ON SOME GEOMETRIC RELATIONS OF A TRIANGLE

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Abstract. For a triangle ABC we consider the circles passing through a vertex of the triangle and tangent to the oppisite side as well as to the circumcircle. We prove that $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{2}{R}+\frac{1}{r}$, where r_a, r_b, r_c are the radii of the these three circles, and R, r are the circumradius and the inradius of the triangle ABC, respectively. This equation generalizes the main result from [1].

1. The main results

Let R and r be the circumradius and the inradius of an arbitrary triangle ABC. Denote the center of the circumcircle of ABC by O. Consider the circle tangent to BC and to the circumcircle of ABC at the vertex A. We denote its center by O_A. Denote by r_a the radius of this circle ($r_a = AO_A$). Analogously, we define r_b and r_c. We will also use the standard notation for lengths and angles of the triangle: $AB = c$, $AC = b$, $BC = a$, $\angle CAB = \alpha$, $\angle ABC = \beta$, $\angle BCA = \gamma$, $p = \frac{a+b+c}{2}$. It is not hard to see that $r_a = \frac{rp}{a\cos^2\left(\frac{\beta-\gamma}{2}\right)}$ [1, Lemma 2].

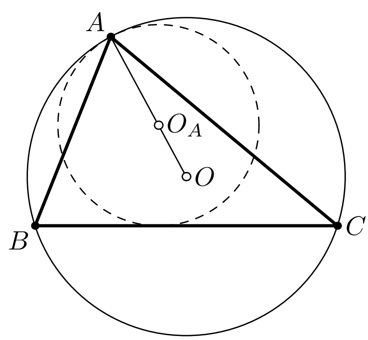

Fig. 1.

In [1], the following inequalities have been proved

$$\frac{4}{R} \le \frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c} \le \frac{2}{r}. \tag{1}$$

So the authors get a new interpritation for the well-known Euler's inequality $R \ge 2r$. In this paper, we prove the following

Theorem 1.

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c} = \frac{2}{R}+\frac{1}{r}.$$

Before proving the theorem, let us consider a supplementary lemma.

Lemma 2. *Suppose A_0 is the midpoint of the arc BC not containing the point A. The points B_0 and C_0 are defined analogously (see Fig. 2). Then the area of the hexagon $AC_0BA_0CB_0$ is equal to pR.*

Proof. First, we note that $S_{AC_0BA_0CB_0} = S_{OBA_0C} + S_{OCB_0A} + S_{OAC_0B}$.

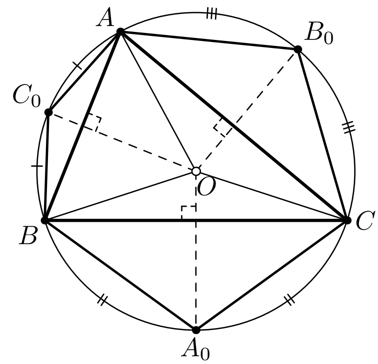

Fig. 2.

However $S_{OBA_0C} = \dfrac{aR}{2}$, $S_{OCB_0A} = \dfrac{bR}{2}$ and $S_{OAC_0B} = \dfrac{cR}{2}$. Thus $S_{AC_0BA_0CB_0} = pR$. □

Now we can prove Theorem 1.

Proof of Theorem 1. Let ω be the circumcircle of the triangle ABC, ω_A be the tangent circle at A to the circumcircle of the triangle ABC and to the side BC, and A_1 be the tangent point of ω_A and the side BC (see Fig. 3).

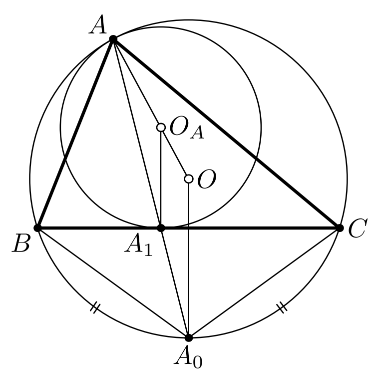

Fig. 3.

