ONE PROPERTY OF THE JERABEK HYPERBOLA AND ITS COROLLARIES

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Abstract. We study the locus of the points P having the following property: if $A_1B_1C_1$ is the circumcevian triangle of P with respect to the given triangle ABC, and A_2, B_2, C_2 are the reflections of A_1, B_1, C_1 in BC, CA, AB, respectively, then the triangles ABC and $A_2B_2C_2$ are perspective. We show that this locus consists of the infinite line and the Jerabek hyperbola of ABC. This fact yields some interesting corollaries.

We start with the following well-known fact [1, page 4.4.5].

Statement 1. *Let the tangents to the circumcircle of ABC at A and B meet in C_0. The line CC_0 meets the circumcircle of ABC for the second time in C_1, and C_2 is the reflection of C_1 in AB. Then CC_2 is a median in ABC.*

Proof. Let C' be the common point of CC_1 and AB, and A'', B'' be the common points of AC_2, BC_2 with BC, AC, respectively. Since $\angle C'CB = \angle C_1AB = \angle BAA''$, the triangles BCC' and BAA'' are similar; therefore, $BA'' = \frac{AB}{BC} \cdot BC'$. But CC' is a symedian in ABC, so $BC' = \frac{BC^2}{BC^2+AC^2} \cdot AB$. Therefore, $\frac{BA''}{BC} = \frac{AB^2}{AC^2+BC^2}$. Analogously, the ratio $\frac{AB''}{AC}$ has the same value, yielding the claim. □

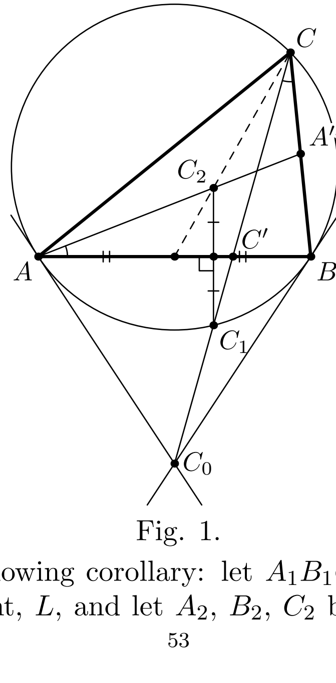

Fig. 1.

This fact yields the following corollary: let $A_1B_1C_1$ be the circumcevian triangle of the Lemoine point, L, and let A_2, B_2, C_2 be the reflections of A_1, B_1,

C_1 in BC, CA, AB, respectively. Then the triangles ABC and $A_2B_2C_2$ are perspective.

Our goal is to find the locus of the points sharing this property. To this end, first we formulate the following assertion.

Lemma 1. *Let CC_1 divide AB in ratio $x : y$. Then CC_2 divides AB in ratio*

$$x(b^2(x+y) - c^2x) : y(a^2(x+y) - c^2y).$$

In order to prove this, it suffices to repeat the argument by which we demonstrated the previous assertion and then to apply Ceva's theorem.

Now let P be the point with barycentric coordinates $(x : y : z)$. Using Lemma 1 and Ceva's theorem, we see that a point P has the property in question iff it lies on some cubic c. From the following assertion, we infer that c is degenerated.

Statement 2. *Let three parallel lines passing through the vertices of ABC meet its circumcircle in A_1, B_1, C_1. The points A_2, B_2, C_2 are the reflections of A_1, B_1, C_1 in BC, CA, AB, respectively. Then the lines AA_2, BB_2, CC_2 are concurrent.*

Proof. Consider the three lines which pass through A_1, B_1, C_1 and are parallel to BC, CA, AB, respectively. It is easy to see that they meet at a point on the circumcircle of ABC. The points A_2, B_2, C_2 are the reflections of this point in the midpoints of the sides of ABC. Therefore, the triangles ABC and $A_2B_2C_2$ are centrosymmetric. □

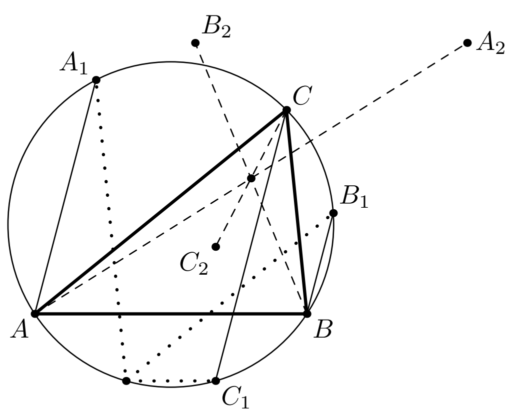

Fig. 2.

