ON GENERALIZED BROCARD ELLIPSE

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Abstract. For a fixed point P and a fixed circle Ω consider a conic (the generalized Brocard ellipse) that touches lines XY inclined at a fixed angle to PX, where $X \in \Omega$. For this construction, we prove some facts that allow to obtain more properties of harmonic quadrilaterals.

1. Introduction

Consider next construction (*the Brocard construction*).

Let Ω be a circle with center O and radius R, let P be a fixed point, $P \neq O$, $P \notin \Omega$; and let φ be a fixed (oriented) angle, $0 < \varphi < \frac{\pi}{2}$. For an arbitrary point $X \in \Omega$ take $Y \in \Omega$ such that $\angle(YX, XP) = \varphi$.

For this construction, we recall the following facts from [1, section 4.6] or [3] (see also some proofs in Appendix).

(1) Lines XY are tangent to a conic $\varepsilon = \varepsilon(\Omega, P, \varphi)$. If P lies inside Ω, ε could be considered as *the generalized Brocard ellipse.*

(2) Point P is one of two foci of ε, and the second one is a point Q such that $OP = OQ$ and $\angle(OQ, OP) = 2\varphi$ (see Fig. 1).

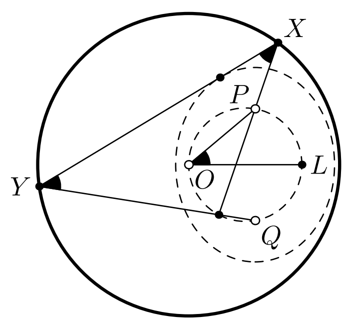

Fig. 1.

(3) The length of the great axis of generalized Brocard ellipse $\varepsilon(\Omega, P, \varphi)$ is equal to $2R \sin \varphi$.

We can reformulate this fact in the following from. Let us fix Ω and φ, and take points P_1, $L_1, Q_1 \in \Omega$ such that $\angle(OQ_1, OL_1) = \angle(OL_1, OP_1) = \varphi$. Note that $P_1Q_1 = 2R \sin \varphi$. While P and Q move along lines OP_1 and OQ_1 (so that $OP = OQ$) conics $\varepsilon(\Omega, P, \varphi)$ touch fixed lines passing through P_1 and Q_1 parallel to OL_1.

(4) Let K be a limit point of the pencil formed by Ω and *the Brocard circle* (POQ). Let lines XK and YK meet Ω for the second time at X' and Y'. Then all chords $X'Y'$ have equal lengths (Fig. 2).

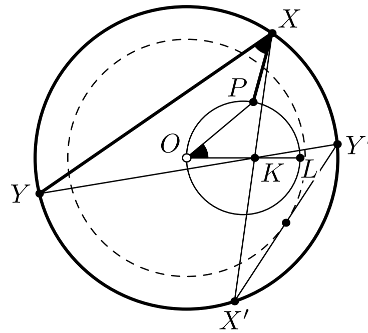

Fig. 2.

From this it follows that there exists a projective transformation π that preserves Ω and takes ε to a circle centered at O. Therefore, ε twice touches Ω. The touching points are points S_1 and S_2 that are common points of circles Ω and (POQ) (the touching points may be complex). Indeed, S_1O is the bisector of $\angle PS_1Q$, and $PS_1 + QS_1 = 2R\sin\varphi$ (one can perform a rotation taking triangle OQS_1 to OPS_1' so that S_1, P, S_1' are collinear; thus $PS_1 + QS_1 = PS_1 + PS_1' = S_1S_1' = 2OS_1\cos\angle OS_1P = 2R\sin\varphi$).

Note that π restricted to Ω is the central projection with center K, thus $\pi(K) = K$. Let L_1L_2 be the diameter of Ω containing K and L. It is easy to show equality of double ratio $(L_1, K, L, L_2) = (L_2, K, O, L_1)$ that gives $\pi(L) = O$.

