

## XII GEOMETRICAL OLYMPIAD IN HONOUR OF I. F. SHARYGIN

### The Correspondence Round

Below is the list of problems for the first (correspondence) round of the XII Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of four elder grades. In Russian school, these are 8-11. In the list below, each problem is indicated by the numbers of Russian school grades, for which it is intended. Foreign students of the last grade have to solve the problems for 11th grade, students of the preceding grade solve the problems for 10th grade etc. However, the participants may solve problems for elder grades as well (solutions of problems for younger grades will not be considered).

A complete solution of each problem costs 7 points. A partial solution costs from 1 to 6 points. A solution without significant advancement costs 0 points. The result of a participant is the sum of all obtained marks.

In your work, please start the solution for each problem in a new page. First write down the statement of the problem, and then the solution. Present your solutions in detail, including all necessary arguments and calculations. Provide all necessary figures of sufficient size. If a problem has an explicit answer, this answer must be presented distinctly. Please, be accurate to provide good understanding and correct estimating of your work !

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

You may note the problems which you liked most (this is not obligatory). Your opinion is interesting for the Jury.

The solutions for the problems (in Russian or in English) must be delivered not earlier than on January 8, 2016 and not later than on April 1, 2016. To upload your work, enter the site <http://geom.informatics.msk.ru> and follow the instructions.

**Attention:** The solutions must be contained in pdf, doc or jpg files. We recommend to prepare the paper using computer or to scan it rather than to photograph it. *In the last two cases, please check readability of the file before uploading.*

If you have any technical problems with uploading of the work, apply to [geomolymp@mccme.ru](mailto:geomolymp@mccme.ru).

The solutions can also be sent by e-mail to the special address [geompapers@yandex.ru](mailto:geompapers@yandex.ru). (*If you send the work to another address the Organizing Committee can't guarantee that it will be received*). In this case the work also

will be uploaded to the server. We recommend the authors to do this by their own. If you send your work by e-mail, please follow a few simple rules:

1. *Each student sends his work in a separate message (with delivery notification).*

2. *If your work consists of several files, send it as an archive.*

3. *In the subject of the message write "The work for Sharygin olympiad", and present the following personal data in the body of your message:*

- *last name;*
- *all other names;*
- *E-mail, phone number, post address;*
- *the current number of your grade at school;*
- *the number of the last grade at your school;*
- *the number and/or the name and the mail address of your school;*
- *full names of your teachers in mathematics at school and/or of instructors of your extra math classes (if you attend additional math classes after school).*

If you have no possibility to deliver the work by Internet, please inform the Organizing Committee to find a specific solution for this case.

Winners of the correspondence round, the students of three grades before the last grade, will be invited to the final round held in Summer 2016 in Moscow region. (For instance, if the last grade is 12 then we invite winners from 9, 10, and 11 grade.) The students of the last grade, winners of the correspondence round, will be awarded by diplomas of the Olympiad. The list of the winners will be published on **www.geometry.ru** at the end of May 2016 at latest. If you want to know your detailed results, please use e-mail **geomolymp@mccme.ru**.

