AN ELEMENTARY PROOF OF LESTER'S THEOREM

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Abstract. In 1996, J. A. Lester discovered that in every scalene triangle the two Fermat-Torricelli points, the circumcenter, and the center of the nine-point circle are concyclic. We give the first proof of this fact to only employ results from elementary geometry.

In 1996, Professor of Mathematics June A. Lester discovered a remarkable new theorem in triangle geometry:

Lester's theorem. *In every scalene triangle, the two Fermat points, the circumcenter and the nine-point center are concyclic.*

The history of this theorem's discovery is almost as peculiar as the result itself. Here is J. A. Lester's own description:

"I discovered the theorem by searching through a large number of special triangle points for quadruples of points which lie on a circle [...] First, I needed a database of special points and their coordinates; I got it from Clark Kimberling's list of triangle centres [...] I next input everything – points, conversion formulas, cross ratio formulas – into an easy-to-use computer math program, *Theorist* [...] Then I input a single numerical shape [of a triangle] and set a search going for quadruples of special points with a real cross ratio."

The computation took several hours to complete and "had to be repeated multiple times to be sure that those real cross ratios found were not a coincidence." Probably for the first time in Euclidian geometry, a theorem was discovered by applying brute force by a person who specifically set out to do so!

Clearly, this approach does not hint to any actual proof of the theorem. June Lester's own initial proof made extensive use of complex numbers as well as extremely laborious calculations. Later on, simpler analytical proofs were discovered – but a *geometrical* proof was still lacking.

The statement of the theorem was communicated to the author of this note by his mathematics teacher Svetlozar Doychev in 2006, and it has been his dream to work out an elementary proof ever since. Finally, one such proof was discovered: it uses nothing more than the properties of similar figures and cyclic quadrilaterals.

First we establish the following

Lemma. *In a $\triangle ABC$ $(AB \neq AC)$, let P be the reflection of B in the line AC and let Q be the reflection of C in the line AB. Let the tangent to the circumcircle* *of* $\triangle APQ$ *at* A *meet the line* PQ *in* T*. Then, the reflection* U *of the point* T *in the point* A *lies in the line* BC *(Fig. 1).*

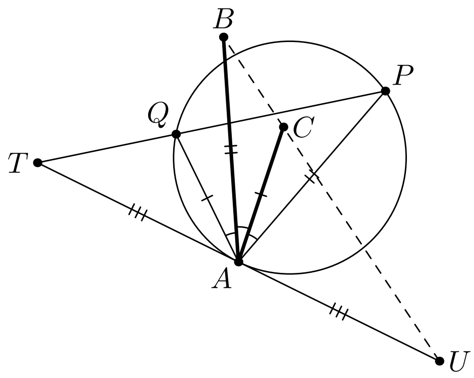

Fig. 1.

Proof. Let K and L be the reflections of the points B and C in the point A. Clearly, it suffices to show that T lies in the line KL (Fig. 2)

We have $AL = AC = AQ$ and $AK = AB = AP$. Besides, $\angle LAQ = 180 - \angle QAC = 180 - 2\angle BAC = 180 - \angle BAP = \angle PAK$. It follows that $\triangle LAQ$ and $\triangle PAK$ are two similar isosceles triangles with base angles equal to $\angle BAC$.

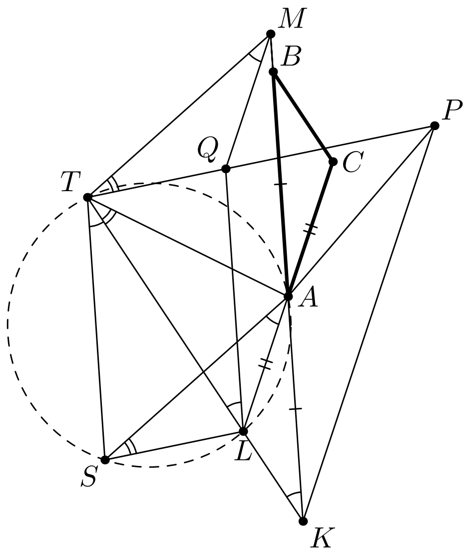

Fig. 2.

By the construction of the point T, the triangles $\triangle TQA$ and $\triangle TAP$ are similar. Therefore, it is possible to construct the point M such that the figures $TQAM$ and $TAPK$ are similar.

Since $\triangle MQA \sim \triangle KAP \sim \triangle QAL \Rightarrow \triangle MQA \simeq \triangle QAL$, the figure $ALQM$ is a parallelogram. Furthermore, $\angle QAM = \angle KPA = \angle BAC = \angle AQB$ shows that the points A, B, K and M are collinear.

Let S be such a point that $\vec{TS} = \vec{QL} = \vec{MA}$ and $\triangle ALS \simeq \triangle MQT$. It follows that $\angle LAS = \angle QMT =$ (as $TQAM \sim TAPK$) $= \angle TKA =$ (as $KA \| LQ$) $= \angle TLQ = \angle LTS$, i.e., that the quadrilateral $LATS$ is cyclic.