(5) Let $A_1 \dots A_n$ be a polygon inscribed into Ω and such that $\angle(A_{i+1}A_i, A_iP) = \varphi$ for $i = 1, \dots, n$ (here $A_{n+1} = A_1$). Then $\angle(A_{i-1}A_i, A_iQ) = -\varphi$, and there exist an infinite set of polygons having these properties. Such polygons are called *Brocard polygons*, and the points P, Q are its *Brocard points*. All sides of the Brocard Polygon touch the same ellipse with foci P, Q. We will call this ellipse *the Brocard ellipse*. The Brocard polygon can be transformed to a regular polygon by an inversion (with center K) or by a projective transformation. In particular, a quadrilateral is a Brocard quadrilateral iff it is harmonic.

2. On points of tangency

In this section we continue using the notation from the Introduction. Projective arguments could be used to obtain the following description of points where ε touches XY.

Proposition 1. *Suppose $X, Y \in \Omega$ are points such that XY touches conic $\varepsilon(\Omega, P, \varphi)$ at point T, and let R be the pole of XY with respect to Ω; then L, T, and R are collinear.*[1]

Proof. Consider a projective transformation π from the fact 4 of the Introduction. We know that $\pi(\Omega) = \Omega$, $\pi(L) = O$, $\pi(\varepsilon) = \omega$, where ω is a circle centered at O. Let $\pi(X) = X' \in \Omega$, $\pi(Y) = Y' \in \Omega$, $\pi(T) = T'$, $\pi(R) = R'$. Note that $X'Y'$ touches ω at T', and R' is the pole of $X'Y'$ with respect to Ω. Obviously O, T', R' are collinear (belong to the perpendicular bisector of $X'Y'$), hence L, T, R are also collinear. $\square$

Below we present two more proofs of the previous Proposition using elementary geometry arguments only. The first of these two proofs was found by Nikita Nesterov.

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[1]This fact reformulated in elementary geometry terms was proposed as a problem at Mathematical competition XVII Kolmogorov Cup [2, Third round, Senior level, Problem 5].

Alternative proof 1. Let $T' = RL \cap XY$. It is sufficient to establish equality $\angle(QT', T'Y) = \angle(XT', T'P)$ (see Fig. 3).

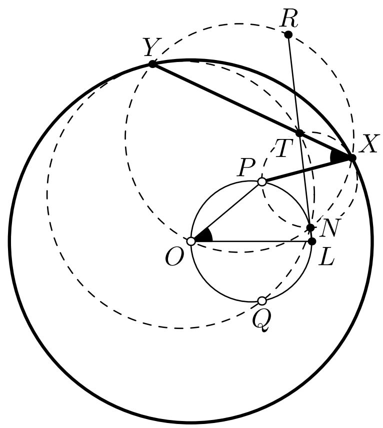

Fig. 3.

Let RL intersect (POQ) for the second time at N. We have $RN = LN \perp ON$, hence N belongs to circle $(OXRY)$ with diameter OR. From this circle $\angle(YN, NR) = \angle(RN, NX)$. Further, $\angle(T'N, NP) = \angle(LN, NP) = \angle(LO, OP) = \varphi = \angle(T'X, XP)$. We obtain that P, X, T', N are concyclic. Similarly, Q, Y, T', N are concyclic. From circles $(PXT'N)$ and $(YQT'N)$ we have $\angle(YQ, QT') = \angle(YN, NT') = \angle(YN, NR) = \angle(RN, NX) = \angle(T'N, NX) = \angle(T'P, PX)$. Note that triangles $QT'Y$ and $PT'X$ have two pairs of equal angles ($\angle(YQ, QT') = \angle(T'P, PX)$ and $\angle(QY, YT') = \angle(T'X, XP) = \varphi$), hence the remaining angles $\angle(QT', T'Y)$ and $\angle(XT', T'P)$ are equal. $\square$

Remark 1. *One could easily check that circles* $(PXT'N)$ *and* $(YQT'N)$ *(from the proof above) are tangent to* Ω*, and* R *is the radical center of circles* $(PXT'N)$*,* $(YQT'N)$*,* Ω*.*

Alternative proof 2. Let Q' be the reflection of Q in XY, and $T' = PQ' \cap XY$. It is sufficient to prove that R, T', L are collinear (see Fig. 4).

