A NEW PROOF OF PTOLEMY'S THEOREM

DASARI NAGA VIJAY KRISHNA

ABSTRACT. In this article we give a new proof of well-known Ptolemy's Theorem of a Cyclic Quadrilaterals.

1. INTRODUCTION

In the Euclidean geometry, Ptolemy's Theorem is a relation between the four sides and two diagonals of a cyclic quadrilateral (a quadrilateral whose vertices lie on a common circle). The Theorem is named after the Greek astronomer and mathematician Ptolemy (Claudius Ptolemaeus). Ptolemy used the Theorem as an aid in creating his table of chords, a trigonometric table that he applied to astronomy.

If the cyclic quadrilateral is given with its four vertices A, B, C, and D in order, then the Theorem states that $AC \cdot BD = AB \cdot CD + BC \cdot AD$.

This relation may be verbally expressed as follows:

If a quadrilateral is inscribed in a circle then the product of the measures of its diagonals is equal to the sum of the products of the measures of the pairs of opposite sides.

Moreover, the converse of the Ptolemy's Theorem is also true:

If the sum of the products of two pairs of opposite sides of a quadrilateral is equal to the product of its diagonals, then the quadrilateral can be inscribed in a circle.

In this short paper we deal with the new proof for this celebrated Theorem. Unfortunately or fortunately what ever the proofs are available in the literature (some of them can be found in [5], [1], [3] and [6]) are just based on constructing some particular lines and applying similarity using little angle chasing between the triangles thus formed. But in our present proof which is quite different from the available proofs, we won't construct any line. In fact we will just try to prove above mentioned result by using the simple consequence of Theorem obtained by the "Stewart Theorem" on the diagonals of convex quadrilateral. Our proof actually follows immediately from Equation (6) of Theorem 2.2. In the end of the article we will also prove two characterizations of a Bicentric Quadrilateral.

2. MAIN THEOREMS

Theorem 2.1. *Let P be the point of intersection of diagonals AC and BD of a convex quadrilateral $ABCD$ and M is an arbitrary point in the plane then*

$$(1) \quad \frac{CP}{AC} \cdot AM^2 - \frac{DP}{BD} \cdot BM^2 + \frac{AP}{AC} \cdot CM^2 - \frac{BP}{BD} \cdot DM^2 = AP \cdot PC - BP \cdot PD,$$

(2) $\text{Area}(BCD) \cdot AM^2 - \text{Area}(ACD) \cdot BM^2 + \text{Area}(ABD) \cdot CM^2-$
$$- \text{Area}(ABC) \cdot DM^2 = \text{Area}(ABC) \cdot (AP \cdot PC - BP \cdot DP).$$

Fig. 1.

Proof. Since PM is a cevian for triangles AMC and BMD by Stewart's Theorem we have

$$PM^2 = \frac{CP}{AC} \cdot AM^2 + \frac{AP}{AC} \cdot CM^2 - AP \cdot PC$$

and

$$PM^2 = \frac{DP}{BD} \cdot BM^2 + \frac{BP}{BD} \cdot DM^2 - BP \cdot PD.$$

By setting the right sides of these two equations equal to each other, we obtain (1).

Using the property that a cevian divides a triangle into two triangles whose ratio between areas is equal to the ratio between corresponding bases we have

$$\frac{AP}{AC} = \frac{\text{Area}(APD)}{\text{Area}(ACD)} = \frac{\text{Area}(APB)}{\text{Area}(ACB)} = \frac{\text{Area}(APD) + \text{Area}(APB)}{\text{Area}(ACD) + \text{Area}(ACB)} = \frac{\text{Area}(ABD)}{\text{Area}(ABCD)}.$$

In the same manner,

$$\frac{CP}{AC} = \frac{\text{Area}(CBD)}{\text{Area}(ABCD)}, \quad \frac{BP}{BD} = \frac{\text{Area}(ACB)}{\text{Area}(ABCD)}, \frac{DP}{BD} = \frac{\text{Area}(ACD)}{\text{Area}(ABCD)}.$$

