SOME PROPERTIES OF THE BROCARD POINTS OF A CYCLIC QUADRILATERAL

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Abstract. In this article we have constructed the Brocard points of a cyclic quadrilateral, we have found some of their properties and using these properties we have proved the problem of A. A. Zaslavsky.

1. The Problem

Alexey Zaslavsky, **Brocard's points in quadrilateral [4].**

Given a convex quadrilateral $ABCD$. It is easy to prove that there exists a unique point P such that $\angle PAB = \angle PBC = \angle PCD$. We will call this point *Brocard point* ($Br(ABCD)$) and the respective angle *Brocard angle* ($\phi(ABCD)$) of broken line $ABCD$. Note some properties of Brocard's points and angles:

– $\phi(ABCD) = \phi(DCBA)$ if $ABCD$ is cyclic;

– if $ABCD$ is harmonic then $\phi(ABCD) = \phi(BCDA)$. Thus there exist two points P, Q such that $\angle PAB = \angle PBC = \angle PCD = \angle PDA = \angle QBA = \angle QCB = \angle QDC = \angle QAD$. These points lie on the circle with diameter OL where O is the circumcenter of $ABCD$, L is the common point of its diagonals and $\angle POL = \angle QOL = \phi(ABCD)$.

Problem. *Let $ABCD$ be a cyclic quadrilateral, $P_1 = Br(ABCD)$, $P_2 = Br(BCDA)$, $P_3 = Br(CDAB)$, $P_4 = Br(DABC)$, $Q_1 = Br(DCBA)$, $Q_2 = Br(ADCB)$, $Q_3 = Br(BADC)$, $Q_4 = Br(CBAD)$. Then $S_{P_1P_2P_3P_4} = S_{Q_1Q_2Q_3Q_4}$.*

I first saw this problem 2-3 years ago in an article by Alexei Myakishev (see [3]). I did not consider solving it then, but my pupils built a structure very similar (seemingly) to the described. Thus Zaslavsky's problem served as a stimulus (accelerator) to develop a good article (related to isogonal conjugate points), which they presented on international events. I sincerely hope that this article will appear on the pages of "Kvant". Now they are university students, but again I saw the problem in the "Journal of Classical Geometry". At first I thought that using the developed article I would figure out the solution quickly, but I did not. A new construction came out.

2. Some Properties Of Brocard Points Of a Triangle.

Let P and Q be the first and the second Brocard points of a $\triangle ABC$. We will prove that the Brocard points and the three vertices A, B and C define three pairs of similar triangles, and three pairs of triangles with equal areas (Fig. 1).

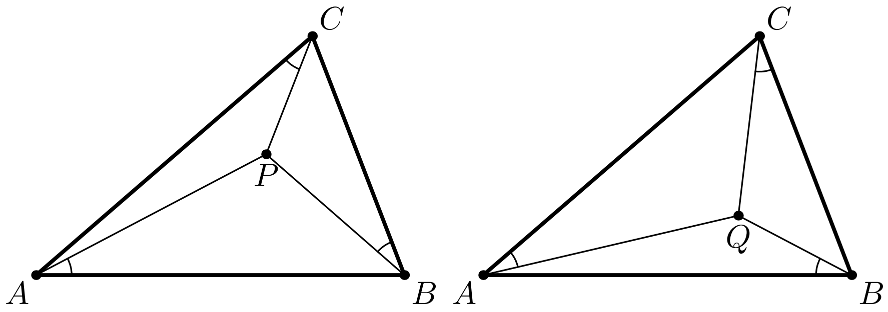

Fig. 1.

Proposition 1. *If P and Q are the first and the second Brocard points of a $\triangle ABC$, then:*

(i) $\triangle ABP \sim \triangle CBQ$, $\triangle BCP \sim \triangle ACQ$ and $\triangle CAP \sim \triangle BAQ$;

(ii) $S_{ABP} = S_{ACQ}$, $S_{BCP} = S_{BAQ}$ and $S_{CAP} = S_{CBQ}$.

