PARABOLAS AND FAMILIES OF CONVEX QUADRANGLES

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Abstract. In this article we study some parabolas naturally associated with a generic convex quadrangle. It is shown that the quadrangle defines, through these parabolas, a one-parameter-family of quadrangles, containing the quadrangle of reference. The members of this family share the same diagonal lines, Newton line and ratio of diagonals. In addition it is shown that this family contains a unique cyclic quadrangle.

1. Introduction

The aim in this article is to show that a given quadrangle $q = ABCD$ can be considered as a member of a one-parameter-family of quadrangles sharing the same diagonals and ratio of diagonals. In addition it is shown that the family contains exactly one cyclic member-quadrangle. The quadrangles of the family have their pairs of opposite sides tangent to two parabolas $\{\kappa_1, \kappa_2\}$, which can be defined directly from the quadrangle but have also another aspect, as envelopes of diagonals of inscribed in q parallelograms. Section 2 discusses in short these parallelograms, since their properties lead to important for the sequel relations between the two parabolas. In section 3 we define the two parabolas as envelopes of certain lines and discuss in some detail their mutual relationship, the discussion culminating with theorem 3.5. In section 4 we discuss the structures involving the directrices of the two parabolas. This is useful for the determination of the particular cyclic member of the one-parameter-family. In section 5 we detect and study a third parabola, the tangents of which parametrize the members of the family. Finally in section 6 we combine all the previous results to our main theorems 6.1 and 6.2.

2. Special inscribed parallelograms

In a generic convex quadrangle $q = ABCD$ we can easily inscribe infinite many parallelograms $q_t = A_tB_tC_tD_t$ with sides parallel to the diagonals $\{AC, BD\}$ of q (See

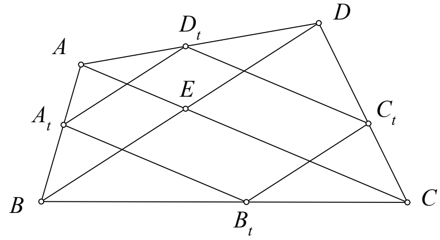

Figure 1. Parallelograms with sides parallel to the diagonals

Figure 1). The vertex A_t of such a parallelogram can be selected on an arbitrary side, AB say. Then, drawing a parallel to the diagonal AC, we define its intersection point B_t with BC. Drawing further from B_t a parallel to BD, we define its intersection C_t with CD and continuing this way, we obtain, using Thales theorem, an inscribed parallelogram $q_t = A_tB_tC_tD_t$ with sides parallel to the diagonals of q. An obvious property of

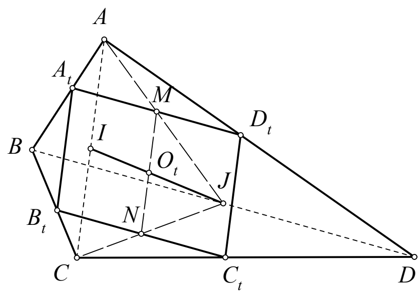

FIGURE 2. Centers of parallelograms with sides parallel to the diagonals

these parallelograms is, that their centers O_t move on the Newton line IJ of q, joining the middles $\{I, J\}$ of the diagonals. The proof can be immediately read from figure 2. Next lemma shows that this is a characteristic property of the parallelograms of this kind.

Lemma 2.1. *A parallelogram $q' = A'B'C'D'$, inscribed in the quadrangle $q = ABCD$, has its sides parallel to the diagonals of q, if and only if its center is on the Newton line of q.*

Proof. The "if" part is the preceding remark. The "only if" part follows from the symmetry of the parallelogram w.r. to its center. By this, the parallelogram $A'B'C'D'$ can

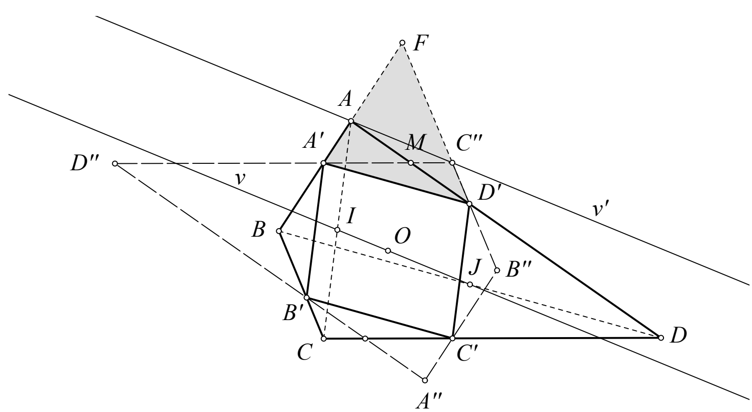

FIGURE 3. Parallelograms with sides parallel to the diagonals II

be considered to be defined by the intersections of corresponding sides of q and its symmetric $q'' = A''B''C''D''$ w.r. to the center O of q' (See Figure 3). It is then easy to see that C'' moves on a parallel ν' of the Newton line ν of q and the triangles $\{FA'D'\}$ have constant angles as O varies on ν. This implies that, for all these points O on the Newton line ν of q, the sides $A'D'$ have a fixed direction, which coincides with BD, when O takes the position of J, thereby proving that $A'D'$ is always parallel to BD. Analogously is seen that $A'B'$ is parallel to the other diagonal AC of q. □

Next theorem shows that this class of parallelograms is unique from another aspect.

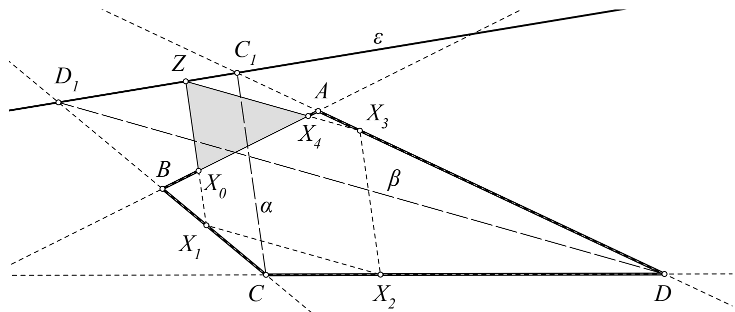

FIGURE 4. Parallelograms with sides parallel to arbitrary directions $\{\alpha, \beta\}$

Theorem 2.2. *Given are a convex quadrangle $q = ABCD$ and two directions $\{\alpha, \beta\}$. For a point $X_0 \in AB$ we draw successively parallels to these directions to obtain respective points on the sides $X_1 \in BC$, $X_2 \in CD$, $X_3 \in DA$. If there are two points $\{X_0 \in AB\}$ such that $X_0X_1X_2X_3$ is a parallelogram, then the two directions are those of the diagonals of q.*

Proof. In fact, consider the additional point $X_4 \in AB$, defined by the intersection of AB with the parallel to β from X_3 and the intersection point $Z = X_3X_4 \cap X_0X_1$ (See Figure 4). Introducing affine coordinates $\{X_0(x), X_4(y)\}$ along the line AB, we see easily ([7]), that they are related linearly $y = ax + b$, that the triangle $\tau_x = X_0X_4Z$ remains similar to itself for all positions of X_0 and its vertex Z moves on a fixed line ε. This line passes through the intersection C_1 of line AD with the parallel to α from C and through the intersection D_1 of line BC with the parallel to β from D.