By replacing these ratios in (1), we get (2).   □

Theorem 2.2. *Let P be the point of intersection of diagonals AC and BD of a cyclic quadrilateral $ABCD$ and M is an arbitrary point in the plane then*

(3) $$AP \cdot PC = BP \cdot PD,$$

(4) $$\frac{CP}{AC} \cdot AM^2 + \frac{AP}{AC} \cdot CM^2 = \frac{DP}{BD} \cdot BM^2 + \frac{BP}{BD} \cdot DM^2,$$

(5) $\text{Area}(BCD) \cdot AM^2 + \text{Area}(ABD) \cdot CM^2 = \text{Area}(ACD) \cdot BM^2 + \text{Area}(ABC) \cdot DM^2,$

(6) $$\frac{AC}{BD} = \frac{BC \cdot CD \cdot AM^2 + AB \cdot AD \cdot CM^2}{AD \cdot CD \cdot BM^2 + AB \cdot BC \cdot DM^2}.$$

Fig. 2.

Proof. From (1), for any point M we have

$$AP \cdot PC - BP \cdot PD = \frac{CP}{AC} \cdot AM^2 - \frac{DP}{BD} \cdot BM^2 + \frac{AP}{AC} \cdot CM^2 - \frac{BP}{BD} \cdot DM^2.$$

Let O be the circumcenter of the quadrilateral $ABCD$ and

$$AO = BO = CO = DO = R.$$

Taking M as O, we have

$$AP \cdot PC - BP \cdot PD = \frac{CP}{AC} \cdot AO^2 - \frac{DP}{BD} \cdot BO^2 + \frac{AP}{AC} \cdot CO^2 - \frac{BP}{BD} \cdot DO^2 =$$

$$= R^2 \cdot \left(\frac{AP + CP}{AC} - \frac{BP + DP}{BD} \right) = 0.$$

Combining (3) with (1) and (2) we get (4) and (5).

Now for (6), we will follow the well known fact that area of a triangle whose sides are a, b and c and circumradius R equals $\frac{abc}{4R}$. We have

$$\text{Area}(BCD) \cdot AM^2 + \text{Area}(ABD) \cdot CM^2 = \text{Area}(ACD) \cdot BM^2 + \text{Area}(ABC) \cdot DM^2,$$

then

$$(7) \quad \frac{BC \cdot BD \cdot CD}{4R} \cdot AM^2 + \frac{AB \cdot BD \cdot AD}{4R} \cdot CM^2 =$$

$$= \frac{AC \cdot AD \cdot CD}{4R} \cdot BM^2 + \frac{AB \cdot BC \cdot AC}{4R} \cdot DM^2,$$

and

$$BD \cdot \left(BC \cdot CD \cdot AM^2 + AB \cdot AD \cdot CM^2 \right) = AC \cdot \left(AD \cdot CD \cdot BM^2 + AB \cdot BC \cdot DM^2 \right).$$

$\square$

3. Main Results

3.1. **Ptolemy's Theorems.** In this section we present a new proof of the famous Ptolemy's Theorem.

Theorem 3.1 (Ptolemy's Theorem)**.** *For any cyclic quadrilateral $ABCD$ with diagonals AC and BD holds*

$$AC \cdot BD = AB \cdot CD + BC \cdot AD.$$

Proof. From Equation (6) we have

$$\frac{AC}{BD} = \frac{BC \cdot CD \cdot AM^2 + AB \cdot AD \cdot CM^2}{AD \cdot CD \cdot BM^2 + AB \cdot BC \cdot DM^2}.$$

Since it is true for any point M, let us take M as A. So

$$\frac{AC}{BD} = \frac{BC \cdot CD \cdot AA^2 + AB \cdot AD \cdot CA^2}{AD \cdot CD \cdot BA^2 + AB \cdot BC \cdot DA^2} = \frac{AC^2}{AB \cdot CD + AD \cdot BC},$$

and

$$AC \cdot BD = AB \cdot CD + BC \cdot AD.$$

$\square$

Now we prove Ptolemy's Second Theorem.