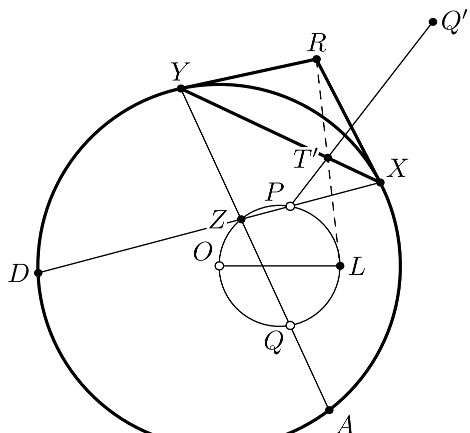

Fig. 4.

Let YZ and XZ intersect Ω for the second time at A and D, respectively. Then $AXYD$ is an inscribed trapezoid having axis of symmetry ZO. Note that O is the midpoint of the arc AZX in the circle (AZX) (since $\angle AZX = \angle YZD = \overset{\frown}{AX} = \angle AOX$).

Now let us fix $AXYD$ (thus points O, Z, R are fixed), and let P and Q move at a constant speed along XD and AY, respectively (with equal values of velocity; thus vectors of velocity of P and Q are symmetric with respect to direction XY). At any fixed moment we have $\angle(OP, PZ) = \angle(OQ, QZ)$, thus P, O, Q, Z are concyclic. Vectors of velocity of P and Q' are equal, hence T' is moving linearly. The center of the circle (POQ) is moving linearly along the perpendicular bisector of OZ, therefore L (that is the intersection point of (POQ) and the bisector ℓ of angle AZX) is moving linearly along ℓ.

Now it is sufficient to specify two cases of location of P, Q, and check that R, T', L are collinear in these cases. One of such cases is $P = X$, $Q = A$. In this cases $T' = X$, and it is easy to see that $L = RX \cap \ell$. The second case is analogous: $P = D$, $Q = Y$. $\square$

Remark 2. *In fact the arguments the last proof of Proposition 1 could be used to prove the existence of the generalized Brocard ellipse. Note that $PQ' = XA = 2R\sin\varphi$ in accordance with fact 3 from the Introduction.*

3. Case of a quadrilateral

3.1. **Inscribed quadrilateral.** We start with the following constructions for an inscribed quadrilateral (further in the case of harmonic quadrilateral we show the connection of this construction and the generalized Brocard ellipse).

From here to the end of this section we use the following notation. Let $ABCD$ be a quadrilateral inscribed to a circle Ω centered at O (see Fig. 5)

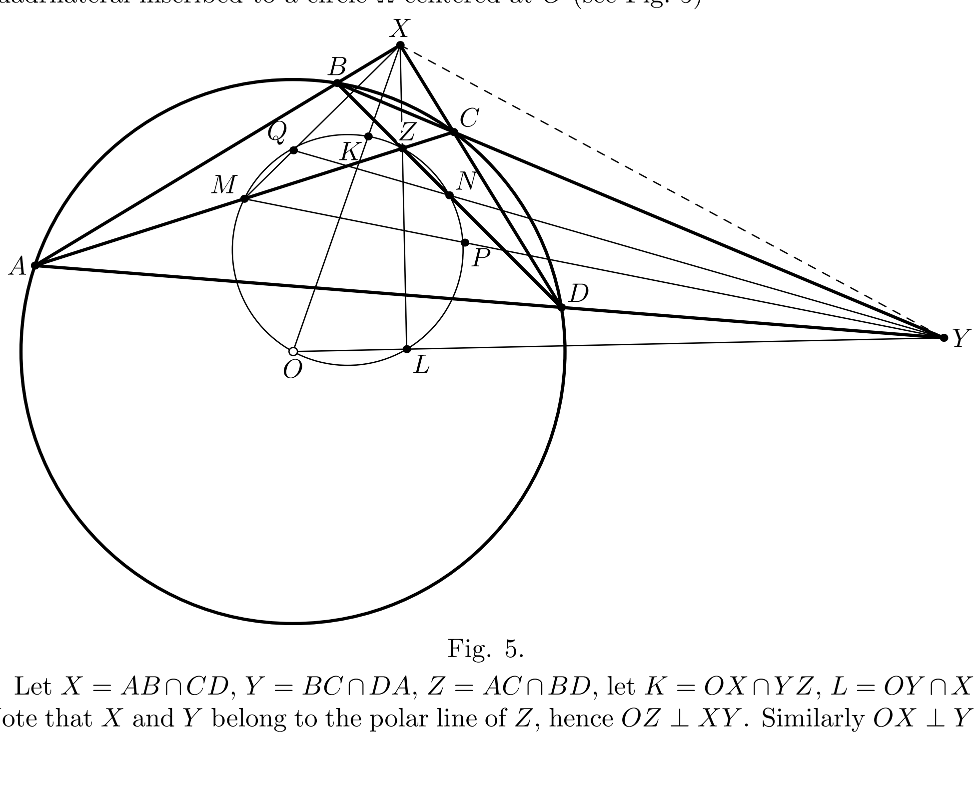

Fig. 5.

