ON SOME PROPERTIES OF CONFOCAL CONICS

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Abstract. We prove two theorems concerning confocal conics. The first one is related to bodies invisible from one point. In particular, this theorem is a generalization of Galperin–Plakhov's theorem. The second one is related to billiards bounded by confocal conics and is used to construct bodies invisible from two points. All the proofs are synthetic.

1. Introduction

The results reported here come from the study of invisibility generated by mirror reflections (see [3, 5, 6, 7, 8]). The construction of a body invisible from a fixed point [5, 6] is based on the following geometric statement concerning confocal conics.

The Galperin–Plakhov Theorem (See [4]). *Consider two different points F_1 and F_2 in the plane and take an ellipse and a hyperbola with foci at F_1 and F_2. We consider only the branch of the hyperbola associated with F_2 (we shall call it the right branch). Let P and Q be the points of intersection of the ellipse with the right branch of the hyperbola. Consider a ray starting at F_1 and intersecting the right branch of the hyperbola. Denote by X, A the intersection points of this ray with the ellipse and with the branch of the hyperbola. Suppose the focus F_2 lies on the line PQ. Then PQ is the bisector of the angle $\angle AF_2X$ (see Fig. 1).*

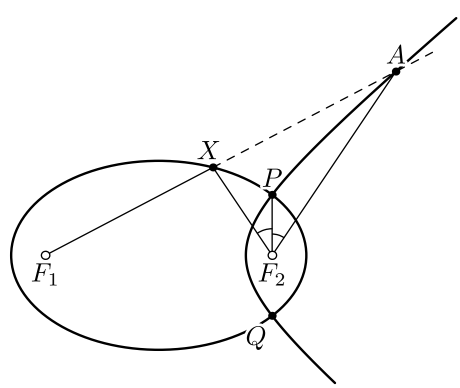


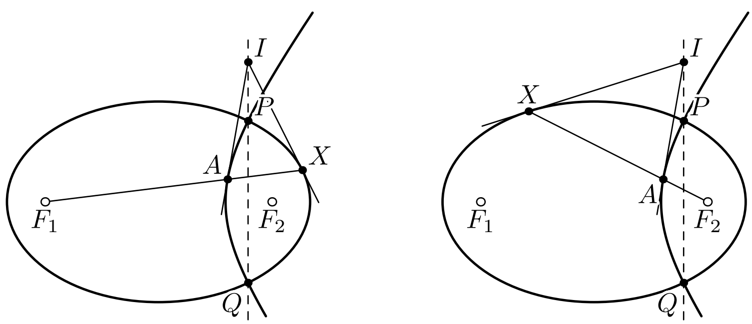
Fig. 1.

In Section 2 we formulate Theorem 1 which is a generalization of the Galperin-Plakhov theorem. In turn, the construction of a body invisible from two points leads to another statements referred to here as Theorem 2. Also Theorem 2 is related to billiards associated with confocal conics (see [9, Chapter 4]).

The paper is organized as follows. In Section 2 we formulate Theorems 1 and 2. In Section 3 we prove and generalize Theorem 1. In Section 4 we prove and generalize Theorem 2.

2. Main results

Theorem 1. *Let confocal ellipse and hyperbola are given. Consider an arbitrary point on the line passing through the intersection points of the ellipse and the right branch of the hyperbola. Draw two tangent lines to the ellipse and to the hyperbola. Then the line through the tangent points passes through a focus (see Fig. 2).*

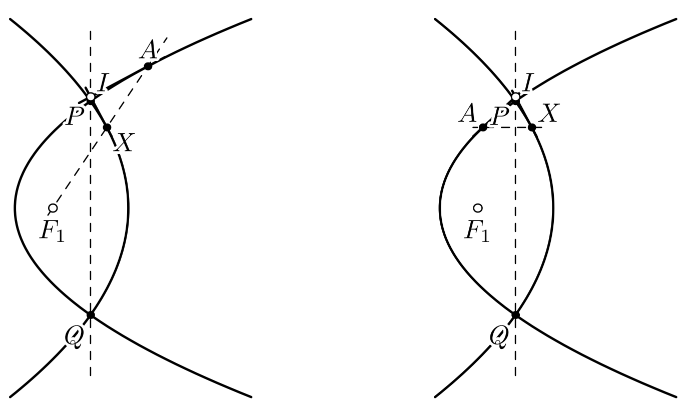


a) b)

Fig. 2.