- (1) (8) A trapezoid  $ABCD$  with bases  $AD$  and  $BC$  is such that  $AB = BD$ . Let  $M$  be the midpoint of  $DC$ . Prove that  $\angle MBC = \angle BCA$ .
- (2) (8) Mark three nodes on a cellular paper so that the semiperimeter of the obtained triangle would be equal to the sum of its two smallest medians.
- (3) (8) Let  $AH_1$ ,  $BH_2$  be two altitudes of an acute-angled triangle  $ABC$ ,  $D$  be the projection of  $H_1$  to  $AC$ ,  $E$  be the projection of  $D$  to  $AB$ ,  $F$  be the common point of  $ED$  and  $AH_1$ . Prove that  $H_2F \parallel BC$ .
- (4) (8) In quadrilateral  $ABCD$   $\angle B = \angle D = 90^\circ$  and  $AC = BC + DC$ . Point  $P$  of ray  $BD$  is such that  $BP = AD$ . Prove that line  $CP$  is parallel to the bisector of angle  $ABD$ .
- (5) (8) In quadrilateral  $ABCD$   $AB = CD$ ,  $M$  and  $K$  are the midpoints of  $BC$  and  $AD$ . Prove that the angle between  $MK$  and  $AC$  is equal to the half-sum of angles  $BAC$  and  $DCA$ .
- (6) (8) Let  $M$  be the midpoint of side  $AC$  of triangle  $ABC$ ,  $MD$  and  $ME$  be the perpendiculars from  $M$  to  $AB$  and  $BC$  respectively. Prove that the distance between the circumcenters of triangles  $ABE$  and  $BCD$  is equal to  $AC/4$ .
- (7) (8–9) Let all distances between the vertices of a convex  $n$ -gon ( $n > 3$ ) be different.

- a) A vertex is called uninteresting if the closest vertex is adjacent to it. What is the minimal possible number of uninteresting vertices (for a given  $n$ )?
- b) A vertex is called unusual if the farthest vertex is adjacent to it. What is the maximal possible number of unusual vertices (for a given  $n$ )?
- (8) (8–9) Let  $ABCDE$  be an inscribed pentagon such that  $\angle B + \angle E = \angle C + \angle D$ . Prove that  $\angle CAD < \pi/3 < \angle A$ .
- (9) (8–9) Let  $ABC$  be a right-angled triangle and  $CH$  be the altitude from its right angle  $C$ . Points  $O_1$  and  $O_2$  are the incenters of triangles  $ACH$  and  $BCH$  respectively;  $P_1$  and  $P_2$  are the touching points of their incircles with  $AC$  and  $BC$ . Prove that lines  $O_1P_1$  and  $O_2P_2$  meet on  $AB$ .
- (10) (8–9) Point  $X$  moves along side  $AB$  of triangle  $ABC$ , and point  $Y$  moves along its circumcircle in such a way that line  $XY$  passes through the midpoint of arc  $AB$ . Find the locus of the circumcenters of triangles  $IXY$ , where  $I$  is the incenter of  $ABC$ .
- (11) (8–10) Restore a triangle  $ABC$  by vertex  $B$ , the centroid and the common point of the symmedian from  $B$  with the circumcircle.
- (12) (9–10) Let  $BB_1$  be the symmedian of a nonisosceles acute-angled triangle  $ABC$ . Ray  $BB_1$  meets the circumcircle of  $ABC$  for the second time at point  $L$ . Let  $AH_A, BH_B, CH_C$  be the altitudes of triangle  $ABC$ . Ray  $BH_B$  meets the circumcircle of  $ABC$  for the second time at point  $T$ . Prove that  $H_A, H_C, T, L$  are concyclic.
- (13) (9–10) Given are a triangle  $ABC$  and a line  $\ell$  meeting  $BC, AC, AB$  at points  $L_a, L_b, L_c$  respectively. The perpendicular from  $L_a$  to  $BC$  meets  $AB$  and  $AC$  at points  $A_B$  and  $A_C$  respectively. Point  $O_a$  is the circumcenter of triangle  $AA_BA_C$ . Points  $O_b$  and  $O_c$  are defined similarly. Prove that  $O_a, O_b$  and  $O_c$  are collinear.
- (14) Let a triangle  $ABC$  be given. Consider the circle touching its circumcircle at  $A$  and touching externally its incircle at some point  $A_1$ . Points  $B_1$  and  $C_1$  are defined similarly.
- a) (9–10) Prove that lines  $AA_1, BB_1$  and  $CC_1$  concur.
- b) (10–11) Let  $A_2$  be the touching point of the incircle with  $BC$ . Prove that lines  $AA_1$  and  $AA_2$  are symmetric about the bisector of angle  $A$ .
- (15) (9–11) Let  $O, M, N$  be the circumcenter, the centroid and the Nagel point of a triangle. Prove that angle  $MON$  is right if and only if one of the triangle's angles is equal to  $60^\circ$ .
- (16) (9–11) Let  $BB_1$  and  $CC_1$  be altitudes of triangle  $ABC$ . The tangents to the circumcircle of  $AB_1C_1$  at  $B_1$  and  $C_1$  meet  $AB$  and  $AC$  at points  $M$  and  $N$  respectively. Prove that the common point of circles  $AMN$  and  $AB_1C_1$  distinct from  $A$  lies on the Euler line of  $ABC$ .
- (17) (9–11) Let  $D$  be an arbitrary point on side  $BC$  of triangle  $ABC$ . Circles  $\omega_1$  and  $\omega_2$  pass through  $A$  and  $D$  in such a way that  $BA$  touches  $\omega_1$  and  $CA$  touches  $\omega_2$ . Let  $BX$  be the second tangent from  $B$  to  $\omega_1$ , and  $CY$  be

