XII GEOMETRICAL OLYMPIAD IN HONOUR OF I. F. SHARYGIN

The Correspondence Round

Below is the list of problems for the first (correspondence) round of the XII Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of four elder grades. In Russian school, these are 8-11. In the list below, each problem is indicated by the numbers of Russian school grades, for which it is intended. Foreign students of the last grade have to solve the problems for 11th grade, students of the preceding grade solve the problems for 10th grade etc. However, the participants may solve problems for elder grades as well (solutions of problems for younger grades will not be considered).

A complete solution of each problem costs 7 points. A partial solution costs from 1 to 6 points. A solution without significant advancement costs 0 points. The result of a participant is the sum of all obtained marks.

In your work, please start the solution for each problem in a new page. First write down the statement of the problem, and then the solution. Present your solutions in detail, including all necessary arguments and calculations. Provide all necessary figures of sufficient size. If a problem has an explicit answer, this answer must be presented distinctly. Please, be accurate to provide good understanding and correct estimating of your work !

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

You may note the problems which you liked most (this is not obligatory). Your opinion is interesting for the Jury.

The solutions for the problems (in Russian or in English) must be delivered not earlier than on January 8, 2016 and not later than on April 1, 2016. To upload your work, enter the site **http://geom.informatics.msk.ru** and follow the instructions.

Attention: The solutions must be contained in pdf, doc or jpg files. We recommend to prepare the paper using computer or to scan it rather than to photograph it. *In the last two cases, please check readability of the file before uploading.*

If you have any technical problems with uploading of the work, apply to **geomolymp@mccme.ru**.

The solutions can also be sent by e-mail to the special address **geompapers@yandex.ru**. (*If you send the work to another address the Organizing Committee can't guarantee that it will be received*). In this case the work also will be uploaded to the server. We recommend the authors to do this by their own. If you send your work by e-mail, please follow a few simple rules:

1. Each student sends his work in a separate message (with delivery notification).

2. If your work consists of several files, send it as an archive.

3. In the subject of the message write "The work for Sharygin olympiad", and present the following personal data in the body of your message:

- last name;

- all other names;

- E-mail, phone number, post address;

- the current number of your grade at school;

- the number of the last grade at your school;

- the number and/or the name and the mail address of your school;

- full names of your teachers in mathematics at school and/or of instructors of your extra math classes (if you attend additional math classes after school).

If you have no possibility to deliver the work by Internet, please inform the Organizing Committee to find a specific solution for this case.

Winners of the correspondence round, the students of three grades before the last grade, will be invited to the final round held in Summer 2016 in Moscow region. (For instance, if the last grade is 12 then we invite winners from 9, 10, and 11 grade.) The students of the last grade, winners of the correspondence round, will be awarded by diplomas of the Olympiad. The list of the winners will be published on **www.geometry.ru** at the end of May 2016 at latest. If you want to know your detailed results, please use e-mail **geomolymp@mccme.ru**.

(1) (8) A trapezoid $ABCD$ with bases AD and BC is such that $AB = BD$. Let M be the midpoint of DC. Prove that $\angle MBC = \angle BCA$.

(2) (8) Mark three nodes on a cellular paper so that the semiperimeter of the obtained triangle would be equal to the sum of its two smallest medians.

(3) (8) Let AH_1, BH_2 be two altitudes of an acute-angled triangle ABC, D be the projection of H_1 to AC, E be the projection of D to AB, F be the common point of ED and AH_1. Prove that $H_2F \parallel BC$.

(4) (8) In quadrilateral $ABCD$ $\angle B = \angle D = 90^\circ$ and $AC = BC + DC$. Point P of ray BD is such that $BP = AD$. Prove that line CP is parallel to the bisector of angle ABD.

(5) (8) In quadrilateral $ABCD$ $AB = CD$, M and K are the midpoints of BC and AD. Prove that the angle between MK and AC is equal to the half-sum of angles BAC and DCA.

(6) (8) Let M be the midpoint of side AC of triangle ABC, MD and ME be the perpendiculars from M to AB and BC respectively. Prove that the distance between the circumcenters of triangles ABE and BCD is equal to $AC/4$.

(7) (8–9) Let all distances between the vertices of a convex n-gon ($n > 3$) be different.

a) A vertex is called uninteresting if the closest vertex is adjacent to it. What is the minimal possible number of uninteresting vertices (for a given n)?

b) A vertex is called unusual if the farthest vertex is adjacent to it. What is the maximal possible number of unusual vertices (for a given n)?

(8) (8–9) Let $ABCDE$ be an inscribed pentagon such that $\angle B + \angle E = \angle C + \angle D$. Prove that $\angle CAD < \pi/3 < \angle A$.

