INSCRIBED EQUILATERAL TRIANGLES

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Abstract. If equilateral triangles are inscribed in an irregular triangle in such a way that each edge (or its extension) of the latter contains a distinct vertex of each equilateral triangle, the centers of the equilateral triangles must lie on one of two lines, each of which is orthogonal to the Euler line of the irregular triangle.

Inscribing a similar triangle using spiral similarity. Given irregular triangle ABC and a point D_a on BC or its extension, we want to construct a triangle $D_aE_bF_c$ such that E_b and F_c lie on CA and AB (or their extensions) respectively, and $D_aE_bF_c$ is similar to a given triangle DEF. This is essentially Problem 30 in [6]. To do this we employ (following Yaglom) *spiral similarity*, a geometric transformation that includes a dilation and rotation, each with the same center. When we apply a spiral similarity to a figure, the resultant figure is similar to the original, since both dilation and rotation preserve similarity.

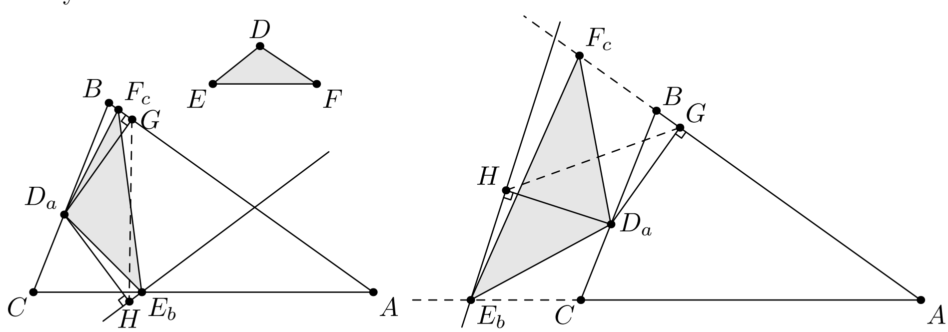

Fig. 1. Given D_a on edge BC of triangle ABC, there are two ways to inscribe a triangle $D_aE_bF_c$ similar to DEF. The one on the left (right) is *properly* (*improperly*) inscribed.

As shown in Fig. 1, we first find the orthogonal projection G of D_a on AB. We construct triangle GD_aH similar to DEF. We can do this in two ways. On the left, the triangles are directly similar; on the right, oppositely similar. The line orthogonal to D_aH at H is spirally similar to the edge on which G lies, and must contain E_b. Then triangle $D_aE_bF_c$ is similar to DEF.

What we learn from this is that given point D_a on edge BC of triangle ABC, there are two similar triangles that we can inscribe. In one case (left side of Fig. 1) the vertices of $D_aE_bF_c$ occur in the same sense as those of ABC as we traverse the perimeter, and we say that that triangle is *properly* inscribed. In the other case (right side of Fig. 1), the vertices of $D_aE_bF_c$ occur in the opposite sense as those of ABC as we traverse the perimeter, and we say that that triangle is *improperly* inscribed.[1] We want to show that

[1]This does not exhaust the possibilities for inscribing a triangle; two vertices could lie on the same edge.

the centers of the properly inscribed triangles (as we vary the location of D_a) all lie on a straight line, and that the same holds true for the improperly inscribed triangles.

Spiral similarity can propagate straight lines. As shown in Fig. 2, let us have a spiral similarity of lines A_1B_1 and A_2B_2, and a second spiral similarity of lines A_2B_2 and A_3B_3 using the same center of rotation O. This means that $\angle A_1OA_2 = \angle B_1OB_2$, $A_1O/A_2O = B_1O/B_2O$, $\angle A_2OA_3 = \angle B_2OB_3$, and $A_2O/A_3O = B_2O/B_3O$. We want to prove that if A_1, A_2, and A_3 are collinear, then B_1, B_2, and B_3 are collinear. This is essentially the first problem in §4.8 of [3]. Using the above, it is easy to show that $\triangle OA_1A_2 \sim \triangle OB_1B_2$ and $\triangle OA_2A_3 \sim \triangle OB_2B_3$ Therefore $\angle OB_2B_1 + \angle B_3B_2O = \angle OA_2A_1 + \angle A_3A_2O$. If the right side is a straight angle, then so is the left side.