Therefore, $\angle ATL = \angle ASL = \angle MTQ =$ (as $TQAM \sim TAPK$) $= \angle ATK \Rightarrow T \in KL$, as needed. This completes the proof of the lemma. □

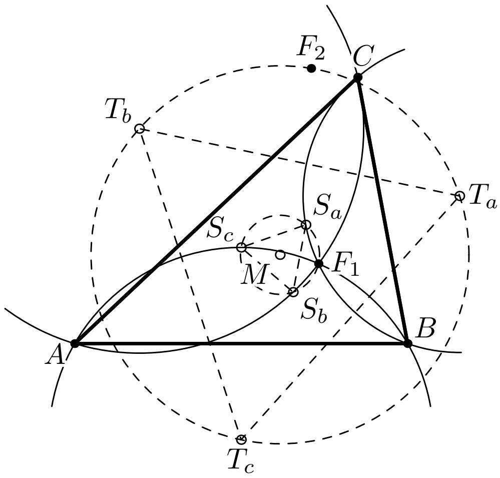

Fig. 3.

Proof of Lester's theorem. Let $\triangle ABC$ be an arbitrary scalene triangle. Construct externally on its sides BC, CA and AB three equilateral triangles δ_a, δ_b and δ_c of centers T_a, T_b and T_c, respectively. Also, construct internally on its sides BC, CA and AB three more equilateral triangles σ_a, σ_b and σ_c of centers S_a, S_b and S_c, respectively.

It is well known that the three circumcircles of the triangles δ_a, δ_b and δ_c meet in the first Fermat point F_1 and that the three circumcircles of the triangles σ_a, σ_b and σ_c meet in the second Fermat point F_2 for the triangle $\triangle ABC$. Also, it is well known that $\triangle T_aT_bT_c$ and $\triangle S_aS_bS_c$ are two equilateral triangles and that the centroid M of $\triangle ABC$ is their common center (Fig. 3).

Notice that $\angle S_bF_1S_c = \angle S_bF_1A + \angle AF_1S_c =$ (as the quadrilaterals AS_bF_1C and BF_1S_cA are both cyclic) $= \angle S_bCA + \angle ABS_c = 30° + 30° = 60° = \angle S_bS_aS_c \Rightarrow F_1$ lies in the circumcircle of $\triangle S_aS_bS_c$.

It follows that S_aF_1 is a common chord of two circles of centers M and T_a, respectively, and that F_1 is the reflection of the point S_a in the line MT_a. Analogously, F_2 is the reflection of the point T_a in the line MS_a.

Now we can apply the lemma to $\triangle MS_aT_a$ ($MS_a \neq MT_a$ as $\triangle ABC$ is non-degenerate). Let the tangent to the circumcircle of $\triangle MF_1F_2$ at M meet the line F_1F_2 in Q and let O' be the reflection of Q in the point M. By the lemma, we conclude that the point O' lies in the line S_aT_a.

Analogous application of the lemma to triangles $\triangle MS_bT_b$ and $\triangle MS_cT_c$ yields that the point O' lies in the lines S_bT_b and S_cT_c as well. But these three lines are the perpendicular bisectors of the sides of $\triangle ABC$! It follows that the point O' coincides with the circumcenter O of $\triangle ABC$.

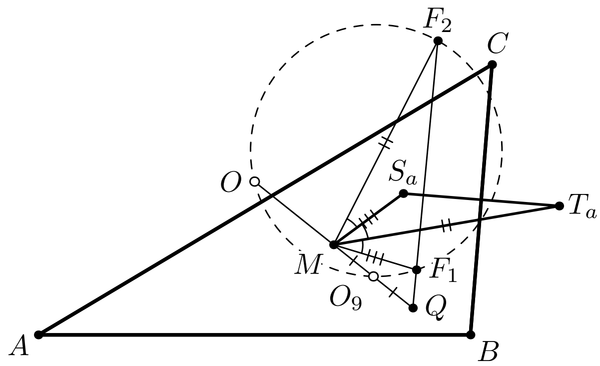

Fig. 4.

Finally, let O_9 be the nine-point center for $\triangle ABC$. It is well-known that the point O_9 lies in the line OM and divides the segment OM externally in ratio $OO_9 : O_9M = 2 : 1$ (Fig. 4).

Therefore, O_9 is the midpoint of the segment MQ. It follows that $QF_1 \cdot QF_2 = QM^2 = (2QM) \cdot (\frac{1}{2}QM) = QO \cdot QO_9$ and the quadrilateral $F_1F_2OO_9$ is cyclic, as needed. This completes the proof of Lester's theorem. □

References

[1] J. A. Lester. The mathematics behind the theorem. *http://thejuniverse.org/PUBLIC/LesterCircle/LCmath.html.*

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[3] P. Yiu. The circles of Lester, Evans, Parry, and their generalizations. In *Forum Geometricorum*, volume 10, pages 175–209, 2010.

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