In Section 3 we prove Theorem 1 and show how the Galperin–Plakhov Theorem follows from Theorem 1. The following corollary is a limiting case of Theorem 1 in which the focus F_2 converges to infinity.

Corollary. *Let two intersecting parabolas with common focus and axis of symmetry are given. Coider an arbitrary point on the line passing through the intersection points of the parabolas. Draw two tangent lines to the parabolas. Then the line through the tangent points either passes through the focus or parallel to the axis of symmetry (see Fig. 3).*



a) b)

Fig. 3.

The point a) of the following theorem was formulated by A. Yu. Plakhov.

Theorem 2. *Let two confocal ellipses $\mathcal{E}_1$ and $\mathcal{E}_2$ with foci F_1 and F_2 are given. Let a ray with the origin at F_1 intersects $\mathcal{E}_1$ and $\mathcal{E}_2$ at A and B, respectively. Let a ray with the origin at F_2 intersects $\mathcal{E}_1$ and $\mathcal{E}_2$ at C and D, respectively. Suppose the points B and C lie on a branch $\mathcal{H}_1$ of the hyperbola with the foci at F_1 and F_2. Then a) the points A and D lie on a branch $\mathcal{H}_2$ of the hyperbola with the foci at F_1 and F_2 (see Fig. 4)*

b) Consider a ray starting at F_1 intersecting the branch $\mathcal{H}_1$ at P_1. Consider the ray F_2P_1 intersecting the ellipse $\mathcal{E}_2$ at P_2. Analogously, we define the points P_3, P_4, P_5. Then $P_5 = P_1$ (see Fig. 5).

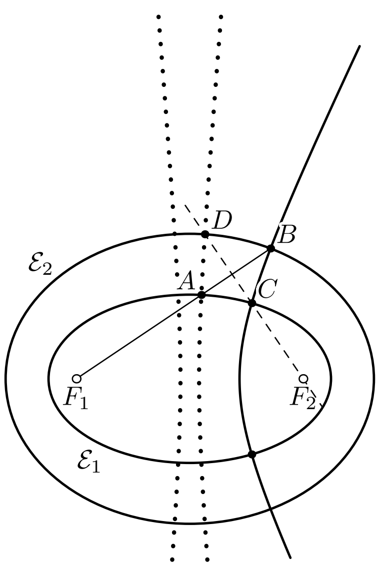


Fig. 4.

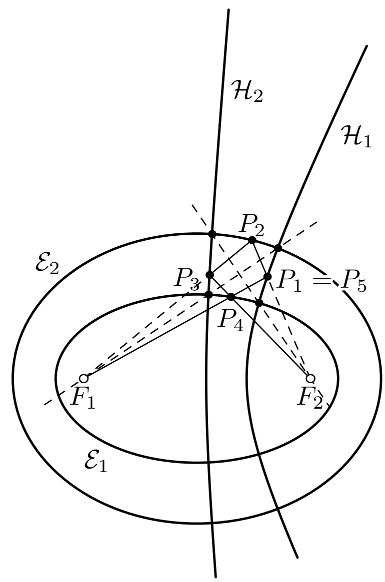


Fig. 5.

In Section 4 we prove Theorem 2 and formulate a generalization of Theorem 2 (b) for an arbitrary number of confocal conics.

3. Proof of Theorem 1

Proof. Denote by P and Q the intersection points the given ellipse $\mathcal{E}$ and the right branch of the hyperbola $\mathcal{H}$. Denote by I the given point on the line PQ and let points X and A be the given tangent points (see Fig. 2). Consider the polar transformation with respect to a circle with the center at F_1. It is well-known that the polar image of each conic with the focus at F_1 is a circle; see [2, the proof of Theorem 3.5]. So the images of $\mathcal{E}$ and $\mathcal{H}$ are the circles $\mathcal{E}'$ and $\mathcal{H}'$, respectively (see Fig. 6).