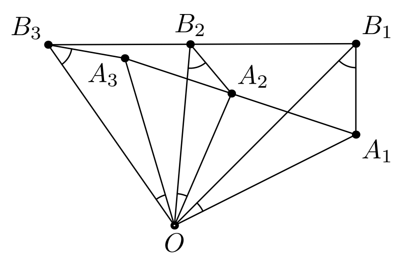

Fig. 2. Applying spiral similarities with center O to A_1B_1 we get A_2B_2 and to A_2B_2, we get A_3B_3. If A_1, A_2, and A_3 are collinear, are B_1, B_2, and B_3 also collinear?

Thus, if we apply a spiral similarity with a single center point to a figure in such a way that any point of the figure (different from the center point) travels on a straight line, then all of the other points of the figure will also travel along straight lines. If we properly inscribe two similar triangles $D_aE_bF_c$ and $D'_aE'_bF'_c$ in triangle ABC, there is a spiral similarity between the two. We can apply other spiral similarities to these two using the same center to obtain triangles such as $D''_aE''_bF''_c$ with D''_a lying on edge BC or its extension. But then E''_b must lie on edge CA or its extension and F''_c must lie on AB or its extension. Furthermore, if we pick an arbitrary triangle center (centroid, incenter, etc.), it will be located similarly in all of the inscribed triangles, and thus these centers for all of the inscribed triangles must be collinear. The same argument applies to triangles that are improperly inscribed. Their centers (whichever choice we make) will also lie on a single straight line.

Pedal triangles of the isodynamic points. We want to locate the line of centers for inscribed equilateral triangles, both for those properly inscribed and those improperly inscribed. In Fig. 3, we show the isodynamic points S_1 and S_2 of triangle ABC. These points are the intersection points of the three Apollonian circles. Each Apollonian circle has a diameter with end points that are the intersection of an edge of ABC with the interior and exterior bisectors of the opposite angle. It is known (see page 53 of [1]) that the pedal triangles of the isodynamic points are equilateral triangles. This will allow us to locate the two lines of equilateral triangle centers that we are seeking.

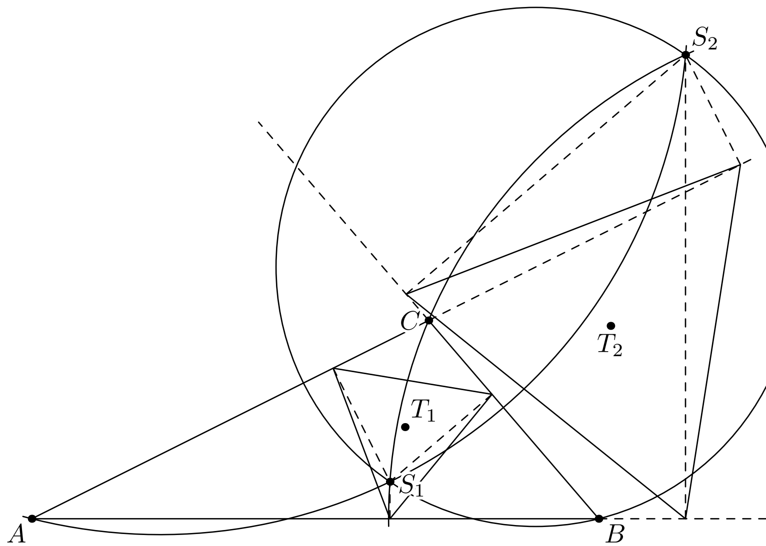

Fig. 3. The isodynamic points S_1 and S_2 are the two intersections of the Apollonian circles. Their pedal triangles are equilateral triangles with centers T_1 and T_2.

In Fig. 4, we do a spiral similarity of triangle $D_aE_bF_c$, the properly inscribed equilateral pedal triangle of the isodynamic point S_1. We use S_1 as the center of the transformation. The angle of rotation is θ, and the ratio of magnification is $\sec\theta$. The result is triangle $D'_aE'_bF'_c$ with center T'_1. Since $\angle D_aS_1D'_a = \theta$ and $S_1D'_a/S_1D_a = \sec\theta$, $\angle D'_aD_aS_1$ must be a right angle. Therefore D'_a lies on edge BC. Similarly E'_b and F'_c also lie on triangle edges. Finally, the same reasoning means that $T_1T'_1$ is orthogonal to S_1T_1. Thus all of the centers of properly inscribed equilateral triangles lie on a line through T_1 orthogonal to S_1T_1. A similar argument shows that all of the centers of improperly inscribed equilateral triangles lie on a line through T_2 orthogonal to S_2T_2.