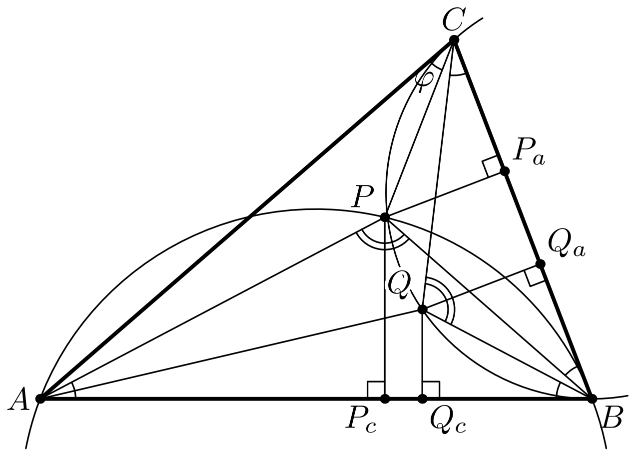

Fig. 2.

Proof. If we denote with φ the Brocard angle (Fig. 2), then $\angle BAP = \angle BCQ = \varphi$, and from the construction ([1]) of the Brocard points of $\triangle ABC$ we have $\angle APB = \angle CQB = 180° - \angle B$. Hence $\triangle ABP \sim \triangle CBQ$.

We construct the altitudes PP_c, PP_a, QQ_c and QQ_a. Then, having $\triangle ABP \sim \triangle CBQ$, $\triangle P_cBP \sim \triangle Q_aBQ$ and $\triangle Q_cBP \sim \triangle P_aBQ$, we have the following equations:

$$\frac{S_{BPC}}{S_{BAQ}} = \frac{BC \cdot PP_a}{AB \cdot QQ_c} = \frac{QQ_a}{PP_c} \cdot \frac{PP_a}{QQ_c} = \frac{BQ}{BP} \cdot \frac{BP}{BQ} = 1.$$

For the other pairs of triangles, the proof is analogous. □

We are going to find, similar to these properties, for the Brocard points of a cyclic quadrilateral.

3. Construction

Let the quadrilateral $ABCD$ be a cyclic, $AB \cap CD = E$ and let for exactitude B is between A and E, $AD \cap BC = F$ and D is between A and F. We denote $\angle BAD = \alpha$ and $\angle ABC = \beta$.

In order to construct the Brocard points of the quadrilateral $ABCD$, we firstly construct the points M_1, M_2, M_3, M_4 and N_1, N_2, N_3, N_4, as shown in Table 1:

M_1	$M_1 \in AD$ and $BM_1 \parallel CD$	N_1	$N_1 \in AD$ and $CN_1 \parallel BA$
M_2	$M_2 \in AB$ and $CM_2 \parallel DA$	N_2	$N_2 \in AB$ and $DN_2 \parallel CB$
M_3	$M_3 \in BC$ and $DM_3 \parallel AB$	N_3	$N_3 \in BC$ and $AN_3 \parallel DC$
M_4	$M_4 \in CD$ and $AM_4 \parallel BC$	N_4	$N_4 \in CD$ and $BN_4 \parallel AD$

Table 1

Now we construct the intersection points of the pairs of circumcircles of the triangles, as shown in Table 2:

	$\triangle BAM_1$	$\triangle DCM_3$
$\triangle BCM_2$	P_1	P_2
$\triangle DAM_4$	P_4	P_3

	$\triangle DCN_1$	$\triangle BAN_3$
$\triangle BCN_4$	Q_1	Q_4
$\triangle ADN_2$	Q_2	Q_3

Table 2

Proposition 2. *The points P_1, P_2, P_3, P_4 and Q_1, Q_2, Q_3, Q_4 are the Brocard points of the quadrangle $ABCD$.*

