THE ARBELOS IN WASAN GEOMETRY, PROBLEMS OF IZUMIYA AND NAITŌ

HIROSHI OKUMURA

Abstract. We generalize two sangaku problems involving an arbelos proposed by Izumiya and Naitō, and show the existence of six non-Archimedean congruent circles.

1. Introduction

In this article we generalize two sangaku problems involving an arbelos proposed by Izumiya and Naitō. Let α, β and γ be the three semicircles with diameters AO, BO and AB, respectively for a point O on the segment AB constructed on the same side of AB. The area surrounded by the three semicircles is called arbelos (see Figure 1). The radical axis of α and β is called the axis. Let a and b be the radii of α and β, respectively, and let δ_α (resp. δ_β) be the incircle of the curvilinear triangle made by α (resp. β), γ and the axis. The two circles δ_α and δ_β have common radius $r_{\mathrm{A}} = ab/(a+b)$ and are called the twin circles of Archimedes.

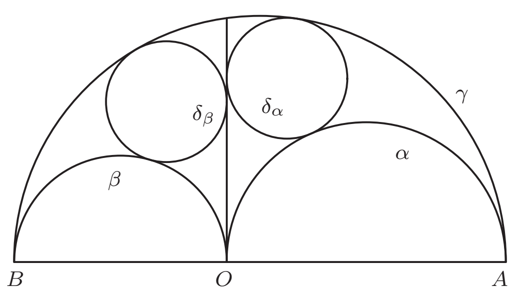

Figure 1.

Izumiya's problems appeared in a sangaku in Saitama hung in 1866, which is as follows [6] (see Figure 2).

Problem 1. If α and β are congruent and the tangent of α from B meets γ in a point C, show that the inradius of the curvilinear triangle formed by α, γ and the perpendicular from C to AB equals $a/9$.

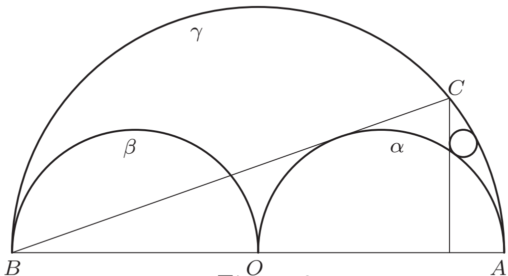

Figure 2.

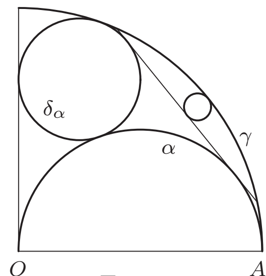

Figure 3.

Naitō's problem appeared in a sangaku in Fukushima hung in 1983 (the sangaku seems to be made in modern day times), which is as follows [1] (see Figure 3).

Problem 2. If α and β are congruent, show that the radius of the circle touching the remaining external common tangent of α and δ_α and the arc of γ cut by the tangent at the midpoint equals $a/9$.

2. Generalization

We now consider the case in which the semicircles α and β are not always congruent. We use the next proposition (see Figure 4).

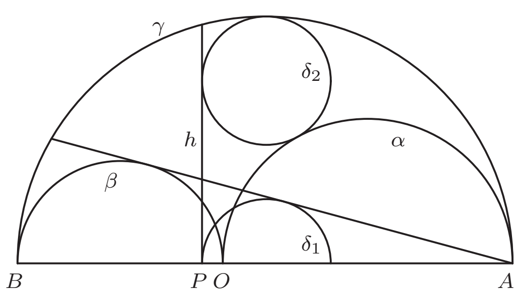

Figure 4.

Proposition 2.1. *For a point P on the segment AB, let h be the perpendicular to AB at P. If δ_1 is the circle touching h at P from the side opposite to B and the tangent of β from A and δ_2 is the circle touching α externally γ internally and h from the same side as δ_1, then δ_1 and δ_2 are congruent.*

Proof. The radius of δ_2 is proportional to the distance between its center and the radical axis of α and γ [2, p. 108], while δ_2 coincides with β if $P = B$. Also the radius of δ_1 is proportional to the distance between its center and the point A, and δ_1 coincides with β if $P = B$. □

