A_1C_1} = \frac{QB_1 - QA_1}{QC_1 - QA_1} = \frac{\frac{QB_1}{QA_1} - 1}{\frac{QC_1}{QA_1} - 1} = \frac{\frac{C_2A}{C_2B} - 1}{\frac{B_2A}{B_2C} - 1} = \frac{C_2A - C_2B}{B_2A - B_2C} \cdot \frac{B_2C}{C_2B} = \frac{AB}{AC} \cdot \frac{CB_2}{BC_2}$$

Likewise, we obtain $\frac{A_2B_2}{A_2C_2} = \frac{AB}{AC} \cdot \frac{CB_1}{BC_1}$, from which we get $\frac{A_1B_1}{A_1C_1} = \frac{A_2B_2}{A_2C_2}$ by Lemma 2.2 then the midpoints of $A_1A_2$, $B_1B_2$, $C_1C_2$ are collinear. This completes the proof of Theorem 1.1. □

**Remark.** Actually, we can prove that the midpoints of $A_1A_2$, $B_1B_2$, $C_1C_2$ belong to the perpendicular bisector of $PQ$ by an easier way. Indeed, follow Lemma 2.1 we have $P$, $Q$ lie on the circle with the diameter $C_1C_2$. It means that midpoint of $C_1C_2$ lie on perpendicular bisector of $PQ$. Similarly, for midpoints of $A_1A_2$ and $B_1B_2$. With this approach we can obtain Theorem 1.1 faster but Lemma 2.2 help us to obtain some more general results.

## 3. Some other extensions

When the orthocenter $H$ coincides with $Q$, bisectors $\ell_1$ and $\ell_2$ can be replaced by any two mutually perpendicular lines passing through $H$. By taking up the idea of using equal ratio in a quadrilateral as above, we can get several other extensions.

**Theorem 3.1.** *Let $ABC$ be a triangle and let $P$ be a point on its circumcircle. Let $\ell_{St}$ be the Steiner line of $P$ with respect to $ABC$ and let $Q$ be any point on $\ell_{St}$. Denote by $\ell_1$ and $\ell_2$ the bisectors of the angle formed by lines $\ell_{St}$ and $PQ$. Let $A_1$, $B_1$, $C_2$ and $A_2$, $B_2$, $C_2$ be points of intersection of lines $\ell_1$ and $\ell_2$ with sides of the triangle $ABC$.*

*Then the points divide the three segments $A_1A_2$, $B_1B_2$, and $C_1C_2$ in the same ratio are collinear.*

Note that, we have an extension of Lemma 2.2

**Lemma 3.2.** *Two triples of collinear points are $A_1$, $B_1$, $C_1$ and $A_2$, $B_2$, $C_2$ satisfying $\frac{B_1A_1}{B_1C_1} = \frac{B_2A_2}{B_2C_2}$. Let $A_0, B_0, C_0$ be points in the plane such that the triangles $A_0A_1A_2$, $B_0B_1B_2$, $C_0C_1C_2$ are directly similar. Then points $A_0$, $B_0$, $C_0$ are collinear.*

*Proof.* Two lines contain two triples of collinear points $A_1$, $B_1$, $C_1$ and $A_2$, $B_2$, $C_2$ which intersect at $L$. The circumcircles of the triangle $LA_1A_2$ and $LC_1C_2$ intersect again at $G$. Then $G$ is the center of spiral similarity which moves $A_1$ to $A_2$ and $C_1$ to $C_2$. Because $\frac{B_1A_1}{B_1C_1} = \frac{B_2A_2}{B_2C_2}$ so in this spiral similarity $B_1$ moves to $B_2$. Now from the directly similar triangles $A_0A_1A_2$, $B_0B_1B_2$, $C_0C_1C_2$, we see that this spiral similarity with the center $G$ which moves $A_1$ to $A_0$, $B_1$ to $B_0$ and $C_1$ to $C_0$. It means that, the triangles $GA_0B_0$ and $GB_0C_0$ are respectively similar to triangles $GA_1B_1$ and $GB_1C_1$. It follows, that $\angle GB_0A_0 = \angle GB_1A_1$ and $\angle GB_0C_0 = \angle GB_1C_1$. From this

$$\angle A_0B_0C_0 = \angle GB_0A_0 + \angle GB_0C_0 = \angle GB_1A_1 + \angle GB_1C_1 = 180^\circ.$$

It means that the points $A_0$, $B_0$, $C_0$ are collinear. □

Now using the main idea from [3] and Lemma 3.2 we can get another extension.

**Theorem 3.3.** *Let $ABC$ be a triangle and let $P$ be a point on its circumcircle. Let $\ell_{St}$ be the Steiner line of $P$ with respect to $ABC$ and let $Q$ be any point on $\ell_{St}$. Denote by $\ell_1$ and $\ell_2$ the bisectors of the angle formed by lines $\ell_{St}$ and $PQ$. Let $A_1$, $B_1$, $C_2$ and $A_2$, $B_2$, $C_2$ be points of intersection of lines $\ell_1$ and $\ell_2$ with sides of the triangle $ABC$. Let $A_1A_2A_3$, $B_1B_2B_3$, and $C_1C_2C_3$ be the directly similar triangles constructed on segments $A_1A_2$, $B_1B_2$, and $C_1C_2$. Then the points $A_3$, $B_3$, and $C_3$ are collinear.*

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