The condition formulated in the statement of the theorem is equivalent with the coincidence of the two lines ε and AB. But this is easily seen to be equivalent with $C_1 \equiv A$ and $D_1 \equiv B$, which proves the claim. □

Corollary 2.3. *Under the notation and conventions of the previous theorem, all the quadrangles $X_0X_1X_2X_3$ are parallelograms, if and only if, the directions $\{\alpha, \beta\}$ are those of the diagonals of $q = ABCD$.*

3. Parabolas enveloping the diagonals

The diagonals of the family of the parallelograms of the previous section envelope two parabolas. This follows from a fundamental property of the parabola, according to which, a linear relation between two lines defines a parabola tangent to these lines. This, in turn, is a special case of the general Chasles-Steiner generation method of conics through homographic relations between the points of two lines ([4, p. 6]). For the diagonal $B'D'$

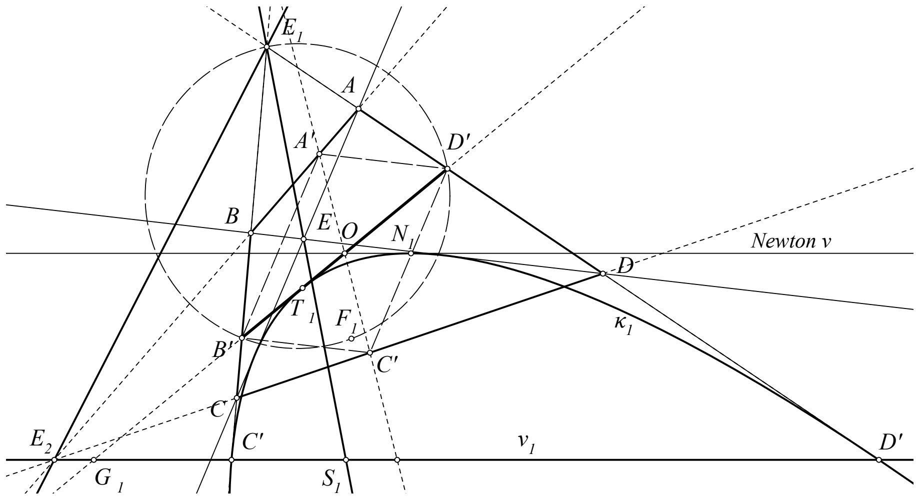

FIGURE 5. Parabola κ_1 tangent to sides $\{AD, BC\}$ and the diagonals

this is seen by introducing coordinates along the opposite sides $\{AD, BC\}$ (See Figure 5). Denoting the coordinates by $\{B(b), C(c), B'(x)\}$ and $\{A(a), D(d), D'(y)\}$, and using the parallelity of the sides to the diagonals of q, we find that

$$y = \frac{(a-d)x - (ab - cd)}{c - b}. \tag{1}$$

The linearity of the relation implies that the conic is a parabola κ_1. This, because for x tending to infinity, the same happens to y, consequently the tangent line $B'(x)D'(y)$ tends to the line at infinity. Thus, the line at infinity is tangent to the conic, which therefore is a parabola ([2, p. 137]). The parabola is identified with the one tangent to the four lines $\{BC, AD, AC, BD\}$, having the well known properties of these kind of conics ([6, p. 54], [3, p. 178], [10]). Among other things, the circles $(E_1B'D')$ pass through the focus F_1 of the parabola and the point of contact T_1 of the tangent $B'D'$ is the harmonic conjugate of the intersection $G_1 = (B'D', \nu_1)$, where $\nu_1 = C'D'$ is the chord of contact points of the lines $\{BC, AD\}$ with the conic. It is also known that the symmedian from E_1 of the triangle $E_1C'D'$ passes through the focus F_1. Another feature of this configurations is also that the triangle $E_1E_2S_1$, with $S_1 = (EE_1, C'D')$ is self-polar relative to κ_1. Regarding the diagonal $B'D'$, we can thus formulate a property of it related to the circle through the focus.

Lemma 3.1. *The diagonals $B'D'$ of the family of inscribed in $ABCD$ parallelograms $A'B'C'D'$, with sides parallel to the diagonals, are the chords of intersection points of the pencil of circles $\mathcal{P}_{E_1,F_1}$ passing through $\{E_1, F_1\}$, with the sides of the angle $\widehat{BE_1A}$.*

Lemma 3.2. *The Newton line of the quadrilateral $ABCD$ is tangent to the parabola at the middle N_1 of the segment I_1J_1 intercepted by the sides $\{AD, BC\}$. It is also parallel to the polar $C'D'$ of E_1 at half the distance of E_1 from that polar. Further, the ratio of the opposite sides $AD/BC = E_1J_1/E_1I_1$.*

Proof. Consider the polar $C'D'$ of E_1 and its parallel ν at half the distance of E_1 from $C'D'$ (See Figure 6). By the well known properties of parabolas, line ν is tangent to the

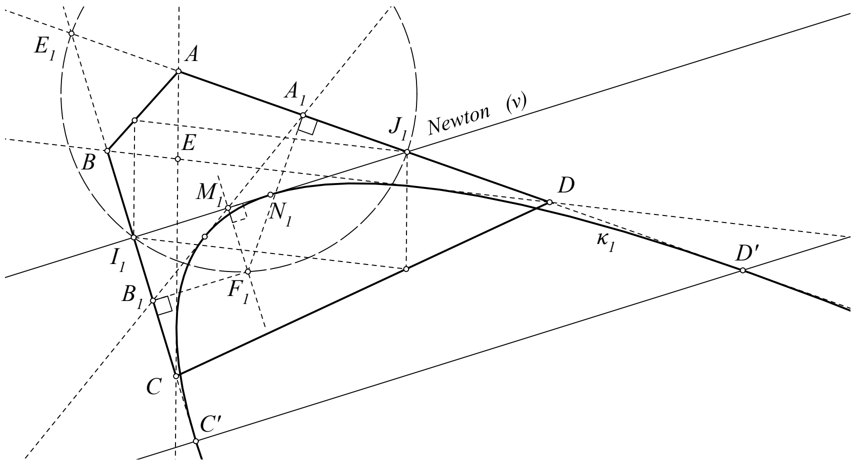

FIGURE 6. The Newton line tangent to the parabola

parabola at its middle N_1. By one of the main properties of parabolas, the ratio of the segments intercepted on a variable tangent by three other fixed tangents to the parabola is constant ([4, p. 52]). Consequently all tangents A_1B_1 to the parabola are intersected by the three tangents $\{AD, \nu, BC\}$ in segments $\{A_1M_1, M_1B_1\}$ having constant ratio A_1M_1/M_1B_1. But this ratio is 1, as is seen by drawing A_1B_1 in the special position $A_1B_1 = E_1C'$. A similar argument, for the diagonals as tangent lines, shows that ν passes through the middles of the diagonals $\{AC, BD\}$ and identifies the line ν with the Newton line and points $\{I_1, J_1\}$ with the respective middles of $\{E_1C', E_1D'\}$. $\square$

Notice that, by an elementary property of the parabola, the projection of the focus on the tangents lies on the tangent at the vertex of the parabola, which thus is the Wallace-Simson line of the focus, with respect to any triangle circumscribed to the parabola ([1, p. 22]). Thus, the tangent A_1B_1 at the vertex of the parabola (See Figure 6) is the Wallace-Simson line of F_1 w.r. to the triangle $E_1I_1J_1$. By well known properties of parabolas, we have also the next corollary.

Corollary 3.3. *Under the previous notation and conventions, the line E_1N_1 is parallel to the axis of the parabola, so that for every point on E_1N_1 the corresponding polar is parallel to the Newton line ν.*

Let now $\{G, H\}$ be the middles of the diagonals on the Newton line, $\{P_1, Q_1\}$ be the contact points of the parabola with the diagonals and $N_2 = (I_1J_1, P_1Q_1)$. Further, let the opposite sides $\{AB, CD\}$ of the quadrangle intersect the Newton line correspondingly at points $\{I_2, J_2\}$. The following lemma establishes a relation between the points $\{N_1, N_2\}$ (See Figure 7).

Lemma 3.4. *The points $\{G, H\}$ are harmonic conjugate to $\{N_1, N_2\}$ and point N_2 is the middle of the segment I_2J_2.*

Proof. For $S_1 = (EE_1, C'D')$, consider the triangle $E_1E_2S_1$, which is self-polar w.r. to the parabola. Hence for all points E on E_1S_1 the polar of E will pass through E_2. This is true in particular for P_1Q_1, so that $\{E_2, P_1, Q_1, N_2\}$ are on the polar of E. Thus N_2 is on the polars of points $\{N_1, E\}$, hence its polar will be line N_1E, so that $(GHN_1N_2) = -1$.

