SOME PROPERTIES OF INTERSECTION POINTS OF EULER LINE AND ORTHOTRIANGLE

DANYLO KHILKO

Abstract. We consider the points where the Euler line of a given triangle ABC meets the sides of its orthotriangle, i.e. the triangle whose vertices are feet of the altitudes of ABC. In this note we study properties of these points and how they relate to the known objects.

A notable construction occured in [2, Problem 3] and [1, Problem G6]. A problem inspired by this construction initiated the research which we present in this paper. While solving this problem, we discovered several facts about intersection points of the Euler line and the sides of the orthotriangle. Further investigation of these points resulted in facts which we find intersting on their own and decided to share them.

The following notation will be used.

Let ABC be an acute triangle. Its altitudes AH_A, BH_B, CH_C intersect at the orthocenter H. Denote the midpoints of the sides AB, BC, CA by M_C, M_A, M_B, respectively, and the circumcenter of ABC by O. Let X_A be the foot of the perpendicular from A to H_BH_C. Define the points X_B, X_C analogously.

Let us remind reader some classical facts first. The points H_A, H_B, H_C, M_A, M_B, M_C lie on a circle (*the nine-point circle*) centered at O_9 which is the midpoint of OH. The lines AX_A, BX_B, CX_C meet at the point O. The next lemma can be used to prove various facts including IMO2013 3 and IMOSL2012 G6.

Lemma 1. *The circumcircles of the triangles $M_AX_BX_C$, $M_BX_AX_C$, $M_CX_AX_B$ intersect at the point O.*

Proof. Consider the circle ω with diameter OH_A. Since AX_A, BX_B, CX_C meet at O, we have $\angle H_AX_BO = \angle H_AX_CO = \angle H_AM_AO = 90°$. Then the points H_A, M_A, X_B, X_C lie on ω. Similarly we have that the circumcircles of the triangles $M_AX_BX_C$, $M_BX_AX_C$, $M_CX_AX_B$ intersect at the point O. □

One might wonder, whether a similar statement holds for the triangles $M_AM_CX_B$, $M_AM_BX_C$, $M_BM_CX_A$.

The following theorem provides the answer.

Theorem 1. *The circumcircles of the triangles $M_AM_BX_C$, $M_AM_CX_B$, $M_BM_CX_A$ have a common point which belongs to the Euler line.*

Before proving Theorem 1, we establish an auxiliary result. It introduces the key object of the proof, which is the main object of this exposition.

Proposition 1. *Let OH intersects the lines H_BH_C, H_AH_C, H_AH_B at the points K_A, K_B, K_C. Then the points M_B, M_C, X_A, K_A are cyclic. The same holds for the fours of points M_A, M_C, X_B, K_B; M_B, M_A, X_C, K_C.*

We need the following lemma which was proposed on the All-Russian Mathematical Olympiad [3, 2004–2005, District round, Grade 11, Problem 4]

Lemma 2. *Let the lines H_BH_C and M_BM_C meet at the point T_A. Then $AT_A \perp OH$.*

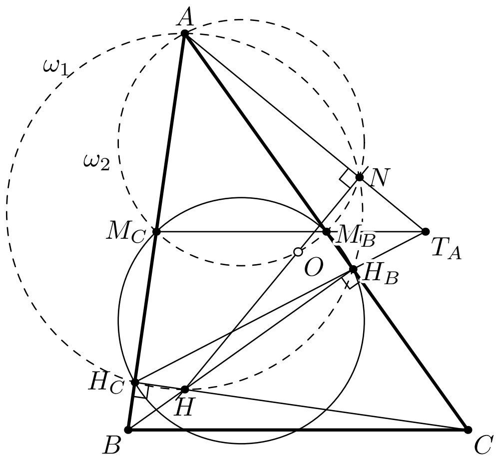

Fig. 1.

Proof. Denote by ω_1 the circumcircle of AH_BH_C (see Fig. 1)). Let OH intersect ω_1 again at the point N. Then $\angle AH_CH = \angle AH_BH = \angle ANH = 90°$. We have that $90° = \angle ANO = \angle AM_BO = \angle AM_CO$. Hence N lies on the circumcircle of AM_BM_C, denoted by ω_2. Consider the circles ω_1, ω_2 and the nine-point circle of ABC. The line AN is the radical axis of ω_1 and ω_2. The line H_BH_C is the radical axis of ω_1 and the nine-point circle. Finally, the line M_BM_C is the radical axis of ω_2 and the nine-point circle. This implies that the line AN passes through T_A. Then $AT_A \perp OH$. □

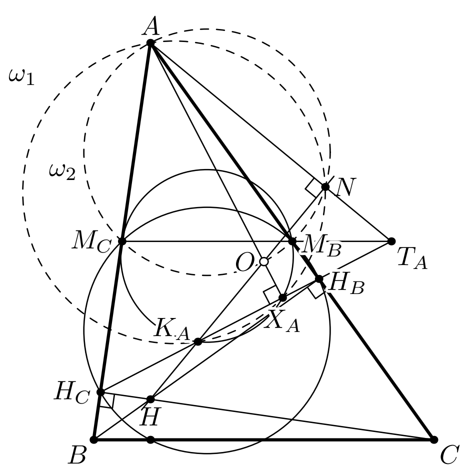

Fig. 2.

Proof of Proposition 1. We will show that $T_AM_B \cdot T_AM_C = T_AX_A \cdot T_AK_A$ (see Fig. 2). By Lemma 2, $\angle ANH = \angle ANK_A = \angle AX_AK_A = 90°$. Then A, N, X_A, K_A are concyclic. We obtain the following equation

$$T_AN \cdot T_AA = T_AX_A \cdot T_AK_A.$$

Also we have $T_AN \cdot T_AA = T_AM_B \cdot T_AM_C$ Hence

$$T_AM_B \cdot T_AM_C = T_AX_A \cdot T_AK_A.$$

Thus the points M_B, M_C, X_A, K_A are concyclic. □

Now we are ready to prove Theorem 1.

