ANOTHER SIMPLE PROOF OF THE GOORMAGHTIGH THEOREM AND THE GENERALIZED GOORMAGHTIGH THEOREM

NGUYEN MINH HA AND PHAM NAM KHANH

Abstract. We introduce a simple proof of the Goormaghtigh theorem and the generalized Goormaghtigh theorem using the concept of cross ratio.

1. Introduction

In 1930, René Goormaghtigh, a French electrical engineer, proposed an interesting theorem named after himself [2].

Theorem 1.1. *Let ABC be a triangle and let P be a point distinct from A, B, C. Let Δ be a line through P. Three points A', B', C' belong to the lines BC, CA, AB respectively such that PA', PB', PC' are the images of PA, PB, PC respectively by reflection with respect to axis Δ. Then, A', B', C' are collinear.*

The proofs of Theorem 1.1 can be found in [1, 3, 4] and [6].

In 2014, Dao Thanh Oai and Tran Quang Hung almost simultaneously and independently expanded Theorem 1.1 as follows [4, 5]:

Theorem 1.2. *Let ABC be a triangle and let P be a point distinct from A, B, C. Let Δ and Δ' be two lines intersecting at P. Three points A', B', C' belong to the lines BC, CA, AB respectively such that*

$$(PA, PA'; \Delta, \Delta') = (PB, PB'; \Delta, \Delta') = (PC, PC'; \Delta, \Delta') = -1.$$

Then, A', B', C' are collinear.

If $\Delta \perp \Delta'$ then Theorem 1.2 is actually coincides with Theorem 1.1. It is possible to find proofs of Theorem 1.2 in [4] and [6]. In this article, we will introduce simple proofs of Theorem 1.1 and Theorem 1.2 using the concept of the cross ratio.

In order to facilitate your observation, please note that in this article, the cross ratio of the pencil (PA, PB, PC, PD) is simply denoted by $P\,(ABCD)$.

2. Preliminary Results

In order to prove Theorem 1.1 and Theorem 1.2, we need to have following 3 lemmas.

Lemma 2.1. *Axial reflection preserves cross ratios.*

Proof. Obviously, axial reflection preserves ratios. Therefore, axial reflection preserves cross ratios. □

Lemma 2.2. *Let $(O_1O_2, O_1A, O_1B, O_1C)$ and $(O_2O_1, O_2A, O_2B, O_2C)$ be two pencils. Then, A, B, C are collinear if and only if*

$$O_1\,(O_2ABC) = O_2\,(O_1ABC)\,.$$

Proof. Let A, B, C are collinear. Denote by O the intersection of O_1O_2 with AB. It is easy to see that

$$O_1(O_2ABC) = O_1(OABC) = (O, A; B, C) = O_2(OABC) = O_2(O_1ABC).$$

Let $O_1(O_2ABC) = O_2(O_1ABC)$. Denote by O, C_1, C_2 the intersections of O_1O_2, O_1C, O_2C with AB. It is easy to see that

$$(O, A; B, C_1) = O_1(OABC_1) = O_1(O_2ABC) = \\ = O_2(O_1ABC) = O_2(OABC_2) = (O, A; B, C_2).$$

Therefore, $C_1 = C_2 = C$. Thus, C belongs to AB. In other words, A, B, C are collinear. □

Remark. Note that we can change in Lemma 2.3 the condition

$$O_1(O_2ABC) = O_2(O_1ABC)$$

to

$$O_1(AO_2BC) = O_2(AO_1BC)$$

and some other conditions.

Lemma 2.3. *Let ten points P, Q, A, A', B, B', C, C', D, D' be collinear and*

$$(A, A'; P, Q) = (B, B'; P, Q) = (C, C'; P, Q) = (D, D'; P, Q) = -1.$$

Then $(A, B; C, D) = (A', B'; C', D')$.

Proof. Let Δ be the line containing the given points. Without loss of generality, assume that Δ is an axis and the points P, Q, A, A', B, B', C, C', D, D' have coordinates 0, q, a, a', b, b', c, c', d, d' respectively.

Since

$$(A, A'; P, Q) = (B, B'; P, Q) = (C, C'; P, Q) = (D, D'; P, Q) = -1,$$

we have

$$2aa' = q(a + a');\ 2bb' = q(b + b');\ 2cc' = q(c + c');\ 2dd' = q(d + d').$$

Therefore,

$$\frac{1}{a} + \frac{1}{a'} = \frac{1}{b} + \frac{1}{b'} = \frac{1}{c} + \frac{1}{c'} = \frac{1}{d} + \frac{1}{d'}.$$

Thus,

$$\frac{a-c}{ac} = -\frac{a'-c'}{a'c'};\ \frac{b-c}{bc} = -\frac{b'-c'}{b'c'};\ \frac{a-d}{ad} = -\frac{a'-d'}{a'd'};\ \frac{b-d}{bd} = -\frac{b'-d'}{b'd'}.$$