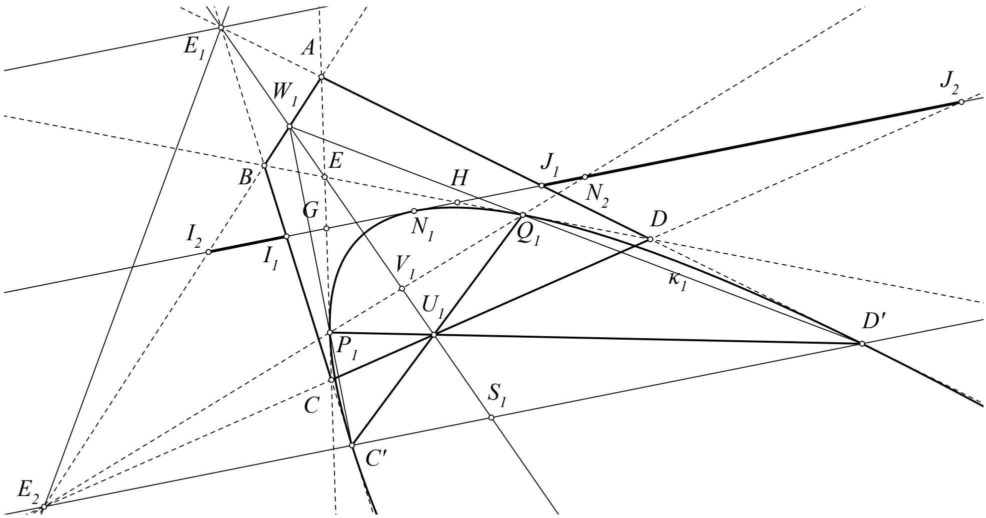

FIGURE 7. Harmonic division $(GHN_1N_2) = -1$ on the Newton line

To show the second claim consider the lines $\{P_1D', C'Q_1\}$ and see that they intersect at a point U_1 on line CD. In fact, if $V_1 = (P_1Q_1, EE_1)$, the two quadruples of collinear points $(P_1Q_1V_1E_2) = (D'C'S_1E_2) = -1$ are harmonic, hence, by well known theorem ([5, p. 90]) the lines $\{P_1D', Q_1C', S_1V_1\}$ are concurrent and the lines $\{P_1C', Q_1D', V_1S_1\}$ are also concurrent at points $\{U_1, W_1\}$ of the polar of E_2. In addition the triangle $E_2U_1W_1$ is self-polar w.r. to the parabola. But P_1D' is the polar of A and $C'Q_1$ is the polar of B. Hence, AB is the polar of of U_1, consequently $W_1 \in AB$. Analogously is seen that U_1 on CD. The previous arguments imply that the pencil $E_2(CBP_1C')$ is harmonic, and since E_2C' is parallel to I_1J_1 the other rays of the pencil intersect I_1J_1 at $\{I_2, J_2\}$ and the middle N_2 of I_2J_2. □

Doing the same work for the second diagonal of the inscribed parallelograms, we find their envelope, which is a second parabola κ_2, tangent to the lines $\{AB, CD, AC, BD\}$. Figure 8 displays the resulting configuration of the two parabolas, denoting with indexed letters points having the same meaning relative to the parabola with the corresponding index. Thus, $\{P_2, Q_2\}$ are the contact points of κ_2 with the diagonals, N_2 is its contact point with the Newton line etc.

Theorem 3.5. *Under the notation and conventions adopted so far, the following are valid properties.*

(1) *The triples $\{(P_1, Q_1, N_2), (P_2, Q_2, N_1)\}$ of contact points of the two parabolas consist of collinear points.*
(2) *The intersection $E' = (P_1Q_1, P_2Q_2)$ of the previous lines defines the lines $\{E_1E', E_2E'\}$, which are parallel to the axes of the parabolas $\{\kappa_1, \kappa_2\}$, hence these lines represent conjugate directions w.r. to both parabolas. The middles of all chords of κ_1 (resp. κ_2) parallel to ν lie on E_1E' (resp. E_2E').*
(3) *The polar ε of $E' = (P_1Q_1, P_2Q_2)$ w.r. to either of the parabolas is the parallel to the Newton line ν through E.*

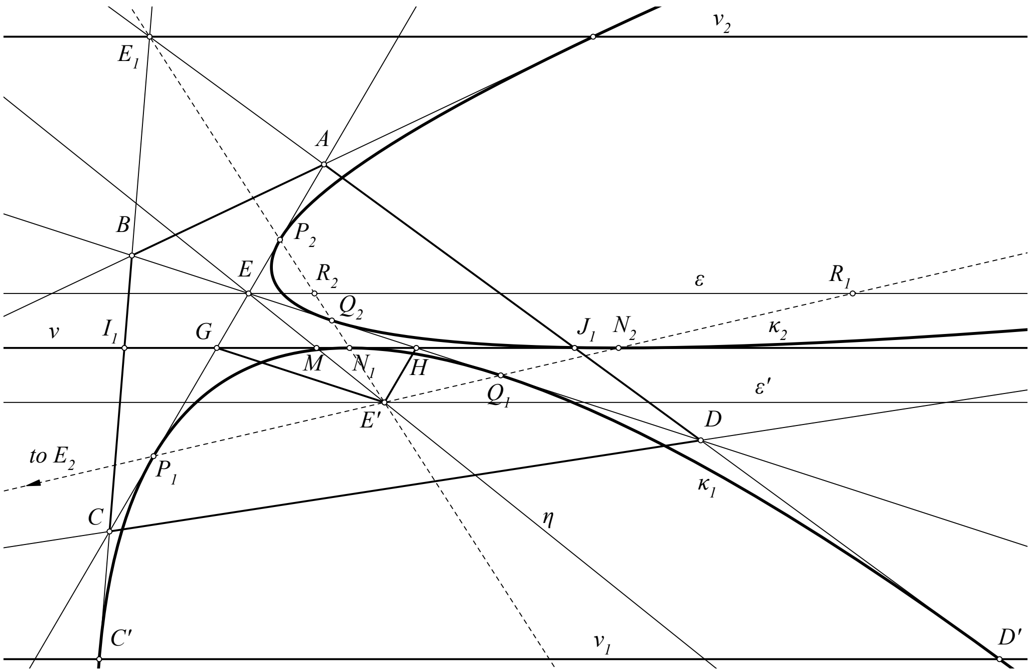

FIGURE 8. The two parabolas enveloping the diagonals of the inscribed parallelograms

(4) *The line ε', which is parallel to the Newton line through E', is the polar of $R_1 = (P_1Q_1, \varepsilon)$ w.r. to κ_2 and also the polar of $R_2 = (P_2Q_2, \varepsilon)$ w.r. to κ_1.*
(5) *The line $\eta = EE'$ passes through the middle M of GH and is the polar of R_1 w.r. to κ_1 and also the polar of R_2 w.r. to κ_2. Points $\{N_1, N_2\}$ are respectively the middles of the segments $\{E'R_2, E'R_1\}$.*
(6) *$EGE'H$ is a parallelogram and $\{G, H\}$ are respectively the middles of $\{P_1P_2, Q_1Q_2\}$.*
(7) *It is $P_1C = AP_2$ and $BQ_2 = Q_1D$.*
(8) *The polars of points $P \in \varepsilon$ w.r. to the two parabolas pass all through E' and there are two points $\{P_H, P_G\}$ on ε, lying symmetric w.r. to E, for which the two polars coincide. These are $P_H = (\varepsilon, E'H), P_G = (\varepsilon, E'G)$ with respective polars GE' and HE'.*
(9) *The triangle P_GP_HE' is anticomplementary of the common, tangential to the two parabolas, triangle EGH and is self-polar w.r. to both parabolas.*

Proof. Nr-1 follows from lemma 3.4, since this implies that N_2 is the middle of segment I_2J_2 intercepted on the Newton line by the opposite sides $\{AB, CD\}$ of the quadrangle, which, by the analogous to lemma 3.2 for κ_2, is the contact point of κ_2 with the Newton line. Analogously point N_1 will lie on the polar P_2Q_2 of E w.r. to κ_1.