Let $X = AB \cap CD$, $Y = BC \cap DA$, $Z = AC \cap BD$, let $K = OX \cap YZ$, $L = OY \cap XZ$. Note that X and Y belong to the polar line of Z, hence $OZ \perp XY$. Similarly $OX \perp YZ$,

$OY \perp ZX$, thus O, X, Y, Z is an orthocentric quadruple. From that it follows that $OX \perp YK$, $OY \perp XL$.

Let M, N, T be midpoints of AC, BD, XY, respectively. Note that M, N, T are collinear (Gauss-Newton line for lines AB, BC, CD, DA), and points O, Z, K, L, M, N belong to the circle ω with diameter OZ.

Let $P = XN \cap YM$, $Q = XM \cap YN$.

Proposition 2. *Line XY is the radical axis of circles Ω and ω.*

Proof. Let U and V be the intersection points of Ω and XZ. Since XZ is the polar line of Y with respect to Ω, we have $OU \perp YU$, $OV \perp YV$, hence U and V belong to circle (OKY) with diameter OY. From that it follows that $XK \cdot XO = XU \cdot XV$, thus X has equal powers with respect to circles Ω and ω. Similarly, Y has equal powers with respect to Ω and ω. □

Proposition 3. *Quadruples $(ABOK)$, $(ABZL)$, $(BCOL)$, $(BCZK)$, $(CDOK)$, $(CDZL)$, $(DAOL)$, $(DAZK)$ are cyclic.*

Proof. From Proposition 2 it follows that $XK \cdot XO = XA \cdot XB = XZ \cdot XL$. This means that quadruples $(ABOK)$, $(ABZL)$ are cyclic. Similarly for the other quadruples. □

Further we need the following

Lemma 1. *Let OXY be a triangle. Suppose that XL, YK are its altitudes, Z is the orthocenter, T is the midpoint of XY (see Fig. 6).*

Let a line through T intersect circle $\omega = (OZKL)$ at M, N. Let $P = XN \cap YM$, $Q = XM \cap YN$. Then lines ON, ZM, KQ, LP are concurrent (or parallel) at some point W, and lines OM, ZN, KP, LQ are concurrent (or parallel) at some point W'. Moreover, $PQ \parallel XY \parallel WW'$; and WW' coincides to XY iff P and Q belong to $(OZKL)$.

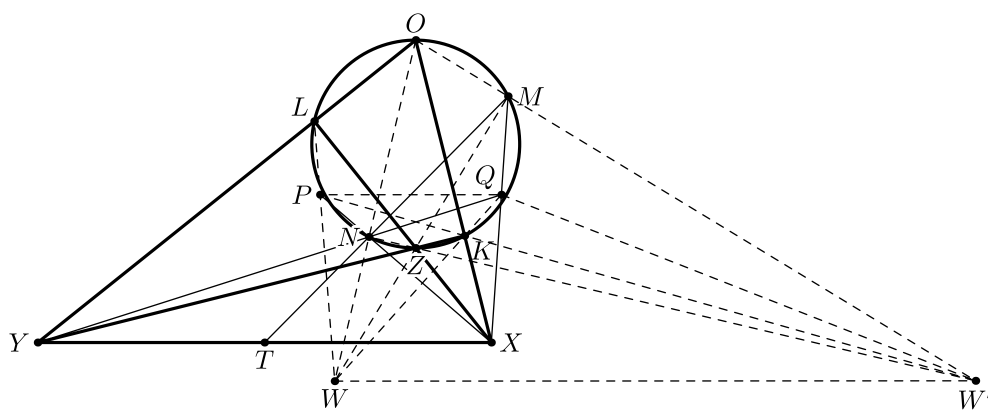

Fig. 6.