Furthermore, we see that the $D'_aE'_b/S_1T'_1 = D_aE_b/S_1T_1$, because they are linked by a spiral similarity centered at S_1. Then the smallest properly inscribed triangle obtains when $S_1T'_1$ is a minimum, i.e., when it is orthogonal to the line of equilateral triangle centers. Therefore T_1 is the center of the smallest properly inscribed equilateral triangle, which is the pedal triangle of S_1. By similar reasoning, T_2 in Fig. 3 is the center of the smallest improperly inscribed equilateral triangle, the pedal triangle of S_2.

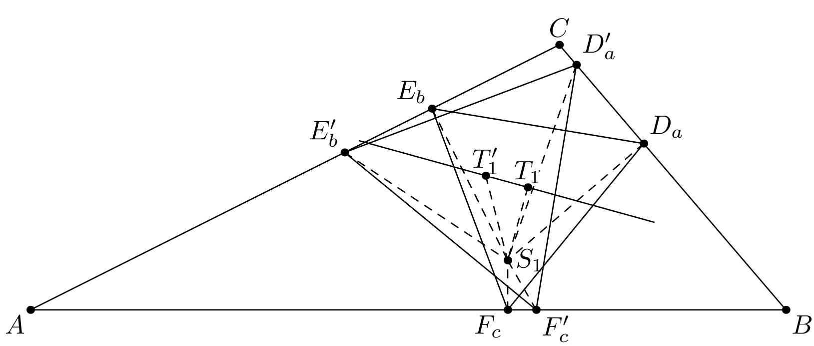

Fig. 4. The isodynamic point S_1 has an equilateral pedal triangle $D_aE_bF_c$ with center T_1.

The locus of equilateral triangle centers is orthogonal to the Euler line. We have shown that the centers of properly inscribed equilateral triangles lie on a line orthogonal to S_1T_1 and that the centers of improperly inscribed equilateral triangles lie on a line orthogonal to S_2T_2. We now want to prove that these lines are orthogonal to the Euler line of the triangle. We need only show that S_1T_1 and S_2T_2 are parallel to the Euler line.

We begin with the fact that the circumcenter, the isodynamic points, and the symmedian (Lemoine) point are collinear, as shown in §602 of [2]. The symmedian point is the concurrence of the symmedians. A symmedian of a triangle is the reflection of a median in the angle bisector through the same vertex. It is known that the distances of the symmedian point from the edges of the triangle are proportional to the lengths of the corresponding edges. See page 59 of [4]. Thus in Fig. 5, we have drawn three vectors from the symmedian point orthogonal to the edges, meeting each edge at a vertex of the pedal triangle of the symmedian point. We want to know the sum of these vectors. We know that

$$\frac{|\vec{a}|}{BC} = \frac{|\vec{b}|}{CA} = \frac{|\vec{c}|}{AB}.$$

If we rotate the vectors by a quarter turn, they are parallel to the respective edges, and we can form them into a triangle that is similar to ABC, and thus their sum is zero.

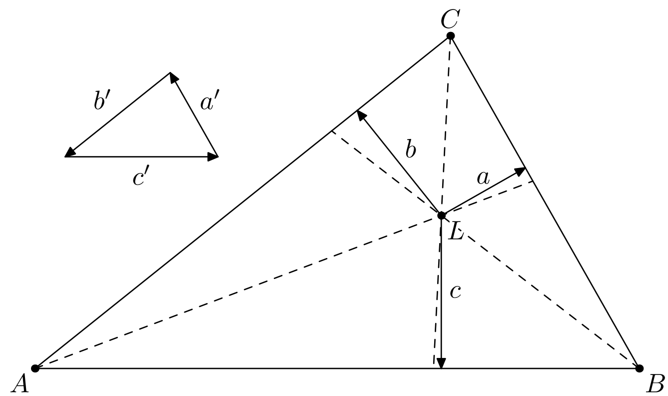

Fig. 5. The dashed lines are symmedians, which concur at the symmedian point L. Vectors from L to the vertices of its pedal triangle add up to zero.