Nr-2 is essentially equivalent to corollary 3.3 and derives from N_1 being the middle of I_1J_1, so that E_1N_1 passes also from the middle of the chord $C'D'$. This shows the stated property for E_1E'. Analogously is proved the property for E_2E'.

For *nr-3* apply corollary 3.3 and take into account *nr-2*. By these the polar ε of E' must go through E, since E' is on the polar of E. Since for all points E' on E_1N_1 the corresponding polars are parallel to the Newton line ν it follows that ε is the parallel from E to ν. Analogously is seen that ε is also the polar of E' w.r. to κ_2.

Nr-4 follows from corollary 3.3 for R_2, which is on the polar ε of E', hence its polar, which is parallel to ν must pass through E'. The same corollary adapted for the parabola κ_2 implies the statement for R_1.

In *nr-5*, the statement for the middle M follows from *nr-3*, since by this $R_1 = (P_1Q_1, \varepsilon)$ is harmonic conjugate to E' w.r. to $\{P_1, Q_1\}$. Thus, $E(P_1Q_1E'R_1)$ is a harmonic pencil and GH is parallel to its ray $\varepsilon = ER_1$, hence the other rays pass through the end points of GH and its middle M. Since R_1 is on the polar of E and also of E' w.r. to κ_1, its polar is EE'. Analogous is the proof for R_2. The last statement is an immediate consequence of the previous statements.

In *nr-6*, the first claim follows by observing that $E(G, H, E', R_1)$ is a harmonic pencil and GH is parallel to its ray ER_1, hence M is the middle of GH. Thus, the diagonals of the quadrangle $EGE'H$ intersect at their middle. Analogous is the proof for the other claims. In fact, by lemma 3.4 the pencil $E'(N_1, N_2, G, H)$ is harmonic and its ray $E'H$ is parallel to AC. Thus the other three rays intersect AC in two equal segments. Analogously is proved also that H is the middle of Q_1Q_2.

Nr-7 follows directly from *nr-6*, since $\{P_1P_2, AC\}$ have G as a common middle etc.

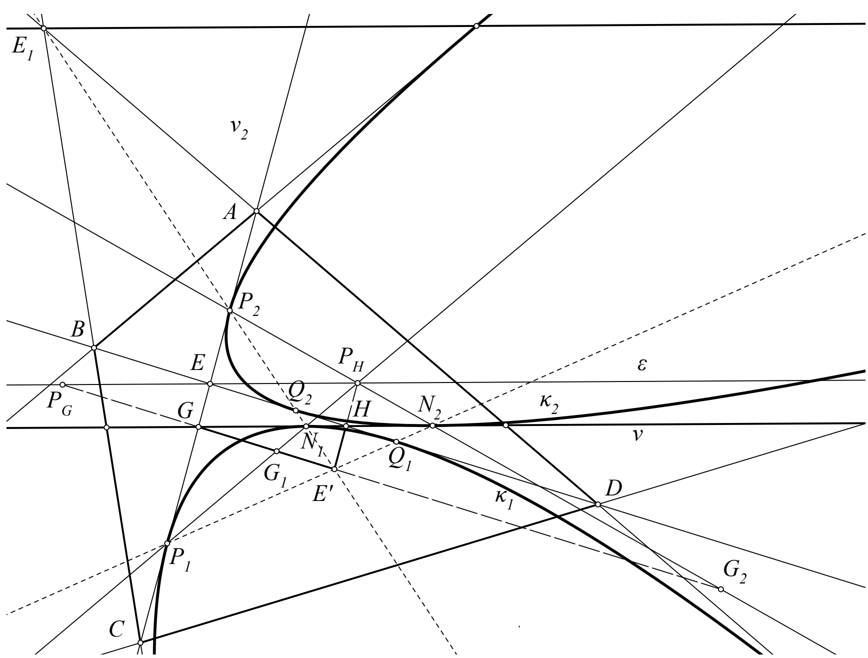

FIGURE 9. Lines $\{GE', HE'\}$ are common polars of $\{P_H, P_G\}$

For *nr-8* start by defining P_H as the intersection $P_H = (P_1N_1, \varepsilon)$ (See Figure 9). Since P_H is on the polars w.r. to κ_1 of G and E', its polar is GE'. From lemma 3.4 follows that the pencil $E'(GHN_1N_2)$ is harmonic, hence intersects on P_1N_1 a harmonic quadruple $(P_1N_1G_1P_H^*) = -1$. Since GE' is the polar of P_H the quadruple $(P_1N_1G_1P_H) = -1$ is also harmonic, hence $P_H^* = P_H$. This completes the proof that $P_H = (E'H, \varepsilon)$ and its polar w.r. to κ_1 is GE'. By replacing in the previous argument points $\{P_1, N_1, G_1\}$, with the points $\{P_2, N_2, G_2 = (P_2N_2, GE')\}$, we show analogously that GE' is the polar of P_H w.r. to κ_2. An analogous reasoning shows that HE' is the polar of P_G.

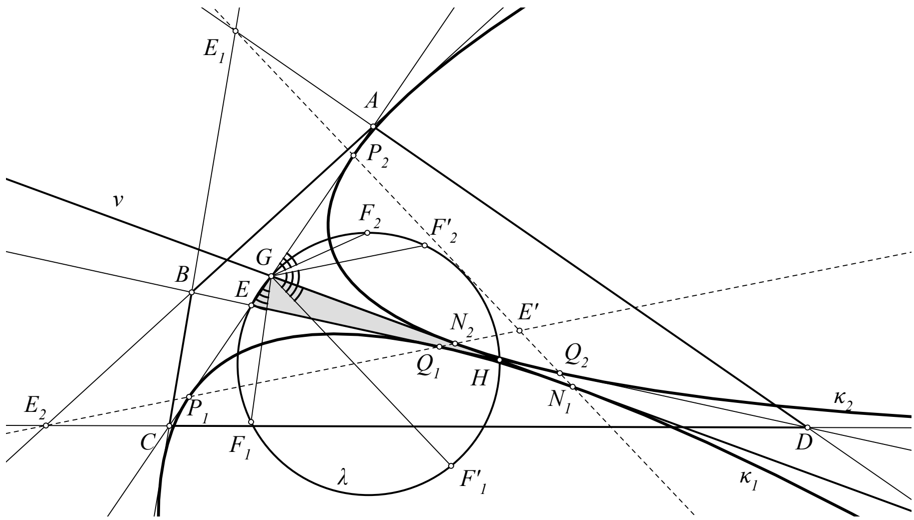

FIGURE 10. The circle λ through the focal points

Nr-9 is an immediate consequence of the previous *nr*. It reflects also the property of parabolas, according to which, the anticomplementary of a tangential triangle of a parabola is a self-polar triangle of the parabola ([2, p. 140], see also [1, p. 74]). □

Corollary 3.6. *With the notation and conventions of this section, the circle λ through the intersection E of the diagonals and the middles $\{G, H\}$ of them, passes also through the focal points $\{F_1, F_2\}$ of the two parabolas $\{\kappa_1, \kappa_2\}$. Further the angle $\widehat{F_1GF_2}$ equals one of the angles formed by the axes of the parabolas.*

Proof. The first claim follows from the well known property of triangles circumscribing a parabola, since the triangle EGH circumscribes both parabolas (See Figure 10). The second claim follows from the equally well known *optical property of conics* ([1, p. 6]). In fact, by theorem 3.5/*nr-2* the lines $\{P_1Q_1, P_2Q_2\}$ are respectively parallel to the axes of $\{\kappa_2, \kappa_1\}$. Drawing GF_2' parallel to P_1Q_1 and GF_1' parallel to P_2Q_2 and applying the aforementioned property we obtain the relations

$$\widehat{F_2GH} = \widehat{P_2GF_2'} \quad \text{and} \quad \widehat{HGF_1} = \widehat{F_1'GP_1},$$

which imply immediately the claim. □

4. A PAIR OF EQUAL CIRCLES

Here we start by introducing the circle λ' on diameter $E'K$, where K is the intersection point of the directrices $\{\delta_1, \delta_2\}$ of the two parabolas $\{\kappa_1, \kappa_2\}$ (See Figure 11). Since, by theorem 3.5/*nr-2*, the lines $\{P_1Q_1, P_2Q_2\}$ are respectively parallel to the axes of the parabolas $\{\kappa_2, \kappa_1\}$, the angles at the intersection points $S_1 = (P_1Q_1, \delta_2)$ and $S_2 = (P_2Q_2, \delta_1)$ are right. Hence points $\{S_1, S_2\}$ are on the circle λ' with diameter $E'K$.

