THE FIVE CONICS PROBLEM

TRAN THU LE AND KIEN TRUNG NGUYEN

Abstract. This paper concerns a generalization of the so-called three conics theorem. We consider the model that consists of five conics. We show that the five conics in the setting has the same property as the three conics theorem and it takes the three conics in the model as a special case. Interesting corollaries are discussed as well.

1. Introduction

Many theorems concerning conics was investigated by some authors. In 1913, a problem was conjectured by J. S. Turner as follows *if three ellipses are such that each pair has one common focus and two real points of intersection, the three chords of visible intersection are concurrent.* In 1936, the problem was firstly proved by E. H. Neville (see [4]). The dual theorem was proposed by A. Akopyan. In addition, Ilya I. Bogdanov considered both the problem on three-dimensional space (see [1]).

In 1974, some theorems related to conics was introduced such as *Double-contact theorem*, *Four conics theorem*, *Incribed octagons* and *Three conics theorem* in a book of Evelyn et al, see Evelyn et al. [2]. The three conics theorem can be stated as follows *If three conics have a common chord, then the opposite common chords of each pair of conics are concurrent.* See Fig. 1 for illustration.

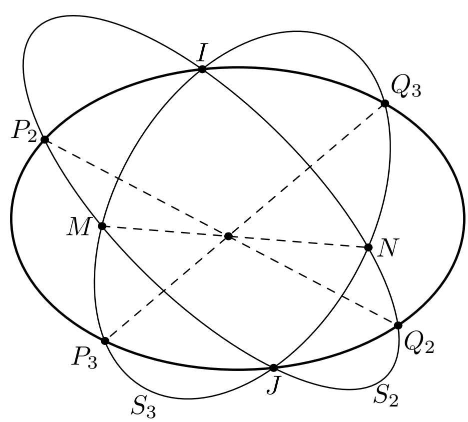

Figure 1. Three conics theorem.

The converse of three conics theorem is also correct. That means, *suppose S_2 intersects S_3 at four points I, J, M, N. Let P_2Q_2 and P_3Q_3 be two chords of S_2 and S_3, respectively. If MN, P_2Q_2, P_3Q_3 concur, then there exists a conic that contains six points I, J, P_2, Q_2, P_3, Q_3.*

In the projective plane, we know that if a conic passes through two cyclic points (see [5]), then the conic is a circle. Therefore if I, J be cyclics points, then these conics in the three conics theorem becomes circles. Thus the point of intersection in the theorem is a radical center of the three circles (see [6]). Moreover, the theorem is also a generalization of the Pascal's theorem. Precisely, if S_2 and S_3 degenerated into two pairs lines $\{IP_2; JQ_2\}$ and $\{IP_3, JQ_3\}$, we obtain the Pascal's theorem (see [3]); see Fig. 2

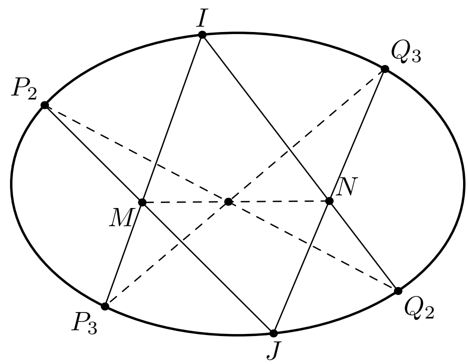

FIGURE 2. Pascal's theorem

2. Five conics theorem

Theorem 2.1 (Five conics theorem)**.** *Given two arbitrary points I and J on the plane and eight points P_1, P_2, P_3, P_4, Q_1, Q_2, Q_3, Q_4 on a conic S. Assume that that three systems $\{P_1, P_2, Q_1, Q_2, I, J\}$, $\{P_2, P_3, Q_2, Q_3, I, J\}$ and $\{P_3, P_4, Q_3, Q_4, I, J\}$ lie on conics S_1, S_2 and S_3, respectively. Then six points $\{P_4, P_1, Q_4, Q_1, I\}$ and J also lie on a conic, say S_4.*