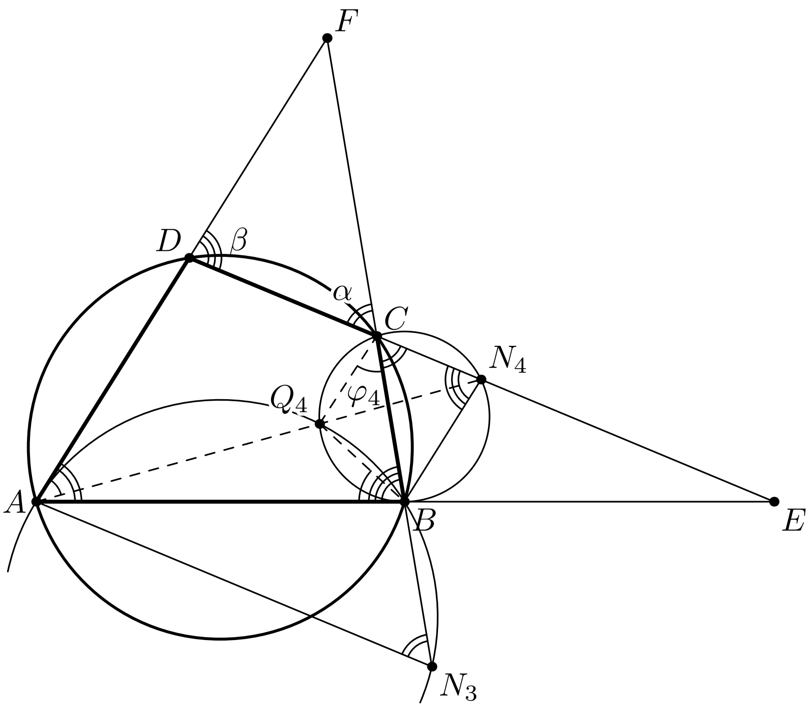

Fig. 3.

Proof. Let Q_4 be the intersection point of the circumcircles of triangles BAN_3 and BCN_4 (Table 2). We denote $\angle Q_4CB = \varphi_4$ (Fig. 3). Since $BN_4 \parallel AD$ and $AN_3 \parallel DC$ (Table 1), then $\angle CN_4B = \beta$ and $\angle BN_3A = \alpha$. So:

– $\angle ABQ_4 = \beta - \angle CBQ_4 = \beta - \angle CN_4Q_4 = \beta - (\beta - \varphi_4) = \varphi_4$,

– $\angle DAQ_4 = \alpha - \angle BAQ_4 = \alpha - (180° - \angle AQ_4B - \varphi_4) = \alpha - (180° - (180° - \alpha) - \varphi_4) = \varphi_4$ □

Hence $Q_4 = Br(CBAD)$ is a Brocard point and $\varphi_4 = \phi(CBAD)$ is a Brocard angle in the quadrilateral $ABCD$. As $ABCD$ is cyclic quadrilateral, hence $\varphi_4 =$

$\phi(DABC)$. For the other points, the proof is analogous. If we denote the Brocard angles in $ABCD$ with φ_1, φ_2, φ_3 and φ_4 then we have:

$$\begin{aligned}
\varphi_1 = \phi(ABCD) = \phi(DCBA) = \angle P_1AB = \angle P_1BC = \\
= \angle P_1CD = \angle Q_1DC = \angle Q_1CB = \angle Q_1BA, \\
\varphi_2 = \phi(BCDA) = \phi(ADCB) = \angle P_2BC = \angle P_2CD = \\
= \angle P_2DA = \angle Q_2AD = \angle Q_2DC = \angle Q_2CB, \\
\varphi_3 = \phi(CDAB) = \phi(BADC) = \angle P_3CD = \angle P_3DA = \\
= \angle P_3AB = \angle Q_3BA = \angle Q_3AD = \angle Q_3DC, \\
\varphi_4 = \phi(DABC) = \phi(CBAD) = \angle P_4DA = \angle P_4AB = \\
= \angle P_4BC = \angle Q_4CB = \angle Q_4BA = \angle Q_4AD.
\end{aligned} \tag{1}$$

4. Some properties of the Brocard points of a cyclic quadrilateral

Proposition 3. *The triads of points C, P_1, M_1; D, P_2, M_2; A, P_3, M_3; B, P_4, M_4; B, Q_1, N_1; C, Q_2, N_2; D, Q_3, N_3 and A, Q_4, N_4 are collinear.*

Proof. $\angle AQ_4B + \angle BQ_4N_4 = (\alpha - 180°) + \alpha = 180°$ (Fig. 3). For the other triads of points, the proof is analogous. □

Proposition 4. *The fours of lines CM_1, DM_2, AM_3, BM_4 and BN_1, CN_2, DN_3, AN_4 are concurrent.*