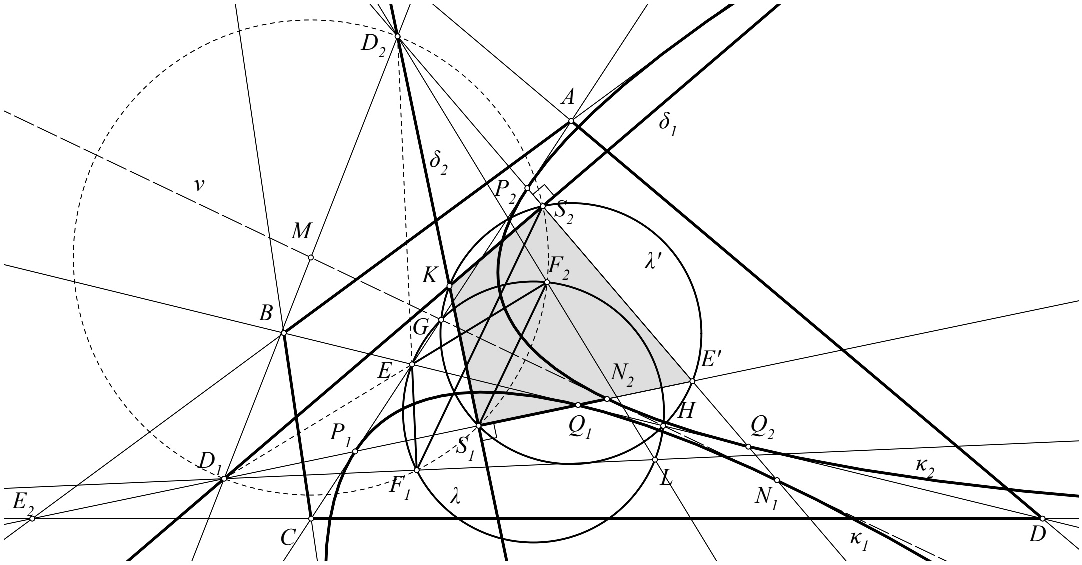

FIGURE 11. The circle λ' on diameter $E'K$

Lemma 4.1. *The pairs of points $\{(F_1, S_2), (F_2, S_1)\}$ are symmetric w.r. to the Newton line ν and the circles $\{\lambda, \lambda'\}$ are congruent and intersect at points $\{G, H\}$ of the Newton line.*

Proof. The first claim follows from the characteristic property of a parabola. In fact, the contact point N_2 of the tangent ν to κ_2 is equidistant from the points $\{F_2, S_1\}$ and similarly, the contact point N_1 of the tangent ν to κ_1 is equidistant from $\{F_1, S_2\}$, therefore the claimed symmetry, implying that ν is the common medial line of the two parallel segments $\{F_1S_2, F_2S_1\}$.

The second claim follows from the first immediately, since $F_1S_2F_2S_1$ is then an isosceles trapezium having equal diagonals $S_1S_2 = F_1F_2$. By corollary 3.6, points $\{K, E\}$ view respectively these equal segments under the same angle, hence the circles $\lambda = (EF_1F_2)$ and $\lambda' = (KS_1S_2)$ are equal.

That point H is on both circles, follows by observing the equal angles $\widehat{F_1HF_2} = \widehat{S_1HS_2}$. Since the first characterizes the points of λ viewing F_1F_2, the second angle will characterize also the points of the equal circle λ' viewing $S_1S_2 = F_1F_2$. Thus H belongs also to λ'. Analogously is seen that G belongs to the circle λ'. □

The circles $\{\lambda, \lambda'\}$ are related to two triangles naturally defined in the configuration of the two parabolas. These are the triangles $\{D_1D_2E', D_1D_2L\}$, for the intersections of lines $\{D_1 = (\delta_1, P_1Q_1), D_2 = (\delta_2, P_2Q_2)\}$ and $L = (D_1F_1, D_2F_2)$ (See Figure 11). The following theorem lists some basic properties of these triangles.

Theorem 4.2. *Under the notation and conventions adopted so far, the following are valid properties.*

(1) *Lines $\{EF_1, EF_2\}$ are polars respectively of D_1 rel. κ_1 and of D_2 rel. κ_2.*
(2) *Lines $\{D_1F_1, D_2F_2\}$ are respectively orthogonal to $\{EF_1, EF_2\}$.*
(3) *The quadrangle $D_1D_2E'L$ is cyclic.*

(4) *Lines* $\{KE', EL\}$ *are parallel and orthogonal to* D_1D_2.
(5) *Points* $\{K, E\}$ *are orthocenters respectively of the triangles* $\{D_1D_2E', D_1D_2L\}$ *and lines* $\{EF_1, EF_2\}$ *pass respectively through points* $\{D_2, D_1\}$.
(6) *The Newton line* ν *passes through the middle* M *of* D_1D_2 *and the circles* $\{\lambda, \lambda'\}$ *are orthogonal to the circle with diameter* D_1D_2, *consequently*

$$MG \cdot MH = MF_1^2 = MF_2^2 = D_1D_2^2/4.$$

Proof. *Nr-1*, the claim for the line EF_1, follows from the definition of D_1 as intersection of P_1Q_1, which is the polar of E rel. κ_1, and of δ_1, which is the polar of F_1. Hence EF_1 is the polar of D_1 rel. κ_1. Analogous is the proof of the claim for EF_2.

Nr-2 follows from the general property of the directrix, according to which the polar EF_1 w.r. to a point D_1 of the directrix is orthogonal to D_1F_1 ([11, p. 183]). Analogously is proved that EF_2 is orthogonal to D_2F_2.

Nr-3 is an immediate consequence of *nr-2*.

Nr-4 follows by observing that the orthogonal triangles $\{KE'H, LGE\}$ are equal. In fact, by lemma 4.1, they have equal hypotenuses $KE' = EL$, which are diameters of equal circles. Also, by theorem 3.5/*nr-6*, $EGE'H$ is a parallelogram, hence the sides $\{EG, HE'\}$ of these triangles are equal. The equality of these triangles implies the equality of the triangles $\{KGH, LHG\}$ and this implies the claimed parallelity of $\{KH, EL\}$. The orthogonality of these two lines to D_1D_2 follows from the fact, noticed at the beginning of this section, that the directrices $\{\delta_1, \delta_2\}$ are respectively orthogonal to $\{D_2H, D_1E'\}$. Hence they are altitudes of the triangle D_1D_2E' and K is the orthocenter of the triangle. This implies that KE' is the third altitude of this triangle, consequently this and its parallel EL are both orthogonal to D_1D_2.

Nr-5, the proof for K was given in the course of proof of *nr-4*. The claim for E follows from the next lemma on triangles with the same circumcircle and same basis.

Nr-6 follows also from the next lemma. □

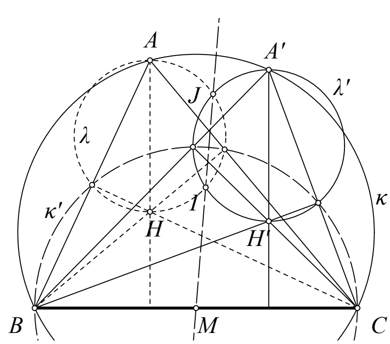

FIGURE 12. Orthocenters $\{E\}$ of circles on same circumcircle κ and base BC

Lemma 4.3. *The orthocenters* $\{H\}$ *of triangles* ABC *with the same circumcircle* κ *and same base* BC *define congruent to each other circles* $\{\lambda\}$ *with equal and parallel to each other diameters* $\{AH\}$. *The circles* $\{\lambda\}$ *are orthogonal to the circle* κ' *on diameter* BC *and the radical axis* IJ *of each pair of them passes through the middle* M *of* BC.