Theorem 3.2 (Ptolemy's Second Theorem). *For any cyclic quadrilateral ABCD with diagonals AC and BD holds*

$$\frac{AC}{BD} = \frac{BC \cdot CD + AB \cdot AD}{AD \cdot CD + AB \cdot BC}.$$

Proof. From Equation (6) we have

$$\frac{AC}{BD} = \frac{BC \cdot CD \cdot AM^2 + AB \cdot AD \cdot CM^2}{AD \cdot CD \cdot BM^2 + AB \cdot BC \cdot DM^2}.$$

Let O be the circumcenter of the quadrilateral $ABCD$ and $R = AO = BO = CO = DO$. Taking M as O we have

$$\frac{AC}{BD} = \frac{BC \cdot CD \cdot R^2 + AB \cdot AD \cdot R^2}{AD \cdot CD \cdot R^2 + AB \cdot BC \cdot R^2},$$

and

$$\frac{AC}{BD} = \frac{BC \cdot CD + AB \cdot AD}{AD \cdot CD + AB \cdot BC}.$$

$\square$

3.2. **Characterizations of bicentric quadrilaterals.**

Now we present the proof of two Theorems about bicentric quadrilaterals.

Let us recall here the definition (for more details see for example [7]).

Definition. *A bicentric quadrilateral* is a convex quadrilateral that has both an incircle and a circumcircle.

Theorem 3.3. *Let ABCD be any bicentric quadrilateral with diagonals AC and BD. If I is incenter of ABCD then*

$$\frac{AI \cdot CI}{BI \cdot DI} = \frac{AC}{BD} = \frac{BC \cdot CD + AB \cdot AD}{AD \cdot CD + AB \cdot BC}.$$

Proof. Since $ABCD$ is a cyclic quadrilateral, hence by Ptolemy's Second Theorem we have

$$(8) \qquad \frac{AC}{BD} = \frac{BC \cdot CD + AB \cdot AD}{AD \cdot CD + AB \cdot BC}.$$

Also from the equation (6) we have

$$\frac{AC}{BD} = \frac{BC \cdot CD \cdot AM^2 + AB \cdot AD \cdot CM^2}{AD \cdot CD \cdot BM^2 + AB \cdot BC \cdot DM^2}.$$

Taking M as I we have

(9)
$$\frac{AC}{BD} = \frac{BC \cdot CD \cdot AI^2 + AB \cdot AD \cdot CI^2}{AD \cdot CD \cdot BI^2 + AB \cdot BC \cdot DI^2}.$$

Using triangle's area formula, we obtain

$$2 \cdot \text{Area}(AIB) = r \cdot AB = AI \cdot BI \cdot \sin\left(\frac{A}{2} + \frac{B}{2}\right),$$

where r is the inradius of the quadrilateral $ABCD$. In the same manner,

$$2 \cdot \text{Area}(BIC) = r \cdot BC = BI \cdot CI \cdot \sin\left(\frac{B}{2} + \frac{C}{2}\right),$$

$$2 \cdot \text{Area}(CID) = r \cdot CD = CI \cdot DI \cdot \sin\left(\frac{C}{2} + \frac{D}{2}\right),$$

and

$$2 \cdot \text{Area}(DIA) = r \cdot AD = DI \cdot AI \cdot \sin\left(\frac{D}{2} + \frac{A}{2}\right).$$

Since $ABCD$ is a cyclic quadrilateral, we have

$$\sin\left(\frac{A}{2} + \frac{B}{2}\right) = \sin\left(\frac{C}{2} + \frac{D}{2}\right)$$

and

$$\sin\left(\frac{B}{2} + \frac{C}{2}\right) = \sin\left(\frac{D}{2} + \frac{A}{2}\right).$$

Then

$$\frac{AI \cdot BI}{CI \cdot DI} = \frac{AB}{CD},$$

and

$$\frac{BI \cdot CI}{AI \cdot DI} = \frac{BC}{AD}.$$

Dividing these two equation on each other, we obtain

$$\left(\frac{AI}{CI}\right)^2 = \frac{AB \cdot AD}{CB \cdot CD},$$

and

$$\left(\frac{BI}{DI}\right)^2 = \frac{BA \cdot BC}{DA \cdot DC}.$$

Now using Equations (8) and (9), we notice that

$$\frac{AC}{BD} = \frac{BC \cdot CD \cdot AI^2 + AB \cdot AD \cdot CI^2}{AD \cdot CD \cdot BI^2 + AB \cdot BC \cdot DI^2} = \frac{BC \cdot CD + AB \cdot AD}{AD \cdot CD + AB \cdot BC}.$$