Note that the focus F_1 is mapped to the line of infinity; the focus F_2 is mapped to the radical axis of the circles $\mathcal{E}'$ and $\mathcal{H}'$; the line PQ is mapped to the intersection point of the common tangent lines to the circles $\mathcal{E}'$ and $\mathcal{H}'$. It easy to see that this intersection point is the internal homothetic center of the circles $\mathcal{E}'$ and $\mathcal{H}'$. Denote by l the image of the point I. Obviously, the line l is passing through the internal homothetic center of the circles $\mathcal{E}'$ and $\mathcal{H}'$; the points X and A are mapped to tangent lines x and a of the circles. Denote by M the intersection point of the lines x and a. It is evident that the point M lies either on the radical axis or on the line of infinity. So the line AX passes through a focus. □

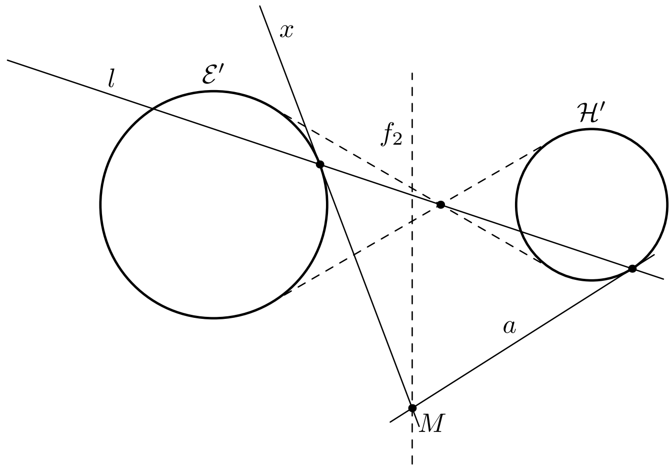


Fig. 6.

Let us show that Theorem 1 is a generalization of the Galperin–Plakhov theorem. We use notations from the proof of Theorem 1. Denote by Y the point of intersection of the ray F_2A with the ellipse $\mathcal{E}$. Denote by B the point of intersection of F_1Y with F_2X (see Fig. 7).

In the sequel we use the following well-known lemma (see [1, problem 11.10]).

Lemma 1. *The quadrangle $AYBX$ is circumscribed and the point B lies on the right branch of the hyperbola $\mathcal{H}$.*

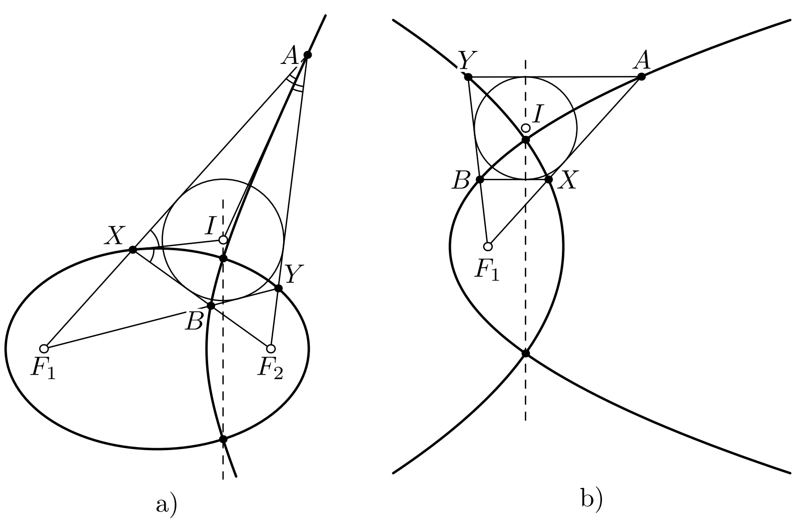


Fig. 7.

From the *optical properties of conics* the point I is the center of the circle inscribed in $AYBX$. Suppose the line PQ passes through the focus F_2. Then PQ is the bissector of the angle AF_2X. So the Galperin-Plakhov theorem is a corollary of Theorem 1.

Let us formulate another related result. Denote by K and L the points of intersection the line XY with F_1F_2 and the line AB with F_1F_2, respectively (see Fig. 8).