Proof. For the proof draw the two segments from M to the intersections of the circles $\{\lambda\}$ with κ' and show that they are tangent to the circles $\{\lambda\}$ and the angle between them is constant and independent of the position of λ. The details are left as an exercise. □

5. Polars relative to the two parabolas

The polars of points relative to the two parabolas have some interesting properties, which we discuss in this section.

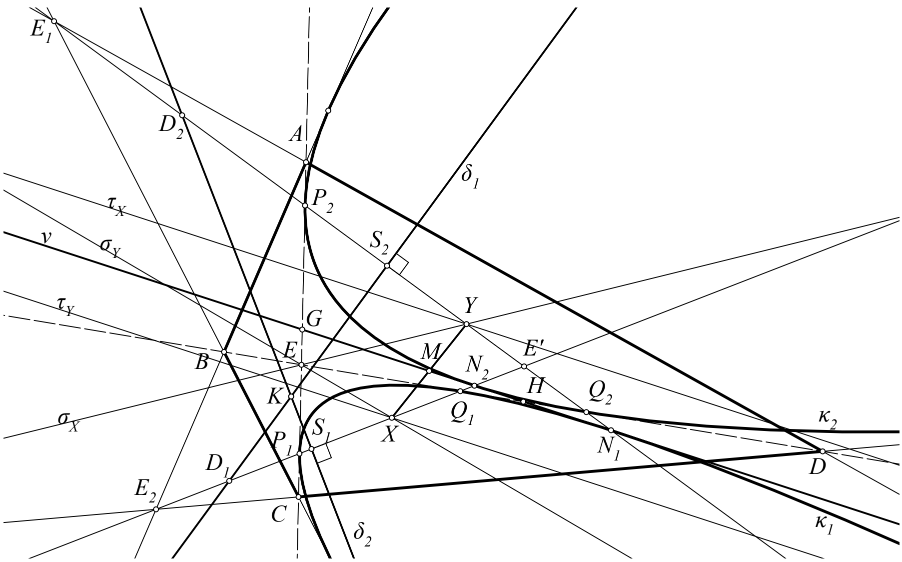

FIGURE 13. Polars of points relative to the two parabolas

Lemma 5.1. *For the points $\{X, Y\}$ lying respectively on the lines $\{P_1Q_1, P_2Q_2\}$ the polars $\{\sigma_X, \tau_X\}$ of X and the polars $\{\sigma_Y, \tau_Y\}$ relative to the parabolas $\{\kappa_1, \kappa_2\}$ have the following properties (See Figure 13).*

1. *All lines $\{\sigma_X, \sigma_Y\}$ pass through $E = (AC \cap BD)$.*
2. *If $Y = (P_2Q_2, \sigma_X)$, then $X = (P_1Q_1, \sigma_Y)$ and vice versa.*
3. *Lines $\{\tau_X, \tau_Y\}$ pass respectively through $\{Y, X\}$ and are parallel to the Newton line ν.*
4. *The middle M of the segment XY is on the Newton line ν.*

Proof. Nr-1 follows from the pole-polar reciprocity. By this, sinse X is on the polar of E rel. κ_1, the polar σ_X will pass through E. Analogous is the proof for σ_Y.

Nr-2 follows again from the same principle, since Y being on the polar σ_X of X implies that the polar σ_Y of Y passes through X.

Nr-3 for line τ_X follows from fact that P_1Q_1 is parallel to the axis of the parabola κ_2 and intersects it at its contact point N_2 with the Newton line ν. Hence this line is in the conjugate direction of P_1Q_1 relative to κ_2 and all polars of $X \in P_1Q_1$ are parallel to ν. Analogous is the proof for τ_Y.

Nr-4 follows from a basic property of parabolas, according to which the polar τ_X of X is parallel and at double the distance of X from the tangent ν at the intersection point

N_2 of the parabola with the parallel P_1Q_1 to the axis of the parabola through X ([13, p. 58]). □

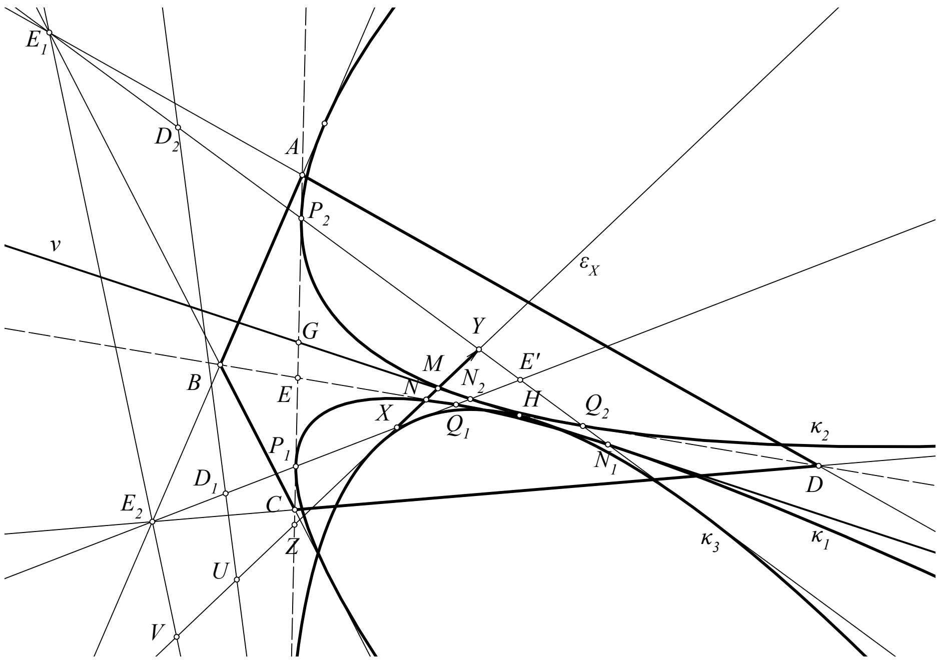

FIGURE 14. Homography $X \mapsto Y$ between two lines

Lemma 5.2. *With the notation and convetions of this section, the following are valid properties.*

(1) *The map $f : X \mapsto Y$ of line P_1Q_1 the line P_2Q_2, defined in the previous lemma, is a linear homographic relation between the two lines.*
(2) *The lines XY envelope a parabola κ_3.*
(3) *The parabola κ_3 is tangent to the 7 lines*: $\{E_1E_2, D_1D_2, AC, BD, P_1Q_1, P_2Q_2, \nu\}$.
(4) *On every tangent of κ_3 the previous 7 lines define respectively points* $\{V, U, Z, N, X, Y, M\}$, *such that the ratios defined by any three of them is constant. In particular* $E_2D_1/D_1P_1 = E_1D_2/D_2P_2$.

Proof. Nr-1 follows from the linearity of the *"polarity"*, mapping a point X to its polar σ_X relative to a conic κ. This is realized by using a system of homogeneous coordinates ([14, p. 174,I]), in which the conic is represented by a symmetric matrix A and the associated bilinear form $x^TAx = 0$. Then, for a point $X = x_0$, the corresponding polar σ_X is represented by the equation $x_0^TAx = 0$ ([14, p. 274,I], [12, p. 188]). And it is easily seen that restricting $X = x_0$ on a line, like $\alpha = P_1Q_1$ and taking the intersections Y of the polars $\sigma_X : x_0^TAx = 0$ with a line line $\beta = P_2Q_2$, defines a linear homography $f : X \mapsto Y$, represented in line coordinates through a function $f(x) = ux + v$, as stated.