Consider the homothetic transformation with center at A such that ω_A is mapped to ω. Then the line BC is mapped to the tangent of ω parallel to BC. Therefore the point A_1 is mapped to A_0, where A_0 is the midpoint of the circular arc BC not containing the point A. In particular, by the Archimedean lemma, AA_1 is a bisector of the angle BAC. Hence we have

$$\frac{R - r_a}{r_a} = \frac{OO_A}{O_AA} = \frac{A_0A_1}{A_1A} = \frac{S_{BA_0A_1}}{S_{AA_1B}} = \frac{S_{CA_0A_1}}{S_{CA_1A}} = \frac{S_{BA_0C}}{S_{ABC}}.$$

By Lemma 2 we get

$$\frac{R-r_a}{r_a}+\frac{R-r_b}{r_b}+\frac{R-r_c}{r_c}=\frac{S_{ABC_0}+S_{BCA_0}+S_{CAB_0}}{S_{ABC}}=\frac{pR-S_{ABC}}{S_{ABC}}.$$

It implies that

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{2}{R}+\frac{1}{r}.$$

□

Using Theorem 1, one can obtain a new proof of inequalities from [1]:

Corollary 3. *For any triangle ABC the following inequalities hold:*

$$\frac{4}{R}\le\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\le\frac{2}{r}.$$

Moreover, we have equality if and only if the triangle ABC is equilateral.

Proof. Since $R\ge 2r$, we have

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}+\frac{2}{R}\le\frac{1}{r}+\frac{2}{2r}=\frac{2}{r}$$

and

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\ge\frac{2}{R}+\frac{1}{R/2}=\frac{4}{R}.$$

It is well-known that $R=2r$ if and only if the triangle ABC is equilateral (see [2]). So we have equalities if and only if the triangle ABC is equilateral. □

Theorem 4. *For any triangle ABC the following relation*

$$\frac{1}{ar_a}+\frac{1}{br_b}+\frac{1}{cr_c}=\frac{p^2+r^2+4R^2+2Rr}{4rpR^2}$$

holds.

Proof. Since $r_a=\dfrac{rp}{a\cdot\cos^2\left(\dfrac{\beta-\gamma}{2}\right)}$, we have

$$\frac{1}{ar_a}=\frac{\cos^2\left(\dfrac{\beta-\gamma}{2}\right)}{rp}=\frac{1+\cos(\beta-\gamma)}{2rp}.$$

Hence,

$$\frac{1}{ar_a}+\frac{1}{br_b}+\frac{1}{cr_c}=\frac{1}{2rp}\left(3+\cos(\beta-\gamma)+\cos(\alpha-\beta)+\cos(\alpha-\gamma)\right)=$$

$$=\frac{1}{2rp}\left(3+(\cos\beta\cos\gamma+\cos\alpha\cos\beta+\cos\alpha\cos\gamma)+(\sin\beta\sin\gamma+\sin\alpha\sin\beta+\sin\alpha\sin\gamma)\right).$$

It is known that

$$\cos\beta\cos\gamma+\cos\alpha\cos\beta+\cos\alpha\cos\gamma=\frac{p^2+r^2-4R^2}{4R^2},$$

$$\sin\beta\sin\gamma+\sin\alpha\sin\beta+\sin\alpha\sin\gamma=\frac{p^2+r^2+4Rr}{4R^2}$$

(see [2]). It implies

$$\frac{1}{ar_a}+\frac{1}{br_b}+\frac{1}{cr_c}=\frac{1}{2rp}\left(3+\frac{p^2+r^2-4R^2}{4R^2}+\frac{p^2+r^2+4Rr}{4R^2}\right)=$$
$$=\frac{1}{8rpR^2}\left(12R^2+2p^2+2r^2-4R^2+4Rr\right)=\frac{1}{4rpR^2}\left(4R^2+p^2+r^2+2Rr\right).$$

This completes the proof of Theorem 4. □

Corollary 5. *For any triangle ABC the following inequalities hold:*

$$\frac{4}{\sqrt{3}R^2}\leq\frac{1}{ar_a}+\frac{1}{br_b}+\frac{1}{cr_c}\leq\frac{1}{\sqrt{3}r^2}.$$

Moreover, we have equality if and only if the triangle ABC is equilateral.