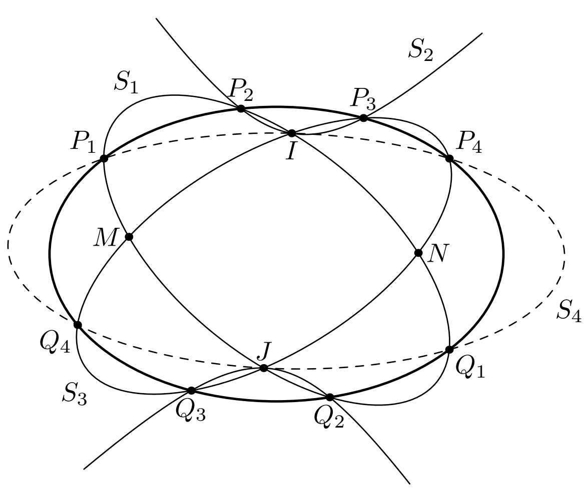

FIGURE 3. Five conics theorem

Proof: Denote by M and N the other intersections of S_1 and S_3. To complete the proof, we only prove that MN, P_1Q_1 and P_4Q_4 are concurrent. Indeed, if that happens, then the six points $\{P_4, P_1, Q_4, Q_1, I, J\}$ lie on the conic S_4 by the converse of three conics theorem. For simplicity, let the equations of IJ, P_2Q_2, P_3Q_3, S_1, S_2, S_3, in a chosen coordinate system, be $L = 0$, $L_2 = 0$, $L_3 = 0$, $S_1 = 0$, $S_2 = 0$, $S_3 = 0$, respectively. As S_1 is a member in the pencil of conics determined by S_2 and the pair $\{IJ, P_2Q_2\}$, its equation can be presented as

$$S_1 = a_1S_2 + b_1LL_2,$$

where a_1 and b_1 are real numbers. Note that if $b_1 = 0$, then S_1 coincide with S_2. Thus, we can suppose that $b_1 \neq 0$, we have

$$\frac{S_1}{b_1} = \frac{a_1}{b_1}S_2 + LL_2.$$

We assign $S_1 := \dfrac{S_1}{b_1}$ and $\lambda_3 := \dfrac{a_1}{b_1}$. The equality becomes

$$S_1 = \lambda_3S_2 + LL_2. \tag{1}$$

By the same argument, the equations of S_3 and S are

$$S_3 = \lambda_2S_2 + LL_3, \tag{2}$$

$$S = \lambda S_2 + L_2L_3, \tag{3}$$

where λ_2, λ are real numbers. By (1) and (3), we obtain $\lambda_2S_1 - \lambda_3S_3 = L(\lambda_2L_2 - \lambda_3L_3)$. This equation represents a conic in the pencil of conics determined by S_1 and S_3. As this conic consists of two lines $L = 0$ and $\lambda_2L_2 - \lambda_3L_3 = 0$, where $L = 0$ is the equation of IJ, the rest, say $\lambda_2L_2 - \lambda_3L_3 = 0$, is the equation of MN. Similarly, we consider the equations

$$\lambda_3S - \lambda S_1 = L_2(\lambda_3L_3 - \lambda L),$$
$$\lambda S_3 - \lambda_2S = L_3(\lambda L - \lambda_2L_2),$$

with $\lambda_3L_3 - \lambda L$ and $\lambda L - \lambda_2L_2$ being the equations of P_1Q_1, P_4Q_4. Note that

$$(\lambda_2L_2 - \lambda_3L_3) + (\lambda_3L_3 - \lambda L) + (\lambda L - \lambda_2L_2) = 0.$$

This linear combination of the equations shows that MN, P_4Q_4, P_1Q_1 are concurrent. Thus, the five conics theorem, has been proved. □

We next discuss some special cases of this problem.