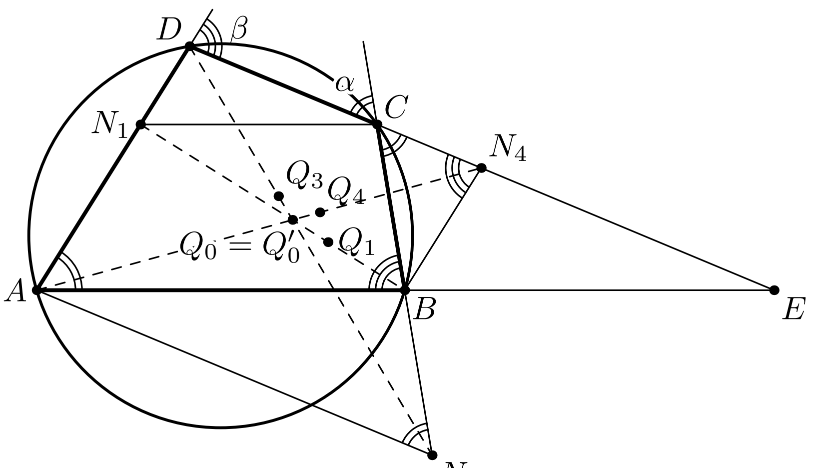

Fig. 4.

Proof. Let $AN_4 \cap BN_1 = Q_0$ (Fig. 4). Then using $\triangle AQ_0N_1 \sim \triangle N_4Q_0B$, $\triangle AED \sim \triangle BEN_4$ and $\triangle N_1CD \sim \triangle BEN_4$ (see Table 1) we have the following equations:

$$\frac{AQ_0}{Q_0N_4} = \frac{AN_1}{BN_4} = \frac{AD}{BN_4} - \frac{DN_1}{BN_4} = \frac{DE}{EN_4} - \frac{DC}{EN_4} = \frac{CE}{EN_4}.$$

Now, let $AD \cap DN_3 = Q_0'$. From the similarity of $\triangle AN_3Q_0' \sim \triangle N_4DQ_0'$, $\triangle AN_3B \sim \triangle ECB$ and also because $AD \parallel BN_4$ we have consecutively the following equations:

$$\frac{AQ_0'}{Q_0'N_4} = \frac{AN_3}{DN_4} = \frac{CE.AB}{BE.DN_4} = \frac{CE}{BE} \cdot \frac{BE}{EN_4} = \frac{CE}{EN_4}.$$

Then $\dfrac{AQ_0}{Q_0N_4} = \dfrac{AQ_0'}{Q_0'N_4}$ and $Q_0 \equiv Q_0'$. In the same way $Q_0 \in CN_2$. So the lines BN_1, CN_2, DN_3, AN_4 intersect in a point Q_0, and similarly the lines CM_1, DM_2, AM_3, BM_4 have a common point P_0. □

Because $\triangle BEN_4 \sim \triangle CEB$ and the Law of Sines used for $\triangle CEB$, we attain:

$$\frac{AQ_0}{Q_0N_4} = \frac{CE}{EN_4} = \frac{CE^2}{BE^2} = \frac{sin^2\beta}{sin^2\alpha}.$$

In the same way we have the following equations:

$$(2)\qquad \begin{array}{c} \dfrac{AQ_0}{Q_0N_4} = \dfrac{CQ_0}{Q_0N_2} = \dfrac{AP_0}{P_0M_3} = \dfrac{CP_0}{P_0M_1} = \dfrac{\sin^2\beta}{\sin^2\alpha} \\ \text{and} \\ \dfrac{BQ_0}{Q_0N_1} = \dfrac{DQ_0}{Q_0N_3} = \dfrac{BP_0}{P_0M_4} = \dfrac{DP_0}{P_0M_2} = \dfrac{\sin^2\alpha}{\sin^2\beta} \end{array}$$

Now we will show that the points P_0, Q_0 and the vertices A, B, C and D define four pairs of triangles with equal areas (Fig. 5).