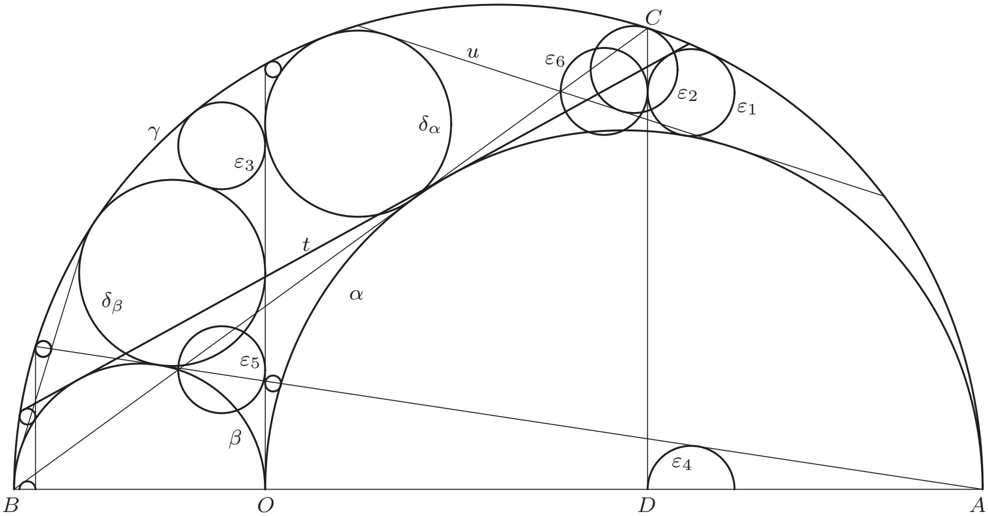

Figure 5.

Theorem 2.2. *Let C be the point of intersection of γ and the tangent of α from B and let D be the foot of perpendicular from C to AB. The incircle of the curvilinear triangle made by α, γ and CD is denoted by ε_1. Let u be the remaining external common tangent of α and δ_α. The circle touching u and the arc of γ cut by u at the midpoint is denoted by ε_2. The incircle of the curvilinear triangle made by γ, δ_β and the axis is denoted by ε_3. The circle touching the tangent of β from A and CD at D from the side opposite to B is denoted by ε_4. The smallest circle passing through the point of intersection of β and BC and touching the axis is denoted by ε_5. The smallest circle passing through the point of intersection of BC and u and touching the line CD is denoted by ε_6. Then the following statements hold.*
(i) *The six circles $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_6$ are congruent and have common radius*

$$\frac{a^2b}{(a+2b)^2}.$$

(ii) *The circle ε_1 touches the line t, and the circle ε_2 touches γ at C.*

Proof. We assume that r_i is the radius of ε_i, $d = a + 2b$, E is the point of intersection of BC and β, F is the foot of perpendicular from E to the axis, G is the point of tangency of α and BC, H is the center of α, and BC meets the axis and u in points J and K, respectively (see Figure 6).

Since the three segments CA, GH and EO are parallel and H is the midpoint of AO, G is the midpoint of CE. While the line BC is the internal common tangent of α and δ_α [3, p. 212]. Therefore G is also the midpoint of JK. Hence $|EJ| = |CK|$, i.e., the circles ε_5 and ε_6 are congruent. Since the triangles BGH, BEO and OFE are similar, $a/d = |OE|/(2b) = |EF|/|OE|$. Therefore $|OE| = 2ab/d$ and $|EF| = 2a^2b/d^2$. Hence $r_5 = a^2b/d^2 = r_6$, and $|OF| = 4ab\sqrt{(a+b)b}/d^2$ from the right triangle OFE.

The last equation implies $|EF| = \frac{a|OF|}{2\sqrt{(a+b)b}}$. Let $x = |BD|$. Then $|CD| = \frac{ax}{2\sqrt{(a+b)b}}$ from the similar triangles OFE and BDC. Therefore we have

$$x(2(a+b)-x) = |CD|^2 = \frac{a^2x^2}{4(a+b)b}.$$

Solving the equation for x, we get $x = 8b(a+b)^2/d^2$. Therefore

$$|AD| = 2(a+b) - x = 2a^2(a+b)/d^2.$$

Therefore $r_4 = b|AD|/|AB| = a^2b/d^2 = r_1$ by Proposition 2.1. Meanwhile ε_3 and the incircle of the curvilinear triangle made by α, γ and t have radius a^2b/d^2 [5, Theorem 9]. Therefore the last circle coincides with ε_1, i.e., ε_1 touches t. While we have also shown that ε_1 and ε_2 are congruent in [4]. This proves (i) and the first half part of (ii).