Proof. Let $\Pi(O)$, $\Pi(Z)$ be the pencils of lines passing through O and Z respectively. Then the correspondence $F : \Pi(O) \to \Pi(Z)$ transforming ON to MZ save the cross-ratios. Hence by Sollertinsky lemma (see lemma 4.4 in [1] or the Appendix) $W = ON \cap MZ$ lies in a conic Γ passing through O and Z. Point $W' = OM \cap NZ$ also belongs to Γ. It is easy to show that T is the pole of KL (with respect to ω), hence $K, L \in \Gamma$. Applying Pascal theorem to six points O, Z, W, W', K, L obtain that $P = WL \cap W'K$ belongs to XN and to YM, while $Q = WK \cap W'L$ belongs to XM and to YN.

In triangle XYM cevians XP and YQ intersect at N that belongs to median MT, hence $PQ \parallel XY$, and MN bisects the segment PQ. Since T is the pole of KL with respect to ω, PQ, KL, and MN are concurrent at some point U (both lines PQ, KL intersect line MNT at a point S such that quadruple M, S, N, T is harmonic).

Note that WW' is the polar line of $OZ \cap MN$, hence $XY \parallel WW'$. Moreover, $XY = WW'$ iff MN passes through $U = OZ \cap KL$. In this case P and Q are symmetric in OZ, and quadruple $LP \cap XY$, $LQ \cap XY$, X, Y is harmonic. By these conditions the pair P, Q is defined uniquely. But points of intersection of PQ and ω also satisfy these conditions, this means that $P, Q \in \omega$. $\square$

Remark 3. *From the proof we see that P, Q belong to a conic E passing through K, L.*

Let us sketch one another proof of Lemma that works if circle $(OZKL)$ does not intersect line XY.

Proof. Consider a projective transformation s preserving circle ω and taking XY to infinity. Then $O^*K^*Z^*L^*$ and $M^*P^*N^*Q^*$ are rectangles with parallel sides (here for images of points we use the same letters provided with a star). Further, O^*, Z^*, K^*, L^*, M^*, $N^* \in \omega$, and $M^*N^* \perp K^*L^*$ ($MKNL$ is harmonic, and the same is true for $M^*K^*N^*L^*$). Points $O^*L^* \cap M^*Q^*$, $O^*K^* \cap M^*P^*$, $K^*L^* \cap M^*N^*$ are collinear since they are projections of M^* to lines O^*L^*, O^*K^*, K^*L^* (Simson line). By Desargue's Theorem triangles $O^*K^*L^*$ and $M^*P^*Q^*$ are perspective. Similarly, triangles $Z^*K^*L^*$ and $N^*P^*Q^*$ are perspective, thus O^*M^*, Z^*N^*, K^*P^*, L^*Q^* are concurrent at a point W'^*. Analogously O^*N^*, Z^*M^*, K^*Q^*, L^*P^* are concurrent at a point W^*.

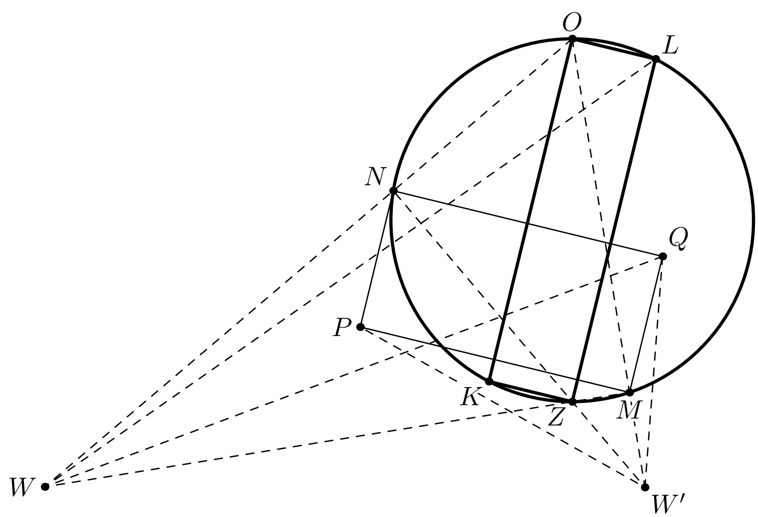

Fig. 7.