Proposition 1. *The lines XY and AB are altitudes in the triangle $\triangle IKL$.*

Proof. Note that the polar line of K with respect to the ellipse $\mathcal{E}$ is the line PQ. It is well-known that AB is a polar line of K with respect to the circle inscribed in $AYBX$. So IK is orthogonal to AB. Analogously IL is orthogonal to XY. □

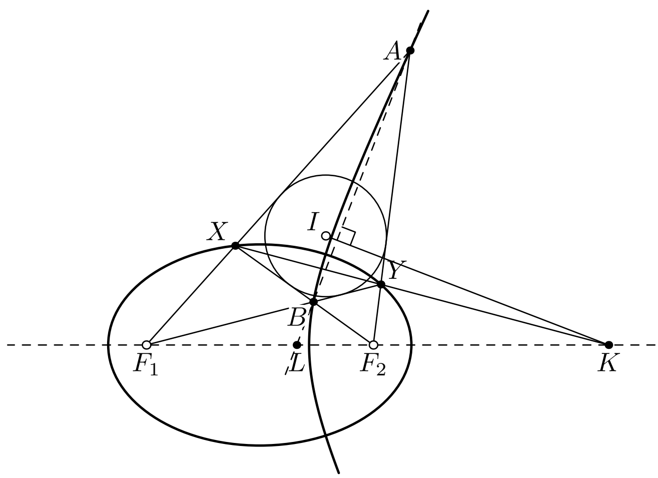


Fig. 8.

Remark. We want to emphasize that two polar lines of the point I with respect to $\mathcal{E}$ and $\mathcal{H}$ intersect on the line PQ, because PQ, XY and AB are altitudes in $\triangle IKL$.

Let us give you Projective generalization of Theorem 1. It is well-known that confocal conics (with foci F_1 and F_2) form a dual pencil of conics (see [2]). Actually, each confocal conic is tangent to 2 pairs of fixed conjugate imaginary lines. The lines of each pair pass through the corresponding focus (F_1 or F_2).

Consider quadrilateral $ABCD$ with opposite sides intersecting at the points F_1, F_2. Consider the dual pencil of consics inscribed in quadrilateral $ABCD$. Consider any two conics from the pencil. Consider a ray from F_1 intersecting both conics and denote by X and Y the intersection points of the ray with conics. Let I be the intersection of tangent lines at X and Y to the conics. Apply a projective trnsformation to the construction of Theorem 1. Then we get the following proposition.

Proposition 2. *a) There exists a straight line passing through I and two intersection points of the conics (see Fig. 9).*

b)the polar lines of I with respect to the conics intersect at the straight line.

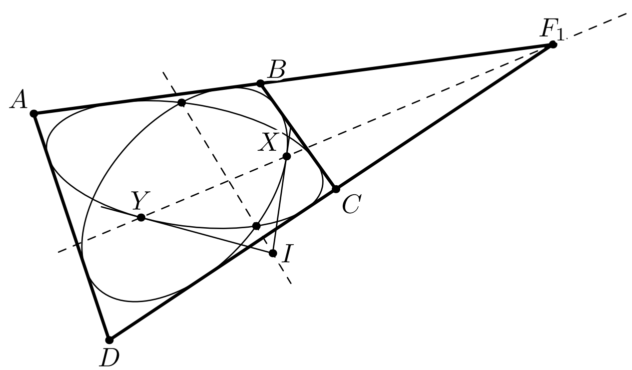


Fig. 9.

4. Proof of Theorem 2

Proof. a) Denote by X the intersection point of F_1D and F_2B; denote by Y the intersection point of F_1B and F_2D; denote by Z the intersection point of F_1C and F_2A; denote by M the intersection point of F_1X and F_2A and denote by N the intersection point of F_1C and F_2X (see Fig. 10).

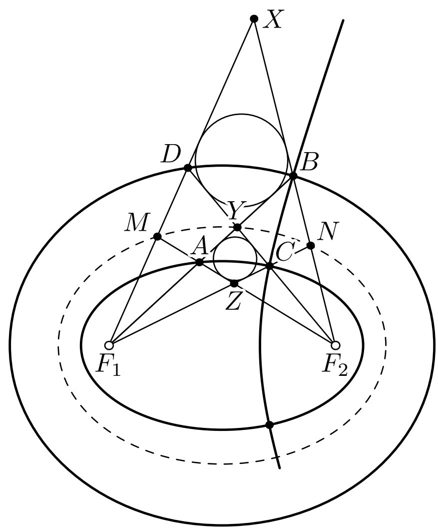


Fig. 10.