So

$$\frac{AC}{BD} = \frac{AD \cdot CI^2}{BC \cdot DI^2},$$

and

$$\frac{AC}{BD} = \frac{BC \cdot AI^2}{AD \cdot DI^2}.$$

By multiplying these, we get

$$\frac{AI \cdot CI}{BI \cdot DI} = \frac{AC}{BD}.$$

Hence it follows the required result. □

Theorem 3.4. *Let $ABCD$ be any bicentric quadrilateral with diagonals AC and BD. If t_a, t_b, t_c, and t_d be the lengths of the tangents to its incircle from the vertices A, B, C, and D respectively then*

$$\frac{AC}{BD} = \frac{t_a + t_c}{t_b + t_d}.$$

Proof. From Equation (6) we have

$$\frac{AC}{BD} = \frac{BC \cdot CD \cdot AM^2 + AB \cdot AD \cdot CM^2}{AD \cdot CD \cdot BM^2 + AB \cdot BC \cdot DM^2}.$$

Let I be the incenter of quadrilateral $ABCD$ and r be its inradius. Taking M as I, we have

$$(10) \qquad \frac{AC}{BD} = \frac{BC \cdot CD \cdot AI^2 + AB \cdot AD \cdot CI^2}{AD \cdot CD \cdot BI^2 + AB \cdot BC \cdot DI^2}.$$

Since $AI = \frac{r}{\sin\left(\frac{A}{2}\right)}$, $BI = \frac{r}{\sin\left(\frac{B}{2}\right)}$, $CI = \frac{r}{\sin\left(\frac{C}{2}\right)}$ and $DI = \frac{r}{\sin\left(\frac{D}{2}\right)}$

$$r^2 = \frac{t_a t_b t_c + t_b t_c t_d + + t_c t_d t_a + t_d t_a t_b}{t_a + t_b + t_c + t_d}.$$

By replacing AI, BI, CI, and DI with their equivalent expressions in terms of t_a, t_b, t_c, and t_d (can be found in [2], [4]), and $AB = t_a + t_b$, $BC = t_b + t_c$, $CD = t_c + t_d$ and $DA = t_d + t_a$ in (10) and by some computation, the required result follows. □

In order to prove the same result in other way, we deal with the following proof.

Another proof. Let $m := \cot\left(\frac{A}{2}\right)$ and $n := \cot\left(\frac{B}{2}\right)$.

Observe that $t_a = r \cdot m$ and $t_b = r \cdot n$.

We have

$$\frac{t_c}{r} = \cot\left(\frac{C}{2}\right) = \tan\left(\frac{A}{2}\right) = \frac{1}{m},$$

hence $t_c = \frac{r}{m}$ and similarly $t_d = \frac{r}{n}$.

So

$$(11) \quad BC \cdot CD \cdot AI^2 = (t_a + t_b)(t_c + t_d)\left(r^2 + t_a^2\right) =$$

$$= \frac{r^4}{mn}(1 + mn)(m + n)\left(m + \frac{1}{m}\right) = AB \cdot AD \cdot CI^2.$$

and

$$AD \cdot CD \cdot BI^2 = \frac{r^4}{mn}(1 + mn)(m + n)\left(n + \frac{1}{n}\right) = AB \cdot BC \cdot DI^2.$$

Now we have

$$\frac{AC}{BD} = \frac{\left(m + \frac{1}{m}\right)}{\left(n + \frac{1}{n}\right)} = \frac{t_a + t_c}{t_b + t_d}.$$

Hence the result is alternatively proved.

□

4. Acknowledgement

References

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DEPARTMENT OF MATHEMATICS, NARAYANA EDUCATIONAL INSTITUTIONS, MACHILIPATNAM, BANGALORE, INDIA

E-mail address: vijay9290009015@gmail.com