Note that O, Z, W, W' is an orthocentric quadruple, hence $WW' \perp OZ$, $OZ \perp PQ$, and s preserves this orthogonality.

Finally, s takes XY to the line at infinity, thus WW' coincides to XY iff $O^*M^* \parallel N^*Z^*$ and $O^*N^* \parallel M^*Z^*$ iff M^*N^* is a diameter of ω iff PQ is a diameter of ω. $\square$

Remark 4. *One can find more properties of the construction considered in Lemma, for example, the following. Let us rotate MN around T. It occurs that the locus of points P*

(or Q) is a conic Δ passing through K, L centered at the midpoint of KL (again it could be proved by Sollertinsky lemma). Note that Γ and Δ touch each other at K and L, and thier tangents at K and L are parallel to X and Y (one can show this considering the case when M and N tends to K or L). The asymptotes of Γ coincide to the axes of Δ and are parallel to the bisectors of angles between OZ and TF, where F is the midpoint of OZ (one can show this considering the case when MN tends to a diameter of ω, in this case W tends to infinity, and $KPLQ$ tends to a rectangle whose sides are parallel to asymptotes of Γ).

Now continue working with quadrilateral $ABCD$. By Lemma, lines ON, $ZM = AC$, KQ, LP are concurrent (or parallel) at some point W, and lines OM, $ZN = BD$, KP, LQ are concurrent (or parallel) at some point W'.

Proposition 4. *$W \in XY$ iff $P, Q \in \omega$ iff $ABCD$ is a harmonic quadrilateral.*

Proof. The first equivalence follows directly from the previous Lemma.

The pole of BD with respect to Ω lies in XY (since XY is the polar line of $Z \in BD$). We have $ON \perp BD$, hence $ON \cap XY$ is the pole of BD. Thus $W \in XY$ iff W is the pole of BD iff the pole of BD belongs to AC iff $ABCD$ is harmonic. □

3.2. **The Brocard points of a harmonic quadrilateral.** Further assume that $ABCD$ is harmonic. By Proposition 4, in this case P, $Q \in \omega$, P and Q are symmetric in OZ, $W \in XY$. Let us mention that equality of arcs PZ and QZ means that $\angle(MX, AC) = \angle(AC, MY)$.[2]

Proposition 5. *Quadruples $(ABQM)$, $(ABPN)$, $(BCQN)$, $(BCPM)$, $(CDQM)$, $(CDPN)$, $(DAQN)$, $(DAPM)$ are cyclic.*

Proof. Now $P, Q \in \omega$, thus from Proposition 2 we have $XQ \cdot XM = XA \cdot XB = XP \cdot XN$. This means that quadruples $(ABQM)$, $(ABPN)$ are cyclic. Similarly for other quadruples. □

Proposition 6. *Points P and Q are the Brocard points for $ABCD$, and each of angles $\angle(XM, AC)$, $\angle(AC, YM)$, $\angle(BD, XN)$, $\angle(YN, BD)$ equals to the Brocard angle.*

Proof. Denote $\varphi = \angle(QO, OZ) = \angle(OZ, OP)$. From $(ABQM)$ we have $\angle(QB, BA) = \angle(QM, MZ) = \varphi$. Similarly we get $\varphi = \angle(QC, CB) = \angle(QD, DC) = \angle(QA, AD) = \angle(PA, AB) = \angle(PB, BC) = \angle(PC, CD) = \angle(PD, DA)$. □

Now the generalized Brocard ellipse $\varepsilon = \varepsilon(\Omega, P, \varphi)$ with foci P, Q is inscribed into quadrilateral $ABCD$. Let us mention that MN passes through the midpoint R of PQ that is the center of $\varepsilon(\Omega, P, \varphi)$ (OZ, PQ, KL, MN are concurrent at R). [3]

Proposition 7. *The ellipse ε of $ABCD$ touches BC at point $BC \cap XZ$.*

Proof. Line BC passes through Y, hence the pole of BC with respect to Ω lies in XZ. From Proposition 1 it follows that the tangency point lies in XZ. □

Remark 5. *From 7 we see that Y is the pole of XZ with respect to ε. Similar statemets are true for X and Z. Thus X, Y, Z is an autopolar triple with respect to ε.*

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[2]This is in accordance with a known property of a harmonic quadrilateral: MX and MY are equally inclined to AC.