Nr-2 is an immediate consequence of *nr-1*, the arguments and the references being the same with those used in section 3 for the establishment of the existence of the two parabolas $\{\kappa_1, \kappa_2\}$.

Nr-3 follows by driving X to special places on the line P_1Q_1 and considering the intersection Y of its polar σ_X with P_2Q_2. In this respect it is useful, inspecting again figure 13, to notice the coincidences:

$$\text{If } X = E_2, \text{ then } Y = E_1, \ \sigma_X = EE_1, \ \sigma_y = EE_2, \ \ \tau_X || \nu \text{ through } E_1.$$
$$\text{If } X = D_1, \text{ then } Y = D_2, \ \sigma_X = ED_2, \ \sigma_y = ED_1, \ \ \tau_X || \nu \text{ through } D_2.$$
$$\text{If } X = P_1, \text{ then } Y = P_2, \ \sigma_X = \sigma_Y = P_1P_2 = AC, \ \ \tau_X || \nu \text{ through } P_2.$$
$$\text{If } X = Q_1, \text{ then } Y = Q_2, \ \sigma_X = \sigma_Y = Q_1Q_2 = BD, \ \ \tau_X || \nu \text{ through } Q_2.$$
$$\text{If } X = N_2, \text{ then } Y = N_1, \ \tau_X = \tau_Y = \nu.$$

For example, when $X = E_2$, then, as noticed at the beginning of section 3, the polar σ_X being in that case the line E_1E, we see that $Y = E_1$, so that E_1E_2 is tangent to κ_3. Analogous are the proofs for the other lines of this list of coincidences.

Nr-5 is again a consequence of the fundamental property of parabolas for the segments intercepted on a tangent by three other tangents of it ([4, p. 52]). □

6. One-parameter-family of quadrangles

The structure of the three parabolas $\{\kappa_i\}$, discussed in the previous section, allows the determination of a one-parameter-family of quadrangles, containing the quadrangle of reference and having these parabolas playing the same role for all its members. To see this, we continue with the notation of the previous section and use the various tangents ε_X of the parabola κ_3. These lines define through their intersections with the lines $\{P_1Q_1, P_2Q_2\}$ the points $\{X = E_2, Y = E_1\}$ (See Figure 15). By lemma 5.2, the paral-

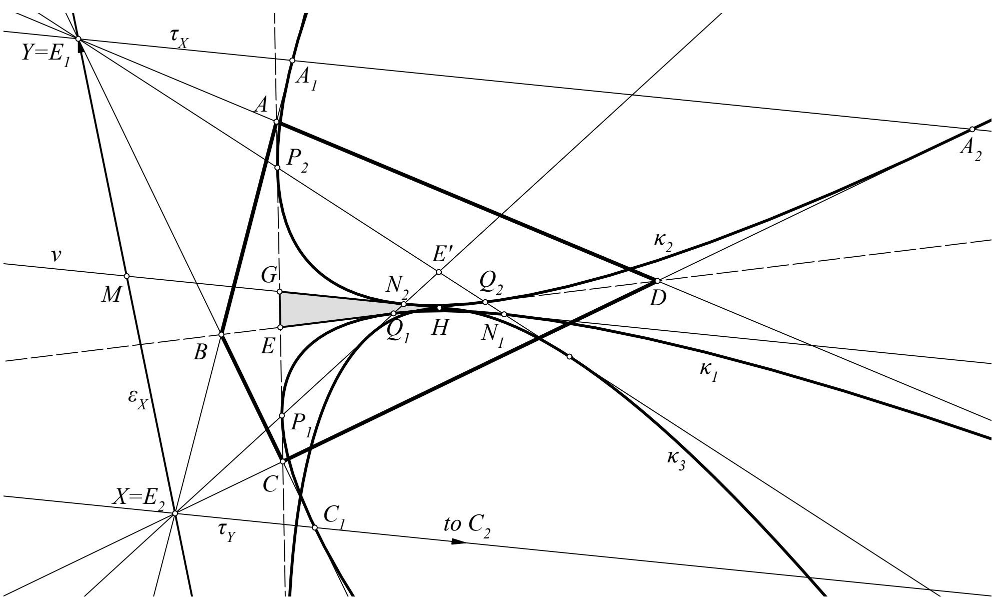

Figure 15. One-parameter-family of quadrangles containing $ABCD$

lels $\{\tau_X, \tau_y\}$ to the Newton line ν, respectively from $\{E_1, E_2\}$, are the polars of $\{E_2, E_1\}$ relative to $\{\kappa_2, \kappa_1\}$ and intersect these parabolas at pairs of points $\{(A_1, A_2), (C_1, C_2)\}$.

By the aforementioned lemma, the quadrangle $q_X = ABCD$ with vertices at the intersections of the four lines $\{E_1C_1, E_1C_2, E_2A_1, E_2A_2\}$ has the three parabolas $\{\kappa_i\}$ playing the roles analyzed in the previous sections. We summarize these results in the following main theorem.

Theorem 6.1. *The diagonals $\{AC, BD\}$ of the quadrangle of reference define through their contact points with the parabolas $\{\kappa_1, \kappa_2\}$ respectively lines $\{P_1Q_1, P_2Q_2\}$ and the tangents $\{\varepsilon_X\}$ to the parabola κ_3 define on these lines the intersection points $\{E_1, E_2\}$ of opposite sides of a one-parameter-family of quadrangles $\{q_t\}$ with the following properties:*

(1) *The family contains the quadrangle of reference q as a special member.*
(2) *All the quadrangles q_t have their two pairs of opposite sides tangent respectively to the two parabolas $\{\kappa_1, \kappa_2\}$.*
(3) *The quadrangles $\{q_t\}$ besides the common diagonal lines, share in addition the same Newton line and the same ratio of diagonals.*

Notice that the construction of the tangent ε_X of the parabola κ_3, from which depends the whole configuration of the family, can be done by starting with an arbitrary point $X = E_2$ on P_1Q_1. In fact, by lemma 5.2/*nr-4*, the equality of the ratios $E_2P_1/P_1Q_1 = E_1P_2/P_2Q_2$, allows the definition of the corresponding point $Y = E_1$ on P_2Q_2 and with this the tangent $\varepsilon_X = E_2E_1$ of κ_3.

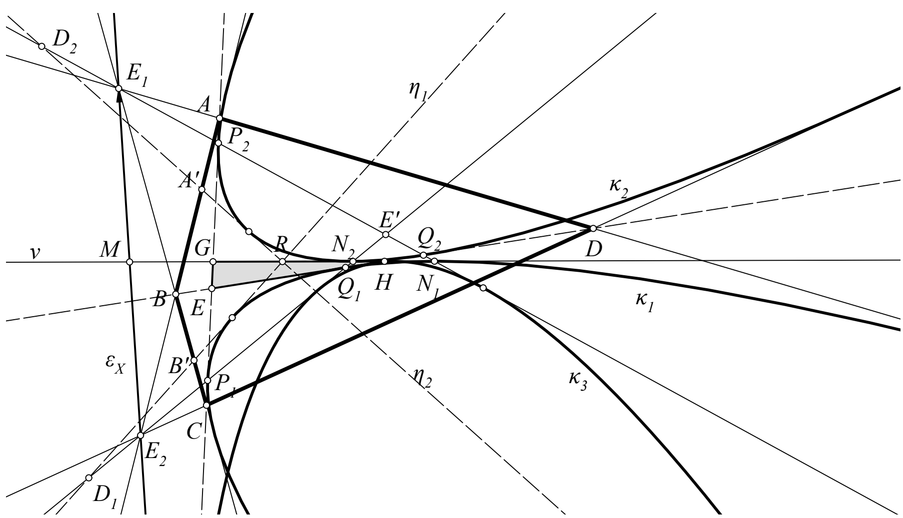

FIGURE 16. Family of quadrangles defined by lines E_1E_2 tangent to κ_3

Figure 16 emphasizes the common elements to all members of the family. Among these elments are: the three parabolas, their common circumscribing triangle EGH, the diagonal lines and the ratio $r = AC/BD$, the Newton line ν, the lines $\{P_1Q_1, P_2Q_2\}$ carrying the contact points of the parabolas with the sides of the triangle EGH and the points $\{D_1, D_2\}$, which are intersections of the directrices of the parabolas with the lines $\{P_1Q_1, P_2Q_2\}$. By lemma 5.2/*nr-3*, the line D_1D_2 is also tangent to κ_3, thus defining a particular member of the family. Next theorem shows that this is the unique cyclic member of the family.