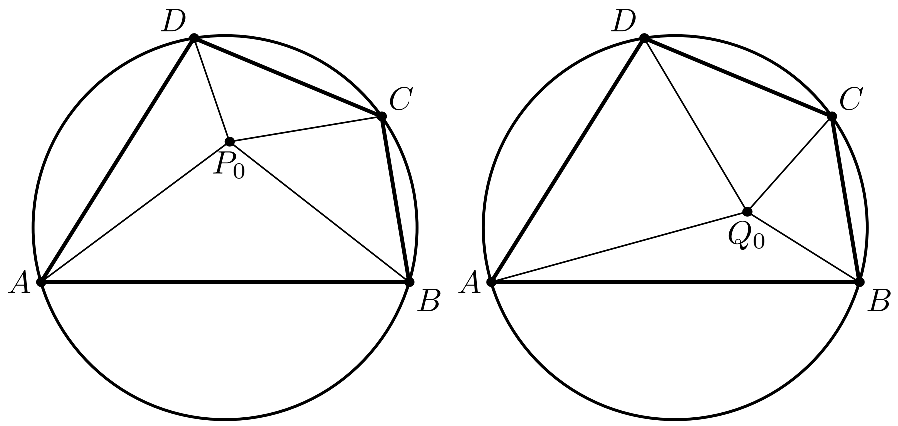

Fig. 5.

Proposition 5. *If P_0 and Q_0 are the intersection points of the fours of lines CM_1, DM_2, AM_3, BM_4 and BN_1, CN_2, DN_3, AN_4 (see Proposition 4), then:*

$$S_{ABP_0} = S_{DAQ_0},\ S_{BCP_0} = S_{ABQ_0},\ S_{CDP_0} = S_{BCQ_0} \text{ and } S_{DAP_0} = S_{CDQ_0}.$$

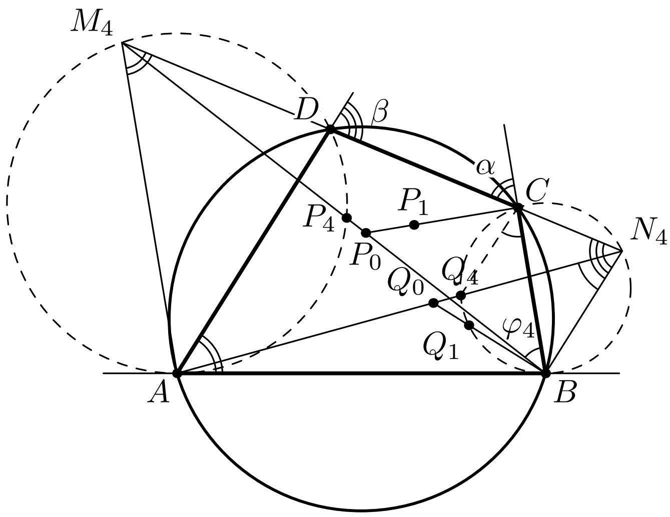

Fig. 6.

Taking into account that the fours of points B, P_0, P_4, M_4 and A, Q_0, Q_4, N_4 (Fig. 6) are collinear (Proposition 3 and Proposition 4) and equations (2) we attain:

$$S_{BCP_0} = \frac{BP_0}{BM_4} S_{BCM_4} = \frac{\sin^2\alpha}{\sin^2\alpha + \sin^2\beta} S_{BCM_4}, \tag{3}$$

$$S_{ABQ_0} = \frac{AQ_0}{AN_4} S_{BN_4A} = \frac{\sin^2\beta}{\sin^2\alpha + \sin^2\beta} S_{BN_4A},$$

Since $\angle BCM_4 = \angle ABN_4 = 180^\circ - \alpha$ (see Table 1) and $\angle CBM_4 = \angle BN_4A = \varphi_4$ (see (1)), then $\triangle BCM_4 \sim \triangle BN_4A$ and

$$\frac{S_{BCM_4}}{S_{BN_4A}} = \frac{BC^2}{BN_4^2} = \frac{\sin^2\beta}{\sin^2\alpha} \text{ (Law of Sines for } \triangle CBN_4) \tag{4}$$

Using equations (3) and (4) we calculate

$$\frac{S_{BCP_0}}{S_{ABQ_0}} = \frac{\sin^2\alpha}{\sin^2\beta} \cdot \frac{S_{BCM_4}}{S_{BN_4A}} = 1.$$

The proof, of the areas equality, of the other triangle pairs is similar.