[3]This is in accordance with Newton's theorem: Gauss-Newton's line MN passes through centers of all conics inscribed to $ABCD$.

Proposition 8. *Quadruples* $(ACON)$, $(ACKQ)$, $(ACLP)$, $(BDOM)$, $(BDKP)$, $(BDLQ)$ *are cyclic (Fig. 8).*

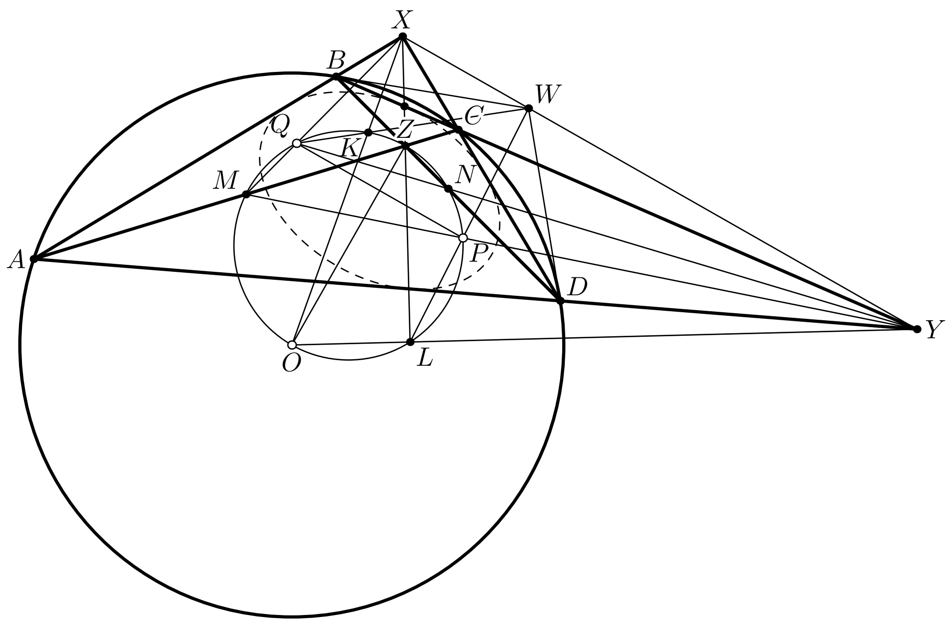

Fig. 8.

Proof. By Propositions 2 and 4, W has equal powers with respect to ω and Ω. Therefore, $WA\cdot WC = WO\cdot WN = WK\cdot WQ = WL\cdot WP$, hence quadruples $(ACON)$, $(ACKQ)$, $(ACLP)$ are cyclic. Similarly for quadruples $(BDOM)$, $(BDKP)$, $(BDLQ)$. □

It is known that in a harmonic quadrilateral there exists an inscribed ellipse $\varepsilon(M,N)$ with foci M, N (it could be shown from angle equalities $\angle(BM,BA) = \angle(BC,BN)$, $\dots$).

Proposition 9. *Ellipses* $\varepsilon(M,N)$ *and* ε *are similar.*

Proof. Consider a composition of symmetry in the bisector of angle AYB and homothety with center Y taking Q, P to M, N. (Such a transformaton exists since $\angle(BY,YN) = \angle(MY,YA)$ and $\angle(YM,MN) = \angle(PQ,QY)$.) This transformation takes ε with foci P, Q to the ellipse ε' with foci M and N toching YA and YB. From the uniqueness of such an ellipse it follows that $\varepsilon' = \varepsilon(M,N)$. □

4. Appendix

4.1. **Proofs or the properties of the Brocard construction. Assertion 1.** Let XY be the base of an isosceles triangle XYZ with sideline XZ passing through P. Then Z lies on circle POQ and sideline YZ passes through Q (Fig. 9).

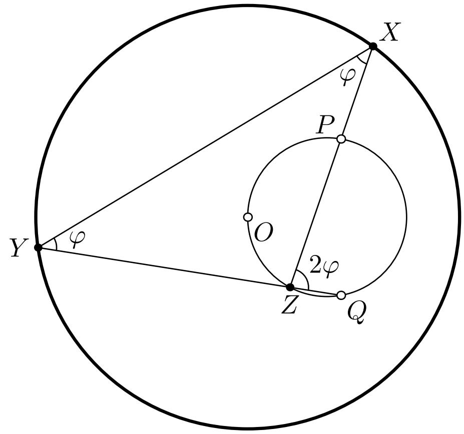

Fig. 9.