The points B and C lie on the same branch $\mathcal{H}_1$. So from Lemma 1 it follows that quadrilateral $CYBN$ is circumscribed and there exists the ellipse $\mathcal{E}$ with foci at F_1 and F_2 passing through the points Y and N. Analogously $YAZC$, $XDYB$ are circumscribed. Hence, the points Y and Z lie on the same branch of a hyperbola with foci F_1 and F_2 and the points X and Y lie on the same branch of a hyperbola with foci F_1 and F_2. So there exists a branch of a hyperbola $\mathcal{H}$ with foci at F_1 and F_2 passing through the points X, Y and Z. So the quadrilateral $XMZN$ is circumscribed. In view of the aforesaid, points M, Y, N lie on the ellipse $\mathcal{E}$. Thus the points A and D lie on the same branch of a hyperbola with foci at F_1, F_2.

b) denote by P_4' the intersecrion point of the ray F_1P_1 with the ellipse $\mathcal{E}_1$. We need to prove that $P_4 = P_4'$. So it is sufficiant to prove that the points P_3, P_4', and F_2 are collinear.

Denote by ϕ_4 and ϕ_2 the hyperbolas with foci at F_1 and F_2 passing through P_4' and P_2, respectivelly. Denote by ϕ_1 and ϕ_3 the ellipses with foci at F_1 and F_2 passing through P_1 and P_3, respectivelly. From Theorem 2 (a) it follows that the intersection point of ϕ_4 with ϕ_1 lies on the straigt line F_2D. Analogously, the intersection point of ϕ_1 with ϕ_2 lies on the line F_1B and the intersection point ϕ_2 with ϕ_3 lies on the line F_2D. From Theorem 2 (a) for conics ϕ_1, ϕ_2, ϕ_3, ϕ_4 it follows that the intersection point of ϕ_3 with ϕ_4 lies on the line F_1B. Finally, from Theorem 2 (a) for conics $\mathcal{E}_1$, ϕ_3, ϕ_4, $\mathcal{H}_2$ it follows that points P_3, P_4', F_2 are collinear. □

Remark. Let $F_1P_4+P_4F_2 = a_1, F_1P_2+P_2F_2 = a_2, F_1P_1-P_1F_2 = b_1, F_1P_3-P_3F_2 = b_2$. Notice that $a_2 + b_1 - b_2 - a_1 = P_4P_1 + P_1P_2 + P_2P_3 + P_3P_4$. So the perimeter of the quadrangle $P_1P_2P_3P_4$ is fixed.

Now we formulate a generalization of Theorem 2 (b) for an arbitrary number of confocal conics. Consider two different points F_1 and F_2 in the plane and take $n = 2k+1$ conics $\varphi_1, \varphi_2, \ldots, \varphi_n$ with foci at F_1 and F_2. Take a point P_1 on φ_1 and consider the ray F_1P_1. The ray reflects from the conic φ_1 at the point P_1; then the line containing the reflected ray passes through F_2. Denote by P_2 the intercection point of the reflected ray and φ_2. Analogously we define the points $P_3, ..., P_n$. The last ray P_nF_2 passes through point F_2. Denote by P_0 the intersection point of the line F_1P_1 with the line P_nF_2

Proposition 3. *The point P_0 lies on the fixed conic with foci at F_1 and F_2.*

Proof. The proof for the basis of induction $n = 3$ is analogous to the proof of Theorem 2 (b). Suppose that the statement of the Proposition is true for $n = 2k - 1$. Let $n = 2k + 1$. By the basis of induction for every 3 consecutive conics $\varphi_{i-1}, \varphi_i, \varphi_{i+1}$ there exists the conic ϕ_i such that the construction associated with the conics $\varphi_1, \ldots, \varphi_{i-1}, \varphi_i$, $\varphi_{i+1}, \ldots, \varphi_n$ may be reduced to the construction associated with the conics $\varphi_1, \ldots, \phi_i$, $\ldots, \varphi_n$. So, inductive step is proved. □

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