Now we can consider the quadrangles $P_1P_2P_3P_4$ and $Q_1Q_2Q_3Q_4$. We will prove that the points P_0, Q_0 and the vertices of quadrangles $P_1P_2P_3P_4$ and $Q_1Q_2Q_3Q_4$ define four pairs of triangles with equal areas (Fig. 7).

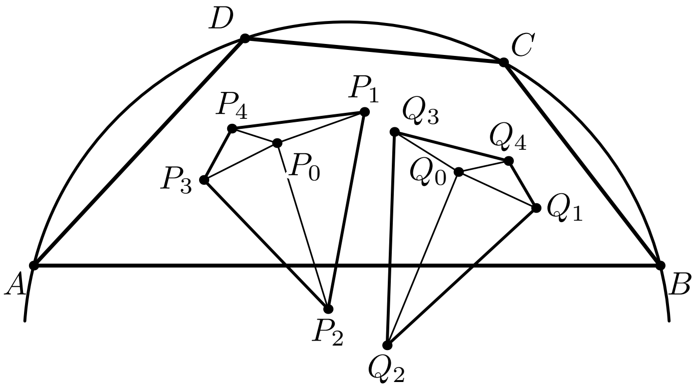

Fig. 7.

Proposition 6. *If P_0 and Q_0 are the intersection points of the fours of lines CM_1, DM_2, AM_3, BM_4 and BN_1, CN_2, DN_3, AN_4 (see Proposition 4), then:*

$$S_{P_1P_2P_0} = S_{Q_1Q_2Q_0}, \quad S_{P_2P_3P_0} = S_{Q_2Q_3Q_0},$$
$$S_{P_3P_4P_0} = S_{Q_3Q_4Q_0}, \quad S_{P_4P_1P_0} = S_{Q_4Q_1Q_0}.$$

Proof. Firstly we will show that the areas of the pairs of triangles CDP_1, BCQ_1 and CDP_2, BCQ_2 are equal (Fig. 8). The proof is similar to that of Proposition 1. Since $\angle P_1BC = \angle Q_1DC = \varphi_1$ and $\angle P_1CB = \angle Q_1CD$ (see equations (1)), hence $\triangle BCP_1 \sim \triangle DCQ_1$. We construct the altitudes P_1P_b, P_1P_c, Q_1Q_b and Q_1Q_c.

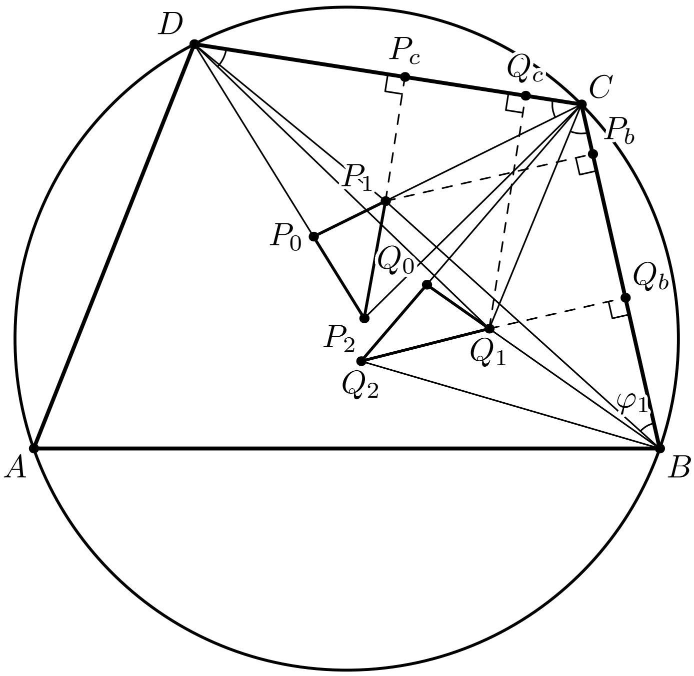

Fig. 8.

Then using $\triangle BCP_1 \sim \triangle DCQ_1$, $\triangle P_1P_bC \sim \triangle Q_1Q_cC$ and $\triangle P_1P_cC \sim \triangle Q_1Q_bC$, we have the following equations:

$$\frac{S_{CDP_1}}{S_{BCQ_1}} = \frac{DC.P_1P_c}{BC.Q_1Q_b} = \frac{Q_1Q_c}{P_1P_b} \cdot \frac{P_1P_c}{Q_1Q_b} = \frac{Q_1C}{P_1C} \cdot \frac{P_1C}{Q_1C} = 1.$$

So we attain: $S_{CDP_1} = S_{BCQ_1}$ and $S_{CDP_2} = S_{BCQ_2}$ (analogous).