Note also that the center of perspective in this claim lies on the Euler circle.

And so, c consists of the infinite line and some conic k. In order to determine k completely, it suffices to indicate five point lying on it. We already know that k contains the Lemoine point, L. Furthermore, k contains the vertices of ABC as well as its orthocenter H (in this case, all of A_2, B_2, C_2 coincide with H). Therefore, k is the Jerabek hyperbola.

Here follow some corollaries of this fact.

Statement 3. *Let P be a point on the Euler line of ABC, $A_1B_1C_1$ be the circumcevian triangle of P, and A_2, B_2, C_2 be the reflections of A_1, B_1, C_1 in the midpoints of BC, CA, AB, respectively. Then the lines AA_2, BB_2, CC_2 are concurrent.*

Proof. The isogonal conjugate, Q, of P lies on the Jerabek hyperbola. Let CQ meet the circumcircle of ABC for the second time in C_3. Then C_2 and C_3 are symmetric with respect to AB. □

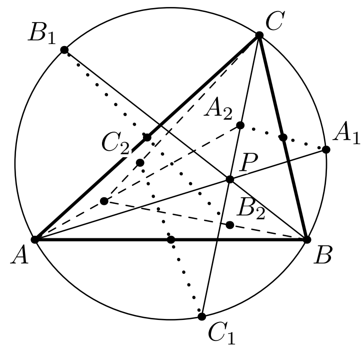

Fig. 3.

Statement 4. *Let P be a point on the Euler line of ABC, A_0, B_0, C_0 be the midpoints of BC, CA, AB, respectively, and A_1, B_1, C_1 be the projections of the circumcenter O of ABC onto AP, BP, CP, respectively. Then the lines A_0A_1, B_0B_1, C_0C_1 are concurrent.*

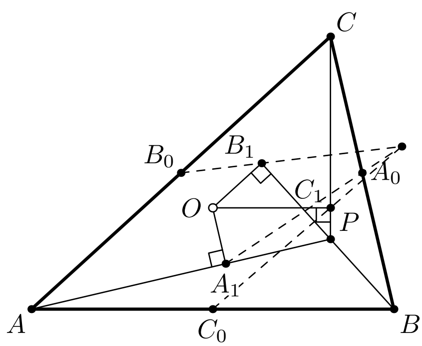

Fig. 4.

Proof. It suffices to apply homothety of center the centroid of ABC and coefficient $-\frac{1}{2}$ to the configuration of the previous claim. □

Statement 5. *Let O, I be the circumcenter and incenter of ABC. An arbitrary line perpendicular to OI meets BC, CA, AB in A_1, B_1, C_1, respectively. Then the circumcenters of the triangles IAA_1, IBB_1, ICC_1 are collinear.*

Proof. Applying an inversion of center I and the previous assertion we see that the circumcircles of the three triangles in question have a common point other than I. □

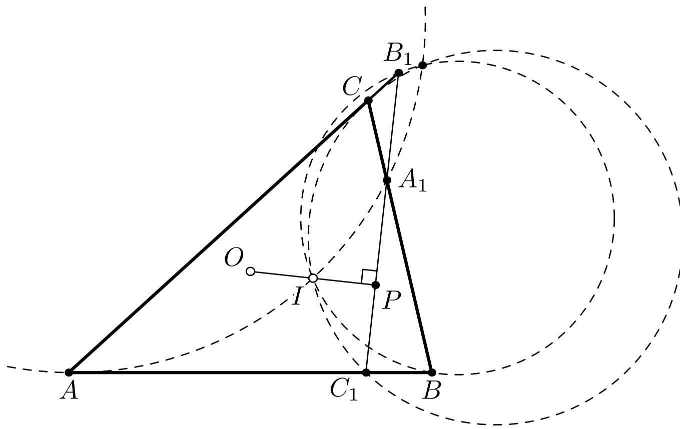

Fig. 5.

References

[1] A. V. Akopyan. *Geometry in figures*. Createspace, 2011.

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