X GEOMETRICAL OLYMPIAD IN HONOUR OF I. F. SHARYGIN

The Correspondence Round

Below is the list of problems for the first (correspondence) round of the X Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of four elder grades. In Russian school, these are 8-11. In the list below, each problem is indicated by the numbers of Russian school grades, for which it is intended. Foreign students of the last grade have to solve the problems for 11th grade, students of the preceding grade solve the problems for 10th grade etc. However, the participants may solve problems for elder grades as well (solutions of problems for younger grades will not be considered).

A complete solution of each problem or of each its item if there are any, costs 7 points. An incomplete solution costs from 1 to 6 points according to the extent of advancement. If no significant advancement was achieved, the mark is 0. The result of a participant is the total sum of marks for all problems.

In your work, please start the solution for each problem in a new page. First write down the statement of the problem, and then the solution. Present your solutions in detail, including all significant arguments and calculations. Provide all necessary figures of sufficient size. If a problem has an explicit answer, this answer must be presented distinctly. Please, be accurate to provide good understanding and correct estimating of your work!

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

You may note the problems which you liked most (this is not obligatory). Your opinion is interesting for the Jury.

The solutions for the problems (in Russian or in English) must be delivered up to April 1, 2014. For this, please apply **since January 2, 2014** to **http://olimpsharygin.olimpiada.ru** and follow the instructions given there. **Attention:** the solution of each problem must be contained in a separate pdf, doc or jpg file. We recommend to prepare the paper using a computer or to scan it rather than to photograph it. In the last two cases, please check readability of the obtained file.

If you have any technical problem, please contact us by e-mail:
geomolymp@mccme.ru.

It is also possible to send the solutions by e-mail to **geompapers@yandex.ru**. In this case, please follow a few simple rules:

1. Each student sends his work in a separate message (with delivery notification). The size of the message must not exceed 10 Mb.

2. If your work consists of several files, send it as an archive.

3. If the size of your message exceeds 10 Mb, divide it into several messages.

4. In the subject of the message write "The work for Sharygin olympiad", and present the following personal data in the body of your message:

- last name;

- all other names;

- E-mail, phone number, post address;

- the current number of your grade at school;

- the last grade at your high school;

- the number of the last grade in your school system;

- the number and/or the name and the mail address of your school;

- full names of your teachers in mathematics at school and/or of instructors of your extra math classes (if you attend additional math classes after school).

If you have no possibility to deliver the work in electronic form, please apply to the Organizing Committee to find a specific solution for this case.

Winners of the correspondence round, the students of three grades before the last grade, will be invited to the final round in Summer 2014 in the city of Dubna, in Moscow region. (For instance, if the last grade is 12, then we invite winners from 9, 10, and 11 grade.) Winners of the correspondence round, the students of the last grade, will be awarded with diplomas of the Olympiad. The list of the winners will be published on **www.geometry.ru** at the end of May 2014. If you want to know your detailed results, please contact us by e-mail **geomolymp@mccme.ru**.

(1) (8) A right-angled triangle ABC is given. Its cathetus AB is the base of a regular triangle ADB lying in the exterior of ABC, and its hypothenuse AC is the base of a regular triangle AEC lying in the interior of ABC. Lines DE and AB meet at point M. The whole configuration except points A and B was erased. Restore the point M.

(2) (8) A paper square with sidelength 2 is given. From this square, can we cut out a 12-gon having all sidelengths equal to 1, and all angles divisible by $45°$?

(3) (8) Let ABC be an isosceles triangle with base AB. Line ℓ touches its circumcircle at point B. Let CD be a perpendicular from C to ℓ, and AE, BF be the altitudes of ABC. Prove that D, E, F are collinear.

(4) (8) A square is inscribed into a triangle (one side of the triangle contains two vertices and each of two remaining sides contains one vertex). Prove that the incenter of the triangle lies inside the square.

(5) (8) In an acute-angled triangle ABC, AM is a median, AL is a bisector and AH is an altitude (H lies between L and B). It is known that $ML = LH = HB$. Find the ratios of the sidelengths of ABC.

(6) (8–9) Given a circle with center O and a point P not lying on it. Let X be an arbitrary point of this circle, and Y be a common point of the bisector of angle POX and the perpendicular bisector to segment PX. Find the locus of points Y.

(7) (8–9) A parallelogram $ABCD$ is given. The perpendicular from C to CD meets the perpendicular from A to BD at point F, and the perpendicular from B to AB meets the perpendicular bisector to AC at point E. Find the ratio in which side BC divides segment EF.

(8) (8–9) Given a rectangle $ABCD$. Two perpendicular lines pass through point B. One of them meets segment AD at point K, and the second one meets the extension of side CD at point L. Let F be the common point of KL and AC. Prove that $BF \perp KL$.