Since the triads of points C, P_1, P_0; D, P_0, P_2; B, Q_1, Q_0 and C, Q_0, Q_2 are collinear (Propositions 3 and 4) and $S_{CDP_0} = S_{BCQ_0}$ (Proposition 5), hence

$$S_{CP_0P_2} = S_{BQ_0Q_2} \text{ and } S_{DP_0P_1} = S_{CQ_0Q_1}.$$

We calculate $S_{P_1P_2P_0} = \dfrac{P_0P_1}{P_0C} \cdot S_{CP_0P_2} = \dfrac{S_{DP_0P_1}}{S_{CDP_0}} \cdot S_{CP_0P_2}$ and $S_{Q_1Q_2Q_0} = \dfrac{Q_0Q_1}{Q_0B} \cdot S_{BQ_0Q_2} = \dfrac{S_{CQ_0Q_1}}{S_{BCQ_0}} \cdot S_{BQ_0Q_2}$, therefore $S_{P_1P_2P_0} = S_{Q_1Q_2Q_0}$.

The proof, of the areas equality, of the other pairs of triangles is similar. □

From Proposition 6 immediately follows

CONSEQUENCE $S_{P_1P_2P_3P_4} = S_{Q_1Q_2Q_3Q_4}$.

We should note that another solution is given by *Chandan Banerjee* in his blog [2].

5. Two additional properties

Above we have proved that $\triangle BCP_1 \sim \triangle DCQ_1$ and $S_{CDP_1} = S_{BCQ_1}$. In fact, we have also $\triangle ABP_1 \sim \triangle CBQ_1$ and $S_{BCP_1} = S_{ABQ_1}$. In general, each pair of Brocard points P_i and Q_i, in a cyclic quadrilateral $ABCD$ and its corresponding vertices, define two pairs of triangles with equal areas (to compare see Proposition 1 and Fig. 1).

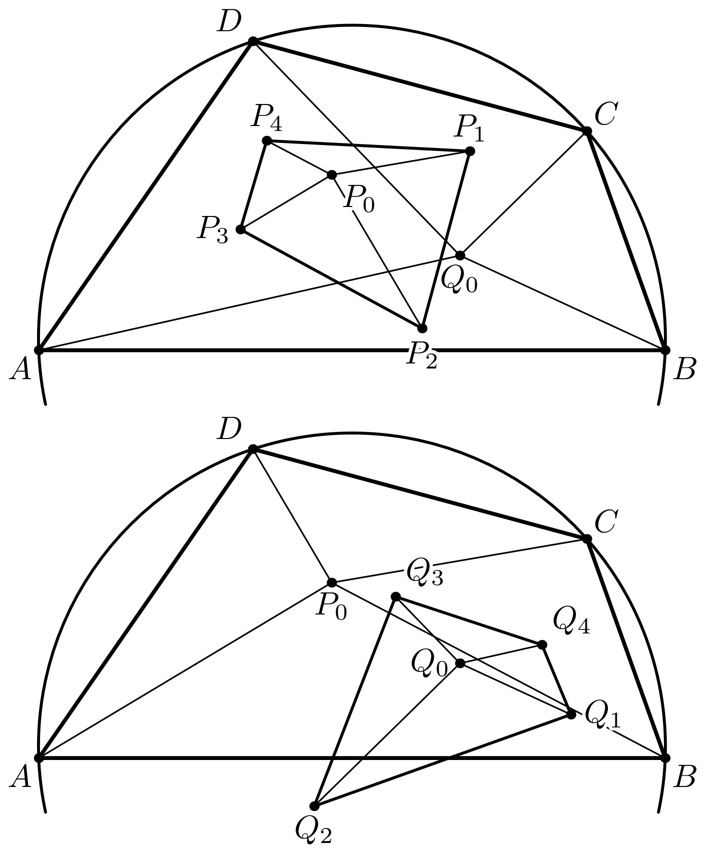

Fig. 9.