Proof. Let L be a point opposite to O in the *Brocard circle* (POQ) (L is the *generalized Lemoine point*). Suppose that PX intersects circle (POQ) for the second time at Z. Note that $\angle(QZ, ZL) = \angle(LZ, ZP) = \varphi$. We have $\angle(QZ, ZL) = \angle(YX, XP) = \varphi$, hence $ZL \parallel XY$. Therefore, $ZO \perp XY$ (since $ZO \perp ZL$). This means that X and Y are symmetric in ZO, hence $\angle(ZY, YX) = \angle(YX, XZ) = \varphi$. Further, $\angle(YZ, ZL) = \angle(ZY, YX) = \varphi = \angle(QZ, ZL)$, thus Y, Z, Q are collinear. □

Assertion 2. All lines XY touch the same ellipse with foci P and Q whose major axis equals $2R\sin\varphi$

Proof. Let P' be the reflection of P in XY (Fig. 10). Then triangle PXP' are POQ similar, thus triangles OPX and QPP' are also similar, i.e.

$$QP' = OX\frac{PQ}{OP} = 2R\sin\varphi$$

do not depend on X. Therefore the common point of lines XY and QP' lies on the ellipse with foci P, Q whose major axis equals $2R\sin\varphi$, and XY touches this ellipse. □

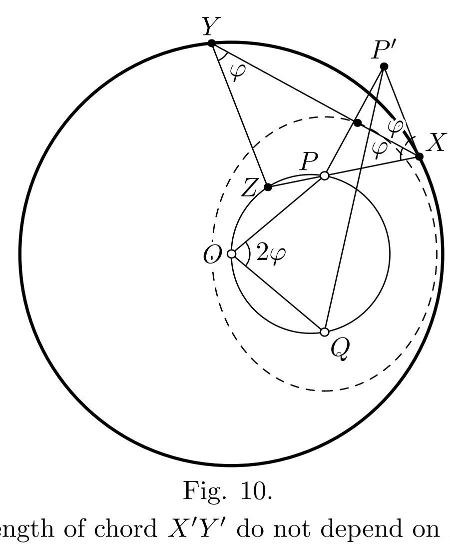

Fig. 10.

Assertion 3. The length of chord $X'Y'$ do not depend on X.

Proof. We have

$$X'Y' = XY\frac{KY'}{KX} = \frac{XY \cdot (R^2 - OK^2)}{KX \cdot KY}.$$

Since K is the limit point of the pencil of circles the ratio of KX^2 and the power of X wrt POQ does not depend on X. Therefore the ratio $XY/(KX \cdot KY)$ is proportional to

$$\frac{XY}{\sqrt{XZ \cdot XP \cdot YZ \cdot YQ}} = \frac{XY}{XZ\sqrt{XP \cdot YQ}} = \frac{2\cos\varphi}{\sqrt{XP \cdot YQ}}.$$

Since XP and YQ are two lines passing through the foci of the ellipse and forming a fixed angle with a tangent to it their product is constant. Therefore the length of $X'Y'$ is also constant. □

4.2. **The Sollertinsky lemma.** Let A, B be two fixed point and let $f : \Pi(A) \to \Pi(B)$ be a transformation conserving the cross-ratios. Then the locus of points $\ell \cap f(\ell)$, $\ell \in \Pi(A)$ is a conic passing through A, B. When $f(AB) = AB$ this conic is degenerated to the union of AB and some other line.

Proof. Take three lines x, y, $z \in \Pi(A)$ and let $X = x \cap f(x)$, $Y = y \cap f(y)$, $Z = z \cap f(z)$. Consider a conic Γ passing through A, B, C, X, Y. For an arbitrary point W of Γ we have $(AX, AY, AZ, AW) = (BX, BY, BZ, BW)$. Therefore $f(AW) = BW$ and W lies on the desired locus. The converse statement (i.e. each point of the locus belongs to Γ) could be proved in the same manner. □

References

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