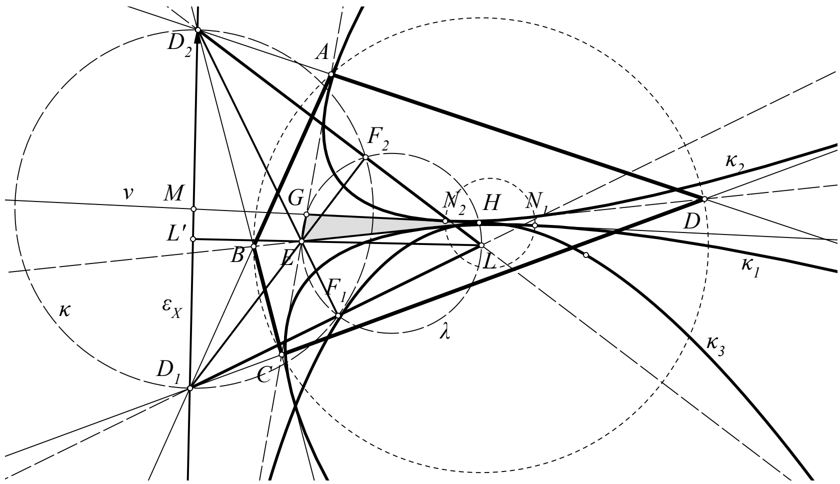

FIGURE 17. The cyclic quadrangle of the one-parameter-family

Theorem 6.2. *The one-parameter-family contains precisely one cyclic quadrangle, defined by that tangent ε_X of the parabola κ_3, which coincides with the line D_1D_2.*

Proof. Most of the work for the proof has been done in section 4. There it was shown that the triangle D_1D_2L, where $L = (D_1F_1, D_2F_2)$, has E as its orthocenter and the circle $\lambda = (EGH)$, carrying also the focals $\{F_1, F_2\}$ of the parabolas, intersects the circle κ, having D_1D_2 for diameter, orthogonally at these two points (See Figure 17). In [8] was shown that this is a necessary and sufficient condition for $ABCD$ to be cyclic. Thus, for $\varepsilon_X = D_1D_2$ the corresponding quadrangle is indeed cyclic.

To show that this is the unique genuine cyclic quadrangle contained in the family, we see first that there is a second position of the tangent ε_X, which has a corresponding circle κ on diameter E_1E_2 also orthogonal to λ. This happens when ε_x coincides with the Newton line ν, in which case $E_1E_2 = N_1N_2$. But, as can easily deduced from lemma 3.4 and theorem 4.2, the circle with diameter N_1N_2 is orthogonal to λ, and the corresponding quadrangle degenerates to a line segment. Thus, taking into account the last cited reference, in order to show that there is no other genuine cyclic quadrangle of the family, it suffices to show that from all these circles with diameter E_1E_2 on corresponding tangents ε_X of κ_3, only the two aforementioned circles with corresponding diameters $\{D_1D_2, N_1N_2\}$ are orthogonal to the circle λ.

This would be easy to show if the circles κ, whose centers are on ν, were members of a pencil ([9, p. 106]). But they are not, and, in order to prove this, we must resort to a short calculation. This can be done as follows. Referring to figure 16, we recall that the diameters E_1E_2 of all these circles, are defined by considering an arbitrary point $X = E_2$ on P_1Q_1, which is a fixed tangent of κ_3 and taking $Y = E_1$ on the other fixed tangent P_2Q_2, such that $E_2P_1/P_1Q_1 = E_1P_2/P_2Q_2$. Considering now the oblique coordinate system of lines $\{\nu, Q_1P_1\}$ intersecting at N_2, we see easily that the coordinates $\{M(x), E_2(y)\}$ of the middle M of E_1E_2 and of E_2 satisfy a linear relation $y = ax + b$, with constants $\{a, b\}$. Let us denote now by κ_x and M_x the variable circle with diameter E_1E_2 and its center M. Let also f denote the inversion relative to the circle $\lambda(O)$. In order to have orthogonality of κ_x on λ, it is necessary and sufficient that the centers $\{M_x, N_x\}$ of the circles $\{\kappa_x, \nu_x = f(\kappa_x)\}$ coincide.

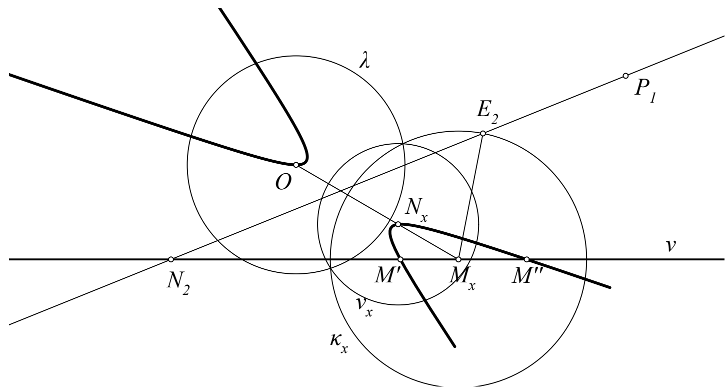

FIGURE 18. The centers of circles ν_x lie on a conic

Figure 18 illustrates the proof idea. The centers N_x of the circles $\nu_x = f(\kappa_x)$ describe a conic. Having found already two intersection points $\{M', M''\}$ of the geometric locus with ν, which are the centers of the two circles with diameters $\{D_1D_2, N_1N_2\}$, if we prove that the centers $\{N_x\}$ describe a conic, then there can be no other intersections with the line ν and the theorem is proved.

That points $\{N_x\}$ describe a conic, can be proved by using the easily deducible relation between the centers M_x and N_x of the circles:

$$N_x = O + r^2 \frac{OM_x}{|OM_x|^2 - r_x^2}, \tag{2}$$

where $\{r, r_x\}$ are respectively the radii of $\{\lambda, \kappa_x\}$. Introducing unit vectors $\{\overline{u}, \overline{v}\}$ along the lines $\{\nu, E'P_1\}$, we can represent the points in the form

$$M_x = x \cdot \overline{u}, \quad E_2 = (ax + b) \cdot \overline{v}, \quad O = s \cdot \overline{u} + t \cdot \overline{v}.$$

Introducing these into equation 2, we find, after simplification, that

$$ON_x/OM_x = \frac{r^2}{Ax^2 + Bx + C}, \tag{3}$$

for some appropriate constants $\{A, B, C\}$. The proof of the theorem is completed by the following lemma, whose proof is left as an exercise. □

Lemma 6.3. *Given a line ε, a point $O \notin \varepsilon$ and a variable point $M_x \in \varepsilon$, the point N_x lying on the line OM_x and satisfying*

$$\frac{ON_x}{OM_x} = \frac{1}{Ax^2 + Bx + C},$$

for some constant numbers $\{A, B, C\}$ and a system of line coordinates $\{x\}$ for ε, describes a conic, passing through O and having there a tangent parallel to ε.

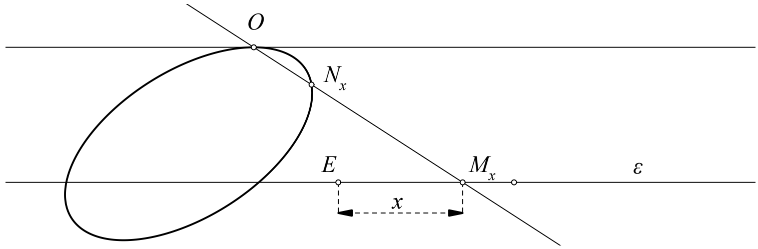

FIGURE 19. Conic generation through a quadratic function

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