(9) (8–9) Let ABC be a right-angled triangle and CH be the altitude from its right angle C. Points O_1 and O_2 are the incenters of triangles ACH and BCH respectively; P_1 and P_2 are the touching points of their incircles with AC and BC. Prove that lines O_1P_1 and O_2P_2 meet on AB.

(10) (8–9) Point X moves along side AB of triangle ABC, and point Y moves along its circumcircle in such a way that line XY passes through the midpoint of arc AB. Find the locus of the circumcenters of triangles IXY, where I is the incenter of ABC.

(11) (8–10) Restore a triangle ABC by vertex B, the centroid and the common point of the symmedian from B with the circumcircle.

(12) (9–10) Let BB_1 be the symmedian of a nonisosceles acute-angled triangle ABC. Ray BB_1 meets the circumcircle of ABC for the second time at point L. Let AH_A, BH_B, CH_C be the altitudes of triangle ABC. Ray BH_B meets the circumcircle of ABC for the second time at point T. Prove that H_A, H_C, T, L are concyclic.

(13) (9–10) Given are a triangle ABC and a line ℓ meeting BC, AC, AB at points L_a, L_b, L_c respectively. The perpendicular from L_a to BC meets AB and AC at points A_B and A_C respectively. Point O_a is the circumcenter of triangle AA_bA_c. Points O_b and O_c are defined similarly. Prove that O_a, O_b and O_c are collinear.

(14) Let a triangle ABC be given. Consider the circle touching its circumcircle at A and touching externally its incircle at some point A_1. Points B_1 and C_1 are defined similarly.

a) (9–10) Prove that lines AA_1, BB_1 and CC_1 concur.

b) (10–11) Let A_2 be the touching point of the incircle with BC. Prove that lines AA_1 and AA_2 are symmetric about the bisector of angle A.

(15) (9–11) Let O, M, N be the circumcenter, the centroid and the Nagel point of a triangle. Prove that angle MON is right if and only if one of the triangle's angles is equal to $60°$.

(16) (9–11) Let BB_1 and CC_1 be altitudes of triangle ABC. The tangents to the circumcircle of AB_1C_1 at B_1 and C_1 meet AB and AC at points M and N respectively. Prove that the common point of circles AMN and AB_1C_1 distinct from A lies on the Euler line of ABC.

(17) (9–11) Let D be an arbitrary point on side BC of triangle ABC. Circles ω_1 and ω_2 pass through A and D in such a way that BA touches ω_1 and CA touches ω_2. Let BX be the second tangent from B to ω_1, and CY be

the second tangent from C to ω_2. Prove that the circumcircle of triangle XDY touches BC.

(18) (9–11) Let ABC be a triangle with $\angle C = 90^\circ$, and K, L be the midpoints of the minor arcs AC and BC of its circumcircle. Segment KL meets AC at point N. Find angle NIC where I is the incenter of ABC.

(19) (9–11) Let $ABCDEF$ be a regular hexagon. Points P and Q on tangents to its circumcircle at A and D respectively are such that PQ touches the minor arc EF of this circle. Find the angle between PB and QC.

(20) (10–11) The incircle ω of a triangle ABC touches BC, AC and AB at points A_0, B_0 and C_0 respectively. The bisectors of angles B and C meet the perpendicular bisector to segment AA_0 at points Q and P respectively. Prove that PC_0 and QB_0 meet on ω.

(21) (10–11) The areas of rectangles P and Q are equal, but the diagonal of P is greater. Rectangle Q can be covered by two copies of P. Prove that P can be covered by two copies of Q.

(22) (10–11) Let M_A, M_B, M_C be the midpoints of the sides of a nonisosceles triangle ABC. Points H_A, H_B, H_C lying on the correspondent sides and distinct from M_A, M_B, M_C are such that $M_AH_B = M_AH_C$, $M_BH_A = M_BH_C$, $M_CH_A = M_CH_B$. Prove that H_A, H_B, H_C are the bases of the altitudes of ABC.

(23) (10–11) A sphere touches all edges of a tetrahedron. Let a, b, c and d be the segments of the tangents to the sphere from the vertices of the tetrahedron. Is it true that that some of these segments necessarily form a triangle? (It is not obligatory to use all segments. The side of the triangle can be formed by two segments)

(24) (11) A sphere is inscribed into a prism $ABCA'B'C'$ and touches its lateral faces $BCC'B'$, $CAA'C'$, $ABB'A'$ at points A_0, B_0, C_0 respectively. It is known that $\angle A_0BB' = \angle B_0CC' = \angle C_0AA'$. a) Find all possible values of these angles.

b) Prove that segments AA_0, BB_0, CC_0 concur.

c) Prove that the projections of the incenter to $A'B'$, $B'C'$, $C'A'$ are the vertices of a regular triangle.