3. Some consequences of the five conics theorem

If M and N (in Fig. 3) are two cyclic points at infinity, then MN is the infinity line; see [5]. As the pairs $\{P_4Q_4, P_1Q_1\}$ and $\{P_2Q_2, P_3Q_3\}$ meet at points on MN – the infinite line, they will be parallel. We can state a consequence as follows.

Corollary 3.1. *Given two circles S_1 and S_3 with two intersections I and J. An arbitrary conic S_2 containing I and J meets S_1 at the other points, say P_2 and Q_2, and meets S_3 at P_3 and Q_3. A conic S passing through P_2, Q_2, P_3, Q_3 meets S_1 again at P_1 and Q_1, and meets S_3 again at P_4 and Q_4. Then P_4, Q_4, P_1, Q_1, I, J lie on a conic S_4 and $\{P_4Q_4, P_1Q_1\}$ and $\{P_2Q_2, P_3Q_3\}$ are two couples of parallel lines.*

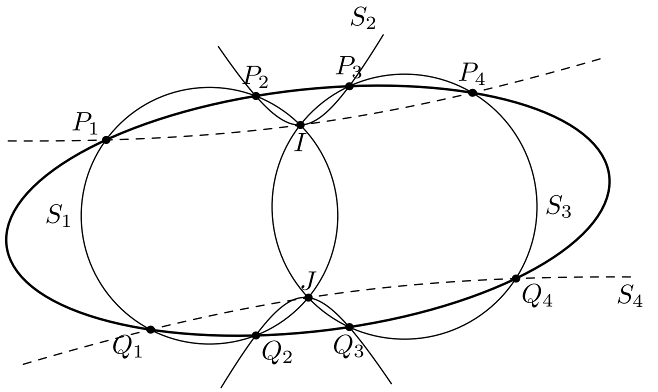

FIGURE 4. A consequence if M and N are two cyclic points as Corollary 3.1.

If I and J are two cyclic points on the infinity, then S_1, S_2, S_3, S_4 are degenerated to circles. It yields a consequence of the *Four conics theorem* (see [2, p.15])

Corollary 3.2. *Given eight points P_1, P_2, P_3, P_4, Q_1, Q_2, Q_3, Q_4 lie on a conic S, such that three systems $\{P_1, P_2, Q_1, Q_2\}$, $\{P_2, P_3, Q_2, Q_3\}$, $\{P_3, P_3, Q_4, Q_4\}$ lie on three circles S_1, S_2, S_3, respectively. Then $\{P_4, P_1, Q_4, Q_1\}$ lie also on a circle S_4.*

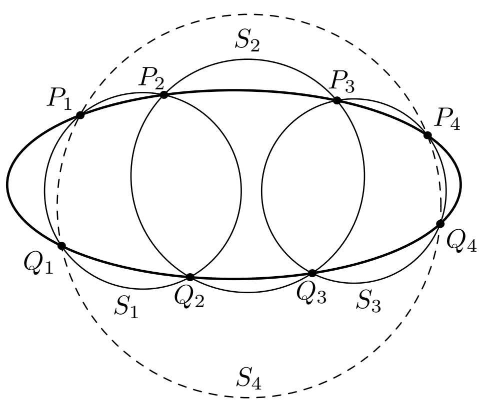

FIGURE 5. A consequence if I and J are two cyclic points as Corollary 3.2.

If P_2, Q_2 are two cyclic points on the infinity, then S_1, S_2 and S are degenerated into circles. Thus, we obtain a result stated as follows.

Corollary 3.3. *Give three circles S_1, S_2 and S. Let IJ, P_1Q_1, P_3Q_3 be the common chords of $\{S_1, S_2\}$, $\{S_1, S\}$ and $\{S_2, S\}$. An arbitrary conic S_3 passes through P_3, Q_3, I, J and cut S at two other points P_4 and Q_4. Then there exists a circle S_4 passing through six point P_4, Q_4, P_1, Q_1, I, J*