(9) (8–9) Two circles ω_1 and ω_2 touching externally at point L are inscribed into angle BAC. Circle ω_1 touches ray AB at point E, and circle ω_2 touches ray AC at point M. Line EL meets ω_2 for the second time at point Q. Prove that $MQ \parallel AL$.

(10) (8–9) Two disjoint circles ω_1 and ω_2 are inscribed into an angle. Consider all pairs of parallel lines l_1 and l_2 such that l_1 touches ω_1, and l_2 touches ω_2 (ω_1, ω_2 lie between l_1 and l_2). Prove that the medial lines of all trapezoids formed by l_1, l_2 and the sides of the angle touch some fixed circle.

(11) (8–9) Points K, L, M and N lying on the sides AB, BC, CD and DA of a square $ABCD$ are vertices of another square. Lines DK and NM meet at point E, and lines KC and LM meet at point F. Prove that $EF \parallel AB$.

(12) (9–10) Circles ω_1 and ω_2 meet at points A and B. Let points K_1 and K_2 of ω_1 and ω_2 respectively be such that K_1A touches ω_2, and K_2A touches ω_1. The circumcircle of triangle K_1BK_2 meets lines AK_1 and AK_2 for the second time at points L_1 and L_2 respectively. Prove that L_1 and L_2 are equidistant from line AB.

(13) (9–10) Let AC be a fixed chord of a circle ω with center O. Point B moves along the arc AC. A fixed point P lies on AC. The line passing through P and parallel to AO meets BA at point A_1; the line passing through P and parallel to CO meets BC at point C_1. Prove that the circumcenter of triangle A_1BC_1 moves along a straight line.

(14) (9–11) In a given disc, construct a subset such that its area equals the half of the disc area and its intersection with its reflection over an arbitrary diameter has the area equal to the quarter of the disc area.

(15) (9–11) Let ABC be a non-isosceles triangle. The altitude from A, the bisector from B and the median from C concur at point K.

a) Which of the sidelengths of the triangle is medial?

b) Which of the lengths of segments AK, BK, CK is medial?

(16) (9–11)Given a triangle ABC and an arbitrary point D.The lines passing through D and perpendicular to segments DA, DB, DC meet lines BC,

AC, AB at points A_1, B_1, C_1 respectively. Prove that the midpoints of segments AA_1, BB_1, CC_1 are collinear.

(17) (10–11) Let AC be the hypothenuse of a right-angled triangle ABC. The bisector BD is given, and the midpoints E and F of the arcs BD of the circumcircles of triangles ADB and CDB respectively are marked (the circles are erased). Construct the centers of these circles using only a ruler.

(18) (10–11) Let I be the incenter of a circumscribed quadrilateral ABCD. The tangents to circle AIC at points A, C meet at point X. The tangents to circle BID at points B, D meet at point Y. Prove that X, I, Y are collinear.

(19) (10–11) Two circles ω_1 and ω_2 touch externally at point P. Let A be a point of ω_2 not lying on the line through the centers of the circles, and AB, AC be the tangents to ω_1. Lines BP, CP meet ω_2 for the second time at points E and F. Prove that line EF, the tangent to ω_2 at point A and the common tangent at P concur.

(20) (10–11) A quadrilateral $KLMN$ is given. A circle with center O meets its side KL at points A and A_1, side LM at points B and B_1, etc. Prove that if the circumcircles of triangles KDA, LAB, MBC and NCD concur at point P, then

a) the circumcircles of triangles KD_1A_1, LA_1B_1, MB_1C_1 and NC_1D_1 also concur at some point Q;

b) point O lies on the perpendicular bisector to PQ.

(21) (10–11) Let $ABCD$ be a circumscribed quadrilateral. Its incircle ω touches sides BC and DA at points E and F respectively. It is known that lines AB, FE and CD concur. The circumcircles of triangles AED and BFC meet ω for the second time at points E_1 and F_1. Prove that $EF \parallel E_1F_1$.

(22) (10–11) Does there exist a convex polyhedron such that it has diagonals and each of them is shorter than each of its edges?

(23) (11) Let A, B, C and D be a triharmonic quadruple of points, i.e

$$AB \cdot CD = AC \cdot BD = AD \cdot BC.$$

Let A_1 be a point distinct from A such that the quadruple A_1, B, C and D is triharmonic. Points B_1, C_1 and D_1 are defined similarly. Prove that

a) A, B, C_1, D_1 are concyclic;

b) the quadruple A_1, B_1, C_1, D_1 is triharmonic.

(24) (11) A circumscribed pyramid $ABCDS$ is given. The opposite sidelines of its base meet at points P and Q in such a way that A and B lie on segments PD and PC respectively. The inscribed sphere touches faces ABS and BCS at points K and L. Prove that if PK and QL are complanar then the touching point of the sphere with the base lies on BD.