XI GEOMETRICAL OLYMPIAD IN HONOUR OF I. F. SHARYGIN

The Correspondence Round

Below is the list of problems for the first (correspondence) round of the XI Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of four elder grades. In Russian school, these are 8-11. In the list below, each problem is indicated by the numbers of Russian school grades, for which it is intended. Foreign students of the last grade have to solve the problems for 11th grade, students of the preceding grade solve the problems for 10th grade etc. However, the participants may solve problems for elder grades as well (solutions of problems for younger grades will not be considered).

The full solution of each problem costs 7 points. The partial solution costs from 1 to 6 points. The solution without significant advancement costs 0 points. The result of the participant is the sum of all obtained marks.

In your work, please start the solution for each problem in a new page. First write down the statement of the problem, and then the solution. Present your solutions in detail, including all necessary arguments and calculations. Provide all necessary figures of sufficient size. If a problem has an explicit answer, this answer must be presented distinctly. Please, be accurate to provide good understanding and correct estimating of your work !

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

You may note the problems which you liked most (this is not obligatory). Your opinion is interesting for the Jury.

The solutions for the problems (in Russian or in English) must be sent not earlier than on January 8, 2015 and not later than on April 1, 2015. For sending your work, enter the site **http://geom.informatics.msk.ru** and follow the instructions.

Attention: The solutions must be contained in pdf, doc or jpg files. We recommend to prepare the paper using computer or to scan it rather than to photograph it. *In the last two cases, please check readability of the file before sending.*

If you have any technical problems with uploading of the work, write to **geomolymp@mccme.ru**.

The solutions can also be sent by e-mail to the special address **geompapers@yandex.ru** (*If you send the work to another address the Organizing Committee can't guarantee that it will be received*). In this case the work also will be loaded to the server. We recommend the authors to do this themselves. If you send your work by e-mail, please follow a few simple rules:

1. Each student sends his work in a separate message (with delivery notification).

2. If your work consists of several files, send it as an archive.

3. In the subject of the message write "The work for Sharygin olympiad", and present the following personal data in the body of your message:

- last name;

- all other names;

- E-mail, phone number, post address;

- the current number of your grade at school;

- the number of the last grade at your school;

- the number and/or the name and the mail address of your school;

- full names of your teachers in mathematics at school and/or of instructors of your extra math classes (if you attend additional math classes after school).

If you have no possibility to send the work by e-mail, please inform the Organizing Committee to find a specific solution for this case.

Winners of the correspondence round, the students of three grades before the last grade, will be invited to the final round in Summer 2015 in Moscow region. (For instance, if the last grade is 12, then we invite winners from 9, 10, and 11 grade.) The students of the last grade, winners of the correspondence round, will be awarded by diplomas of the Olympiad. The list of the winners will be published on www.geometry.ru at the end of May 2015. If you want to know your detailed results, please use e-mail.

(1) (8) Tanya cut out a convex polygon from the paper, fold it several times and obtained a two-layers quadrilateral. Can the cutted polygon be a heptagon?

(2) (8) Let O and H be the circumcenter and the orthocenter of a triangle ABC. The line passing through the midpoint of OH and parallel to BC meets AB and AC at points D and E. It is known that O is the incenter of triangle ADE. Find the angles of ABC.

(3) (8) The side AD of a square $ABCD$ is the base of an obtuse-angled isosceles triangle AED with vertex E lying inside the square. Let AF be a diameter of the circumcircle of this triangle, and G be a point on CD such that $CG = DF$. Prove that angle BGE is less than half of angle AED.

(4) (8) In a parallelogram $ABCD$ the trisectors of angles A and B are drawn. Let O be the common points of the trisectors nearest to AB. Let AO meet the second trisector of angle B at point A_1, and let BO meet the second trisector of angle A at point B_1. Let M be the midpoint of A_1B_1. Line MO meets AB at point N. Prove that triangle A_1B_1N is equilateral.

(5) (8–9) Let a triangle ABC be given. Two circles passing through A touch BC at points B and C respectively. Let D be the second common point

of these circles (A is closer to BC than D). It is known that $BC = 2BD$. Prove that $\angle DAB = 2\angle ADB$.

(6) (8–9) Let AA', BB' and CC' be the altitudes of an acute-angled triangle ABC. Points C_a, C_b are symmetric to C' wrt AA' and BB'. Points A_b, A_c, B_c, B_a are defined similarly. Prove that lines A_bB_a, B_cC_b and C_aA_c are parallel.

(7) (8–9) The altitudes AA_1 and CC_1 of a triangle ABC meet at point H. Point H_A is symmetric to H about A. Line H_AC_1 meets BC at point C'; point A' is defined similarly. Prove that $A'C'||AC$.

