ON ROTATION OF A ISOGONAL POINT

ALEXANDER SKUTIN

Abstract. In this short note we give a synthetic proof of the problem posed by A. V Akopyan in [1]. We prove that if Poncelet rotation of triangle T between circle and ellipse is given then the locus of the isogonal conjugate point of any fixed point P with respect to T is a circle.

We will prove more general problem:

Problem. *Let T be a Poncelet triangle rotated between external circle ω and internal ellipse with foci Q and Q' and P be any point. Then the locus of points P' isogonal conjugates to P with respect to T is a circle.*

Proof. First, prove the following lemma:

Lemma. *Suppose that ABC is a triangle and P, P' and Q, Q' are two pairs of isogonal conjugates with respect to ABC. Let H be a Miquel point of lines PQ, PQ', $P'Q$ and $P'Q'$. Then H lies on (ABC).*

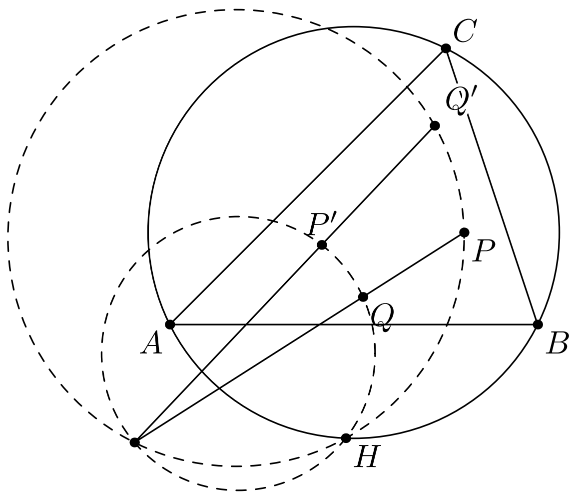

Fig. 1.

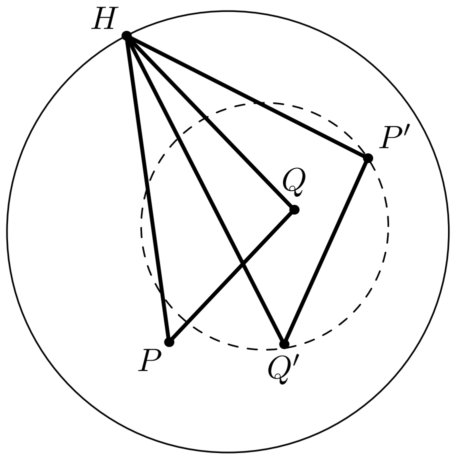

Fig. 2.

Proof. From here, the circumcircle of a triangle XYZ is denoted by (XYZ) and the oriented angle between lines ℓ and m is denoted by $\angle(\ell, m)$. Let A^* and B^* be such points that $A^*AH \sim B^*BH \sim PQH$. It is clear that $HPQ \sim HQ'P'$. From construction it immediately follows that there exists a similarity with center H which maps the triangle QBP' to the triangle PB^*Q'. So $HPB^*Q' \sim HQBP'$, and similarly $A^*PQ'H \sim AQP'H$. From the properties of isogonal conjugation it can be easily seen that $\angle(Q'A^*, A^*P) = \angle(P'A, AQ) = \angle(Q'A, AP)$, hence points A^*, A, P, and Q' are cocyclic. Similarly the quadrilateral PB^*BQ' is inscribed in a circle. Let lines AA^* and BB^* intersect in a point F. Indeed $ABQH \sim A^*B^*PH$, so $\angle(BQ, QA) = \angle(B^*P, PA^*)$. Obviously $\angle(B^*P, PA^*) = \angle(B^*B, BQ') + \angle(Q'A, AF)$. Thus

$$\angle(B^*P, PA^*) + \angle(BQ', Q'A) = \\ = \angle(FB, BQ') + \angle(BQ', Q'A) + \angle(Q'A, AF) = \angle(BF, FA),$$

but we have proved that

$$\angle(B^*P, PA^*) + \angle(BQ', Q'A) = \angle(BQ, QA) + \angle(BQ', Q'A) = \angle(AC, CB),$$

so F is on (ABC). We know that $A^*AH \sim B^*BH$, so $\angle(A^*A, AH) = \angle(B^*B, BH)$, hence $AFHB$ is inscribed in a circle. From that it is clear that H is on (ABC).

□

Now the problem can be reformulated in the following way. Suppose that ω is a circle, P, Q and Q' are fixed points, H is a variable point on ω. Let P' be such a point that $PQH \sim Q'P'H$. We need to prove that locus of points P' is a circle.

It is clear that the transformation which maps H to P' is a composition of an inversion, a parallel transform and rotations. Indeed, denote by z_x the coordinate of a point X in the complex plane. Than this transformation have the following equation:

$$z_h \to z_{q'} + (z_h - z_{q'})\frac{z_q - z_p}{z_h - z_p}.$$

Therefore, the image of the circle ω under this transformation is a circle.

□

Author is grateful to Alexey Pakharev for help in preparation of this text.

References

[1] A. V. Akopyan. Rotation of isogonal point. *Journal of classical geometry*:74, 1, 2012.

Moscow State University
E-mail address: a.skutin@mail.ru