We want to see what a similar vector sum is for the circumcenter of the triangle. In Fig. 6, we have drawn three vectors from the circumcenter O to the vertices of the pedal triangle of O with respect to triangle ABC. This triangle, $A'B'C'$, is also the medial triangle of triangle ABC. Since A' is the midpoint of BC, we know that $\overrightarrow{OA'} = (\overrightarrow{OB} + \overrightarrow{OC})/2$, and similarly for $\overrightarrow{OB'}$ and $\overrightarrow{OC'}$. Since G is the centroid of ABC,

$$\overrightarrow{OG} = (\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})/3 = (\overrightarrow{OA'} + \overrightarrow{OB'} + \overrightarrow{OC'})/3.$$

We know that O, G, and H lie on the Euler line, and that $OH = 3OG$, so that

$$\overrightarrow{OH} = 3\overrightarrow{OG} = \overrightarrow{OA'} + \overrightarrow{OB'} + \overrightarrow{OC'}.$$

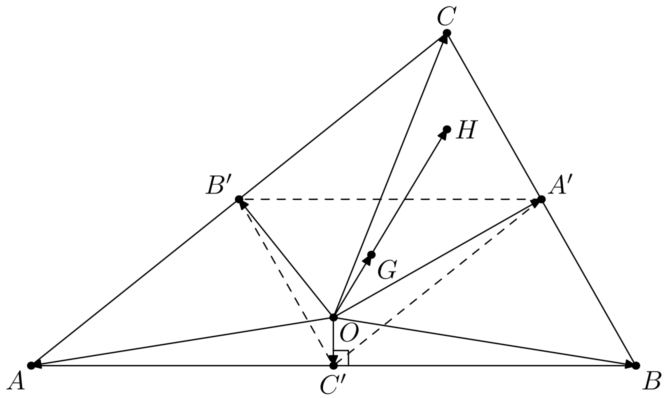

Fig. 6. Point O is the circumcenter of triangle ABC, G is its centroid, and H is its orthocenter. We want to show that the three vectors from O to the vertices of its pedal triangle sum to $\overrightarrow{OH} = \overrightarrow{OA'} + \overrightarrow{OB'} + \overrightarrow{OC'}$.

Now we know that the sum of the vectors from the symmedian point to the vertices of its pedal triangle is zero, and the sum of the vectors from the circumcenter to the vertices of its pedal triangle is the vector from it to the orthocenter, i.e., in the direction of the Euler line, which contains both points. But for any other point on the line through the symmedian point and the orthocenter, this must also be true because of the linear relationships. Then this must also be true of the isodynamic points,which are on this line. And if we sum three vectors from a point to the vertices of an equilateral triangle, the resultant is in the direction of the center of the equilateral triangle. Thus, the lines S_1T_1 and S_2T_2 are parallel to the Euler line, and the lines containing the centers of the inscribed equilateral triangles are orthogonal to the Euler line.

Trilinear coordinates. The trilinear coordinates for the isodynamic points and their pedal triangle centers from the *Encyclopedia of Triangle Centers* are

$$\begin{aligned}
S_1 &= \sin(A+\pi/3) : \sin(B+\pi/3) : \sin(C+\pi/3) && \text{X(15)}\\
&= \cos(A-\pi/6) : \cos(B-\pi/6) : \cos(C-\pi/6) &&\\
S_2 &= \sin(A-\pi/3) : \sin(B-\pi/3) : \sin(C-\pi/3) && \text{X(16)}\\
&= \cos(A+\pi/6) : \cos(B+\pi/6) : \cos(C+\pi/6) &&\\
T_1 &= \cos(B-C) + 2\cos(A-\pi/3) && \text{X(396)}\\
&\quad : \cos(C-A) + 2\cos(B-\pi/3) : \cos(A-B) + 2\cos(C-\pi/3) &&\\
T_2 &= \cos(B-C) + 2\cos(A+\pi/3) && \text{X(395)}\\
&\quad : \cos(C-A) + 2\cos(B+\pi/3) : \cos(A-B) + 2\cos(C+\pi/3). &&
\end{aligned}$$

The trilinear coordinates of the lines containing the centers of the inscribed equilateral triangles are

$$\begin{aligned} T_1T_1' &= \sin(A-\pi/3) : \sin(B-\pi/3) : \sin(C-\pi/3) \\ &= \cos(A+\pi/6) : \cos(B+\pi/6) : \cos(C+\pi/6) \\ T_2T_2' &= \sin(A+\pi/3) : \sin(B+\pi/3) : \sin(C+\pi/3) \\ &= \cos(A-\pi/6) : \cos(B-\pi/6) : \cos(C-\pi/6). \end{aligned}$$

Acknowledgements. The current version of this proof was primarily devised by the editors of this journal, whom I warmly thank. My original version relied heavily on analytics involving trilinear coordinates. It was much longer and more opaque. I learned a great deal from the sage (but succinct) observations of the editors.

References

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