Taking each side of the first equation to be divided by each side of the second equation, each side of the third equation to be divided by each side of the forth equation, we have

$$\frac{a-c}{b-c}\cdot\frac{b}{a} = \frac{a'-c'}{b'-c'}\cdot\frac{b'}{a'};\ \frac{a-d}{b-d}\cdot\frac{b}{a} = \frac{a'-d'}{b'-d'}\cdot\frac{b'}{a'}.$$

Dividing respective sides of the above two equations, we have

$$\frac{a-c}{b-c} : \frac{a-d}{b-d} = \frac{a'-c'}{b'-c'} : \frac{a'-d'}{b'-d'}.$$

In other words, $(A, B; C, D) = (A', B'; C', D')$. □

Remark. Note that in the condition of Lemma 2.2 points A, B, C, D may coincide with points A', B', C', D'.

3. Proof of Theorem 1.2

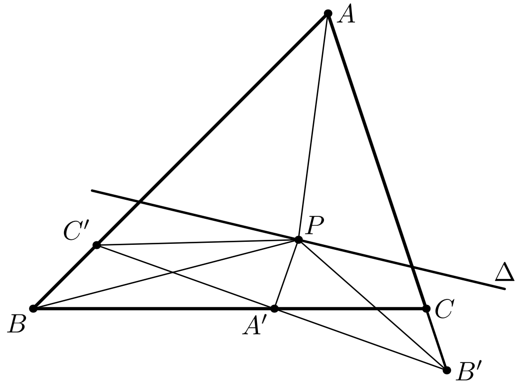

Fig. 1.

By Lemma 2.1 we have $P\,(AA'B'C') = P\,(A'ABC)$. By Lemma 2.2 we have that $P\,(A'ABC) = A\,(A'PBC)$. Since A, B, C' and A, C, B' are collinear, it follows that $A\,(A'PBC) = A\,(A'PC'B')$, and by the symmetry of a cross ration we obtain $A\,(A'PC'B') = A\,(PA'B'C')$. Thus, by Lemma 2.2 again, A', B', C' are collinear.

4. Proof of Theorem 1.2

Consider an arbitrary line ℓ not passing through P and intersecting lines Δ, Δ', PA, PA', PB, PB', PC, PC' at points T, T', X, X', Y, Y', Z, Z' respectively (fig. 2).

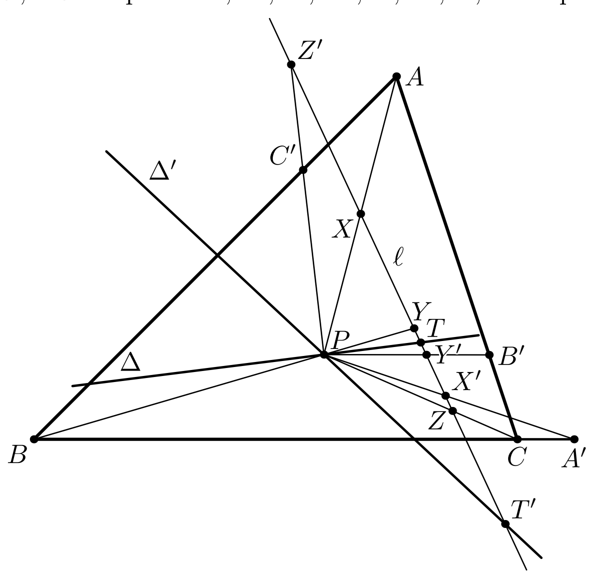

Fig. 2.

Since

$$(PA, PA'; \Delta, \Delta') = (PB, PB'; \Delta, \Delta') = (PC, PC'; \Delta, \Delta') = -1,$$

we have

$$P\left(XX'TT'\right) = P\left(YY'TT'\right) = P\left(ZZ'TT'\right) = -1.$$

Note that since $(A, B; C, D) \cdot (B, A; C, D) = 1$ for any points A, B, C, D. We have

$$P\left(X'XTT'\right) = P\left(Y'YTT'\right) = P\left(Z'ZTT'\right) = -1.$$

According to Lemma 2.3, $(X, X'; Y', Z') = (X', X; Y, Z)$. Similarly to the proof of Theorem 1.1, we can deduce that

$$P\left(AA'B'C'\right) = P\left(XX'Y'X'\right) = P\left(X'XYZ\right) = P\left(A'ABC\right).$$

By Lemma 2.2 $P\left(A'ABC\right) = A\left(A'PBC\right)$. Moreover,

$$A\left(A'PBC\right) = A\left(A'PC'B'\right) = A\left(PA'B'C'\right).$$

Then, by Lemma 2.2, A', B', C' are collinear.

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(N. M. Ha) Hanoi University of Education, Hanoi, Vietnam
E-mail address, N. M. Ha: `minhha27255@yahoo.com`

(P. N. Khanh) High School for Gifted Students Amsterdam, Hanoi, Vietnam
E-mail address, P. N. Khanh: `phamnamkhanh4265@gmail.com`