(8) (8–9) Diagonals of an isosceles trapezoid $ABCD$ with bases BC and AD are perpendicular. Let DE be the perpendicular from D to AB, and let CF be the perpendicular from C to DE. Prove that angle DBF is equal to half of angle FCD.

(9) (8–9) Let ABC be an acute-angled triangle. Construct points A', B', C' on its sides BC, CA, AB such that:
- $A'B' \parallel AB$;
- $C'C$ is the bisector of angle $A'C'B'$;
- $A'C' + B'C' = AB$.

(10) (8–9) The diagonals of a convex quadrilateral divide it into four similar triangles. Prove that is possible to inscribe a circle into this quadrilateral.

(11) (8–10) Let H be the orthocenter of an acute-angled triangle ABC. The perpendicular bisector to segment BH meets BA and BC at points A_0, C_0 respectively. Prove that the perimeter of triangle A_0OC_0 (O is the circumcenter of $\triangle ABC$) is equal to AC.

(12) (8–11) Find the maximal number of discs which can be disposed on the plane so that each two of them have a common point and no three have it.

(13) (9–10) Let AH_1, BH_2 and CH_3 be the altitudes of a triangle ABC. Point M is the midpoint of H_2H_3. Line AM meets H_2H_1 at point K. Prove that K lies on the medial line of ABC parallel to AC.

(14) (9–11) Let ABC be an acute-angled, nonisosceles triangle. Point A_1, A_2 are symmetric to the feet of the internal and the external bisectors of angle A wrt the midpoint of BC. Segment A_1A_2 is a diameter of a circle α. Circles β and γ are defined similarly. Prove that these three circles have two common points.

(15) (9–11) The sidelengths of a triangle ABC are not greater than 1. Prove that $p(1 - 2Rr)$ is not greater than 1, where p is the semiperimeter, R and r are the circumradius and the inradius of ABC.

(16) (9–11) The diagonals of a convex quadrilateral divide it into four triangles. Restore the quadrilateral by the circumcenters of two adjacent triangles and the incenters of two mutually opposite triangles.

(17) (10–11) Let O be the circumcenter of a triangle ABC. The projections of points D and X to the sidelines of the triangle lie on lines l and L such

that $l \parallel XO$. Prove that the angles formed by L and by the diagonals of quadrilateral $ABCD$ are equal.

(18) (10–11) Let $ABCDEF$ be a cyclic hexagon, points K, L, M, N be the common points of lines AB and CD, AC and BD, AF and DE, AE and DF respectively. Prove that if three of these points are collinear then the fourth point lies on the same line.

(19) (10–11) Let L and K be the feet of the internal and the external bisector of angle A of a triangle ABC. Let P be the common point of the tangents to the circumcircle of the triangle at B and C. The perpendicular from L to BC meets AP at point Q. Prove that Q lies on the medial line of triangle LKP.

(20) (10–11) Given are a circle and an ellipse lying inside it with focus C. Find the locus of the circumcenters of triangles ABC, where AB is a chord of the circle touching the ellipse.

(21) (10–11) A quadrilateral $ABCD$ is inscribed into a circle ω with center O. Let M_1 and M_2 be the midpoints of segments AB and CD respectively. Let Ω be the circumcircle of triangle OM_1M_2. Let X_1 and X_2 be the common points of ω and Ω, and Y_1 and Y_2 the second common points of Ω with the circumcircles of triangles CDM_1 and ABM_2. Prove that $X_1X_2||Y_1Y_2$.

(22) (10–11) The faces of an icosahedron are painted into 5 colors in such a way that two faces painted into the same color have no common points, even a vertices. Prove that for any point lying inside the icosahedron the sums of the distances from this point to the red faces and the blue faces are equal.

(23) (11) A tetrahedron $ABCD$ is given. The incircles of triangles ABC and ABD with centers O_1, O_2, touch AB at points T_1, T_2. The plane π_{AB} passing through the midpoint of T_1T_2 is perpendicular to O_1O_2. The planes π_{AC}, π_{BC}, π_{AD}, π_{BD}, π_{CD} are defined similarly. Prove that these six planes have a common point.

(24) (11) The insphere of a tetrahedron $ABCD$ with center O touches its faces at points A_1, B_1, C_1 and D_1.

a) Let P_a be a point such that its reflections in lines OB, OC and OD lie on plane BCD. Points P_b, P_c and P_d are defined similarly. Prove that lines A_1P_a, B_1P_b, C_1P_c and D_1P_d concur at some point P.

b) Let I be the incenter of $A_1B_1C_1D_1$ and A_2 the common point of line A_1I with plane $B_1C_1D_1$. Points B_2, C_2, D_2 are defined similarly. Prove that P lies inside $A_2B_2C_2D_2$.