Let ζ be the circle with center C passing through G. We invert the figure in ζ. Then the circles α and δ_α are orthogonal to ζ, i.e, they are fixed by the inversion. The line u, which intersects ζ, is inverted to a circle intersecting ζ touching α and δ_α passing through C. Therefore γ is the inverse of u. This implies that the points of intersection of γ and u lie on ζ. Hence C is the midpoint of the arc of γ cut by u. Therefore ε_2 touches γ at C. This proves the second half part of (ii). □

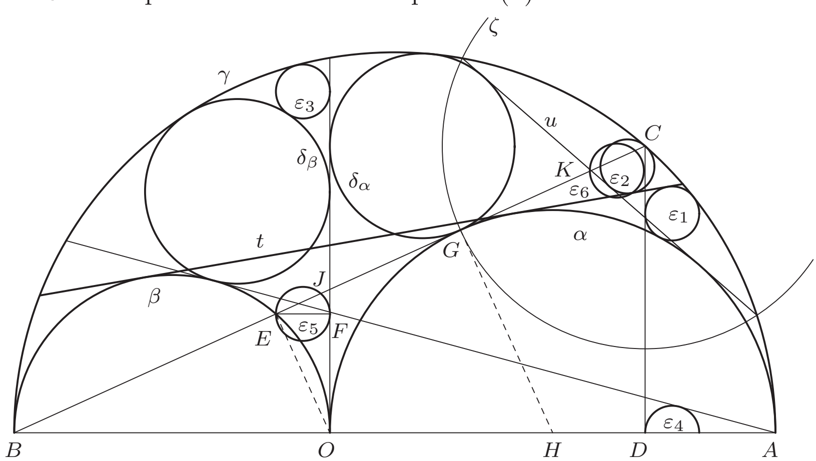

Figure 6.

Circles of radius r_A are called Archimedean circles [3]. Therefore we now have six non-Archimedean congruent circles $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_6$. Exchanging the roles of α and β, we get another six non-Archimedean congruent circles of radius $ab^2/(2a+b)^2$, which are denoted in Figure 5.

3. The circle associated with a point on γ

For a circle δ touching α externally and γ internally, if P is the point of intersection of γ and the internal common tangent of δ and α closer to B, we say that δ is associated with P. As mentioned in the proof of Theorem 2.2, the circle δ_α is associated with the point B (see Figure 6). We can also consider that the point circle A is associated with the point A itself, because the perpendicular to AB at A can be considered as the internal common tangent of the point circle A and α. Let I be the point of intersection of γ and the axis. The next theorem gives the circle associated with the point I.

Theorem 3.1. *The internal common tangent of α and ε_1 passes through I.*

Proof. Let ρ be the circle with center I passing through O. We invert the figure in ρ (see Figure 7). Then α and β are fixed. While t, which intersects ρ, is inverted into the circle with center I touching α and β intersecting ρ. Therefore γ is the inverse of t. Hence the figure consisting of α, γ and t is fixed by the inversion. This implies that ε_1 is also fixed. Since α and ε_1 are orthogonal to ρ, their point of tangency lies on ρ, and their common internal tangent passes through I. □

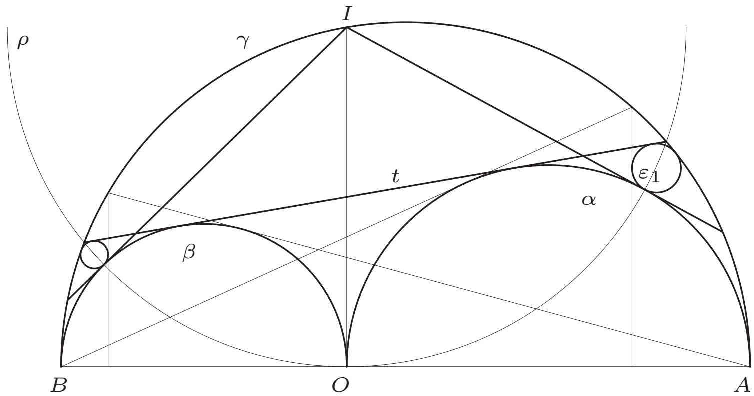

Figure 7.

The proof also shows that the points of intersection of γ and t lies on ρ. Therefore I is the midpoint of the arc of γ cut by t.

References

[1] F. W. S. Association ed. *The sangaku in Fukushima*. Sõju Shuppan, 1989.

[2] J. L. Coolidge. *A treatise on the circle and the sphere*. Chelsea Publishing Co., Bronx, N.Y., 1971, page 602. Reprint of the 1916 edition.

[3] C. W. Dodge, T. Schoch, P. Y. Woo, and P. Yiu. Those ubiquitous Archimedean circles. *Math. Mag.*, 72(3):202–213, 1999. ISSN: 0025-570X. DOI: 10.2307/2690883. URL: https://doi.org/10.2307/2690883.

[4] H. Okumura. Are those Archimedean triplet circles really triplets? *Normat*, 60(1):9–14, 48 (2013), 2012. ISSN: 0801-3500.

[5] H. Okumura and M. Watanabe. The twin circles of Archimedes in a skewed arbelos. *Forum Geom.*, 4:229–251, 2004. ISSN: 1534-1178.

[6] *The Sangaku in Saitama*. Saitama Prefectural Library, 1969.

TAKAHANADAI MAEBASHI GUNMA 371-0123, JAPAN
Email address: hokmr@yandex.com