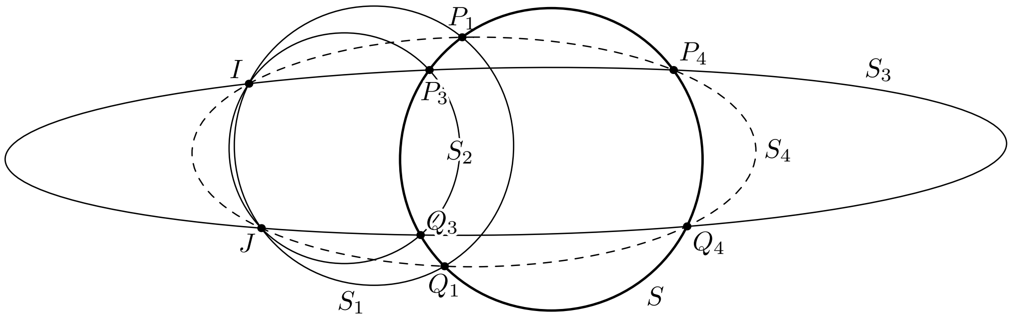

FIGURE 6. A consequence if P_2 and Q_2 are two cyclic points as Corollary 3.3.

If S_1 and S_3 are degenerated into two pairs of lines $\{IP_1, JQ_1\}$ and $\{IP_3, JQ_3\}$, we have a nice result.

Corollary 3.4. *Conic S_2 intersects S at four points P_2, Q_2, P_3, Q_3. Let I and J be two arbitrary points on S_1. The lines IP_2, JQ_2, IP_3, JQ_3 meet S again at P_1, Q_1, P_4, Q_4. Then P_1, Q_1, P_4, Q_4, I, J lie on a conic S_4.*

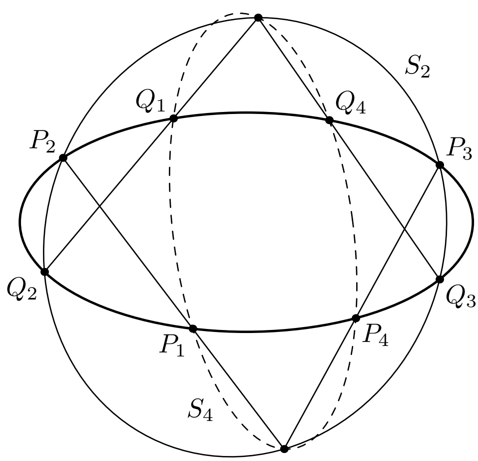

FIGURE 7. A consequence if S_1, S_3 are degenerated into two pairs of lines as Corollary 3.4.

From consequence 3.4, let $P_2 \equiv P_3$, $P_4 \equiv P_1$, $I \equiv J$ we obtain another result.

Corollary 3.5. *Let I be the intersection of two chords PP' and QQ' of the conic S. Conic S_2 passing through I and tangent to S at P and Q. Then there exists a conic S_4 passing through I and tangent to S at P' and Q'.*

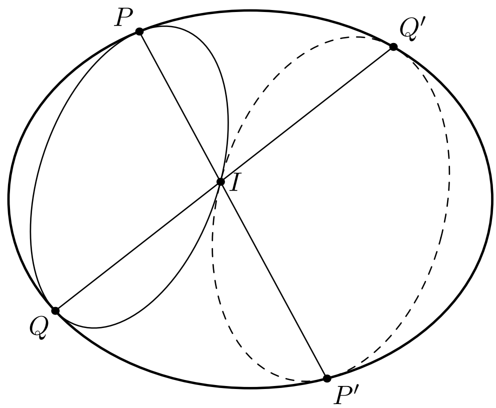

FIGURE 8. A consequence about three conics tangent to the each others as Corollary 3.5.

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MATHEMATICS DEPARTMENT, TEACHER COLLEGE, CANTHO UNIVERSITY, VIETNAM.
E-mail address: letranthu23041996@gmail.com

MATHEMATICS DEPARTMENT, TEACHER COLLEGE, CANTHO UNIVERSITY, VIETNAM.
E-mail address: trungkien@ctu.edu.vn