On the other hand, the points P_0 and Q_0 not only "stole" the pairs of triangles with equal areas related to vertices of the quadrilateral $ABCD$ (see Proposition 5 and Fig. 5), but they have and those in quadrilaterals $P_1P_2P_3P_4$ and $Q_1Q_2Q_3Q_4$ (see Proposition 6 and Fig. 7). Naturally arise the question, where are the pairs of similar triangles associated with P_0, Q_0 and vertices of quadrilaterals $ABCD$, $P_1P_2P_3P_4$ and $Q_1Q_2Q_3Q_4$. It turns out that they appear in a very curious way.

In Fig. 9 we show that the triangles defined by the point P_0 and quadrilateral $P_1P_2P_3P_4$ are similar to the triangles determined by a point Q_0 and quadrilateral $ABCD$. Conversely, the triangles defined by the point Q_0 and quadrilateral $Q_1Q_2Q_3Q_4$ are similar to the triangles determined by a point P_0 and quadrilateral $ABCD$ (Fig. 9).

Proposition 7. *If P_0 and Q_0 are the intersection points of the fours of lines CM_1, DM_2, AM_3, BM_4 and BN_1, CN_2, DN_3, AN_4 (see Proposition 4), then:*

$$\triangle P_1P_2P_0 \sim \triangle CBQ_0, \quad \triangle P_2P_3P_0 \sim \triangle DCQ_0,$$
$$\triangle P_3P_4P_0 \sim \triangle ADQ_0, \quad \triangle P_4P_1P_0 \sim \triangle BAQ_0$$

and

$$\triangle Q_1Q_2Q_0 \sim \triangle DCP_0, \quad \triangle Q_2Q_3Q_0 \sim \triangle ADP_0,$$
$$\triangle Q_3Q_4Q_0 \sim \triangle BAP_0, \quad \triangle Q_4Q_1Q_0 \sim \triangle CBP_0.$$

Proof. We will prove that $\triangle Q_1Q_2Q_0 \sim \triangle DCP_0$ (Fig. 8). The points Q_1 and Q_2 lie on the circumcircle of the triangle $\triangle DCN_1$ (Table 2) therefore $\angle Q_1Q_2C = \angle Q_1DC = \varphi_4$. But $\angle Q_2Q_0Q_1 = \angle Q_0CB + \angle Q_1BC = \beta - \varphi_1 + \varphi_2$ (see equations (1)) and $\angle CP_0D = 180^\circ - (\angle P_0DC + \angle P_1CD) = 180^\circ - (180^\circ - \beta - \varphi_2 + \varphi_1) = \beta - \varphi_1 + \varphi_2$, namely $\angle Q_2Q_0Q_1 = \angle CP_0D$. Hence $\triangle Q_1Q_2Q_0 \sim \triangle DCP_0$. The similarity of the other pairs of triangles we prove similarly. □

From Proposition 7 follows, that $\angle P_1 + \angle P_3 = \angle Q_2 + \angle Q_4 = \angle A + \angle C = 180^\circ$ (see Fig. 9), which means that quadrilaterals $P_1P_2P_3P_4$ and $Q_1Q_2Q_3Q_4$ are cyclic.

6. Appendix

I apply the problems of my students.
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Problem 1. *If in the quadrangle $ABCD$ is inscribed ellipse with foci F_1 and F_2, and the isogonal conjugated points of F_1 and F_2 with respect to $\triangle DAB$, $\triangle ABC$, $\triangle BCD$, $\triangle CDA$ are A_1, B_1, C_1, D_1 and A_2, B_2, C_2, D_2, then the quadrangles $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ are congruent.*

Problem 2. *In the quadrangle $ABCD$ is inscribed a circle with center O. If the isogonal conjugated points of O with respect to $\triangle DAB$, $\triangle ABC$, $\triangle BCD$, $\triangle CDA$ are A_1, B_1, C_1 and D_1, then $A_1B_1C_1D_1$ is a parallelogram. If $AC \perp BD$, then $A_1B_1C_1D_1$ is a rhombus. If around the quadrangle $ABCD$ is described a circle, then $A_1B_1C_1D_1$ is a rectangle.*

The construction, which reveals how quadrangles $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ merged into a parallelogram (when F_1 get near to F_2) is really fascinating. We did it using GeoGebra.

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