Proof. In [2], it is proved that $16Rr-5r^2\leq p^2\leq 4R^2+4Rr+3r^2$ and $3\sqrt{3}r\leq p\leq\frac{3\sqrt{3}}{2}R$. Hence,

$$\frac{1}{ar_a}+\frac{1}{br_b}+\frac{1}{cr_c}=\frac{1}{4rpR^2}(4R^2+p^2+r^2+2Rr)\leq$$
$$\leq\frac{4R^2+\frac{27}{4}R^2+\left(\frac{R}{2}\right)^2+2R\cdot\frac{R}{2}}{4r\cdot 3\sqrt{3}r\cdot R^2}=\frac{4+\frac{27}{4}+\frac{1}{4}+1}{12\sqrt{3}r^2}=\frac{1}{\sqrt{3}r^2}$$

and

$$\frac{1}{ar_a}+\frac{1}{br_b}+\frac{1}{cr_c}=\frac{1}{4rpR^2}(4R^2+p^2+r^2+2Rr)\geq\frac{4R^2+(16Rr-5r^2)+r^2+2Rr}{4r\cdot R^2\cdot\frac{3\sqrt{3}}{2}R}=$$
$$=\frac{4R^2+18Rr-4r^2}{2rR^3\cdot 3\sqrt{3}}=\frac{2R^2+9Rr-2r^2}{rR^3\cdot 3\sqrt{3}}\geq\frac{4}{\sqrt{3}R^2}$$

because the last inequality is equivalent to $2R^2-3Rr-2r^2\geq 0$, i.e. $(R-2r)(2R+r)\geq 0$ (see the Euler's inequality $R\geq 2r$). So

$$\frac{1}{ar_a}+\frac{1}{br_b}+\frac{1}{cr_c}\geq\frac{4}{\sqrt{3}R^2}.$$

Moreover, we have the equality if and only if the triangle ABC is equilateral. □

In the triangle ABC let h_a, h_b, h_c be the lengths of the altitudes from the vertices A, B, C, respectively. The following theorem holds.

Theorem 6. *For any triangle ABC the following relation*

$$\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}=2+\frac{p^2+2Rr+r^2}{2R^2}$$

holds.

Proof. Since $r_a=\dfrac{rp}{a\cos^2\left(\frac{\beta-\gamma}{2}\right)}$, we have

$$\frac{h_a}{r_a}=\frac{a\cos^2\left(\frac{\beta-\gamma}{2}\right)}{rp}\cdot h_a=\frac{2S\cdot\cos^2\left(\frac{\beta-\gamma}{2}\right)}{S}=2\cos^2\left(\frac{\beta-\gamma}{2}\right)=1+\cos(\beta-\gamma).$$

It implies

$$\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}=3+\cos(\beta-\gamma)+\cos(\beta-\alpha)+\cos(\alpha-\gamma)=$$

$$=3+(\cos\beta\cos\gamma+\cos\beta\cos\alpha+\cos\alpha\cos\gamma)+(\sin\beta\sin\gamma+\sin\beta\sin\alpha+\sin\alpha\sin\gamma)=$$

$$=3+\frac{p^2+r^2-4R^2}{4R^2}+\frac{p^2+r^2+4Rr}{4R^2}=3+\frac{2p^2+2r^2+4Rr-4R^2}{4R^2}=2+\frac{p^2+r^2+2Rr}{2R^2}.$$

This completes the proof of Theorem 6. □

Corollary 7. *For any triangle ABC the following inequalities hold:*

$$2+\frac{9Rr-2r^2}{R^2}\le\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\le 4+\frac{3Rr+2r^2}{R^2}\le 7.$$

Moreover, we have equality if and only if the triangle ABC is equilateral.

Proof. It is known that $16Rr-5r^2\le p^2\le 4R^2+4Rr+3r^2$ (see [2]). Hence

$$\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}=2+\frac{p^2+r^2+2Rr}{2R^2}\le 2+\frac{(4R^2+4Rr+3r^2)+2Rr+r^2}{2R^2}=4+\frac{3Rr+2r^2}{R^2}$$

and

$$\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\ge 2+\frac{(16Rr-5r^2)+2Rr+r^2}{2R^2}=2+\frac{9Rr-2r^2}{R^2}.$$

Since $\frac{r}{R}\le\frac{1}{2}$, we have

$$\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\le 4+3\cdot\frac{r}{R}+6\cdot\left(\frac{r}{R}\right)^2\le 4+\frac{3}{2}+\frac{6}{4}=7.$$

□

References

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