- the second tangent from  $C$  to  $\omega_2$ . Prove that the circumcircle of triangle  $XDY$  touches  $BC$ .
- (18) (9–11) Let  $ABC$  be a triangle with  $\angle C = 90^\circ$ , and  $K, L$  be the midpoints of the minor arcs  $AC$  and  $BC$  of its circumcircle. Segment  $KL$  meets  $AC$  at point  $N$ . Find angle  $NIC$  where  $I$  is the incenter of  $ABC$ .
- (19) (9–11) Let  $ABCDEF$  be a regular hexagon. Points  $P$  and  $Q$  on tangents to its circumcircle at  $A$  and  $D$  respectively are such that  $PQ$  touches the minor arc  $EF$  of this circle. Find the angle between  $PB$  and  $QC$ .
- (20) (10–11) The incircle  $\omega$  of a triangle  $ABC$  touches  $BC, AC$  and  $AB$  at points  $A_0, B_0$  and  $C_0$  respectively. The bisectors of angles  $B$  and  $C$  meet the perpendicular bisector to segment  $AA_0$  at points  $Q$  and  $P$  respectively. Prove that  $PC_0$  and  $QB_0$  meet on  $\omega$ .
- (21) (10–11) The areas of rectangles  $P$  and  $Q$  are equal, but the diagonal of  $P$  is greater. Rectangle  $Q$  can be covered by two copies of  $P$ . Prove that  $P$  can be covered by two copies of  $Q$ .
- (22) (10–11) Let  $M_A, M_B, M_C$  be the midpoints of the sides of a nonisosceles triangle  $ABC$ . Points  $H_A, H_B, H_C$  lying on the correspondent sides and distinct from  $M_A, M_B, M_C$  are such that  $M_AH_B = M_AH_C, M_BH_A = M_BH_C, M_CH_A = M_CH_B$ . Prove that  $H_A, H_B, H_C$  are the bases of the altitudes of  $ABC$ .
- (23) (10–11) A sphere touches all edges of a tetrahedron. Let  $a, b, c$  and  $d$  be the segments of the tangents to the sphere from the vertices of the tetrahedron. Is it true that that some of these segments necessarily form a triangle? (It is not obligatory to use all segments. The side of the triangle can be formed by two segments)
- (24) (11) A sphere is inscribed into a prism  $ABC A' B' C'$  and touches its lateral faces  $BCC' B', CAA' C', ABB' A'$  at points  $A_0, B_0, C_0$  respectively. It is known that  $\angle A_0 B B' = \angle B_0 C C' = \angle C_0 A A'$ . a) Find all possible values of these angles.  
 b) Prove that segments  $AA_0, BB_0, CC_0$  concur.  
 c) Prove that the projections of the incenter to  $A' B', B' C', C' A'$  are the vertices of a regular triangle.