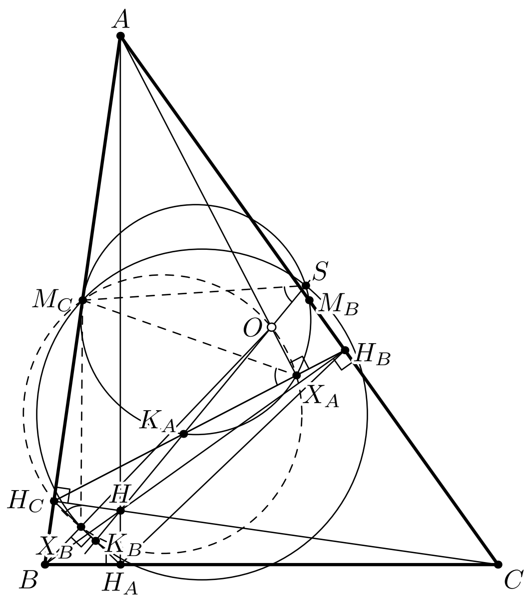

Fig. 3.

Proof of Theorem 1. Let the circumcircle of the quadrilateral $M_CX_BK_BM_A$ meets OH again at S (see Fig. 3). It suffices to show that the point S belongs to the circumcircle of $M_CX_AK_AM_B$, a similar statement for $M_AX_CK_CM_B$ will follow. We will work with oriented angles between lines. Denote by $\angle(l, m)$ the angle of the counterclockwise rotation which maps a line l to one parallel to a line m. See more in [4].

We have

$$\angle(M_CX_B, X_BK_B) = \angle(M_CS, SK_B) = \angle(M_CS, SK_A).$$

From Lemma 2 we obtain that

$$\angle(M_CX_B, X_BK_B) = \angle(M_CX_B, X_BH_C) = \angle(M_CX_A, X_AH_C) = \angle(M_CX_A, X_AK_A).$$

Hence we conclude that

$$\angle(M_CS, SK_A) = \angle(M_CX_A, X_AK_A),$$

and the proof is completed. □

Remark 1. *It is possible to prove the first part of Theorem 1 about three circles by angle chasing using Lemma1, however, this way does not imply that the intersection point of the circles lies on the line OH.*

Having proved Theorem 1, we establish further properties of the points K_A, K_B, K_C.

Theorem 2. *The circumcircles of the triangles $K_AH_AO_9$, $K_BH_BO_9$ and $K_CH_CO_9$ have another common point different from O_9.*

Firstly, we remind that in Lemma 2 we have defined the point T_A as the common point of H_BH_C and M_BM_C. Define the points T_B and T_C analogously.

Proposition 2. *The points K_A, H_A, T_A and O_9 are cyclic.*

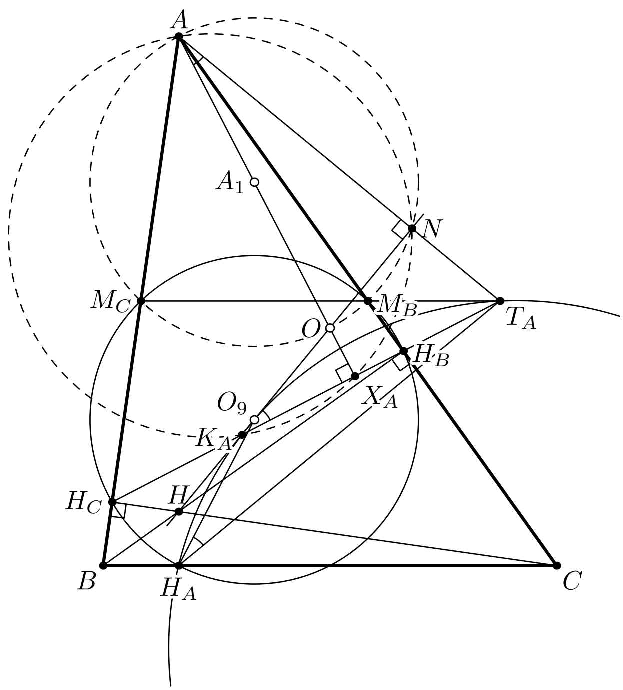

Fig. 4.

Proof. In the proof of Proposition 1 we have obtained that the points A, N, X_A and K_A are cyclic. Therefore,

$$\angle(X_AA, AT_A) = \angle(X_AA, AN) = \angle(X_AK_A, K_AN) = \angle(T_AK_A, K_AO_9).$$

We claim that $\angle(X_AA, AT_A) = \angle(T_AH_A, H_AO_9)$ (see Fig. 4). Indeed, the points A and H_A are symmetric with respect to M_AM_B. Then the nine-point circle and the circumcircle of AM_AM_B are symmetric with respect to M_AM_B. Hence O_9 is symmetric to the point A_1 which is the center of the circumcircle AM_AM_B. As O belongs to this circle and $\angle ANO = 90°$ we have that A_1 lies on AO i. e. AX_A. So we have $\angle(X_AA, AT_A) = \angle(T_AH_A, H_AO_9)$. Then

$$\angle(T_AH_A, H_AO_9) = \angle(X_AA, AN) = \angle(T_AK_A, K_AO_9),$$

and we are done. $\square$

Consider the points T_A, T_B, T_C.

Lemma 3. *The points T_A, T_B, C are collinear.*

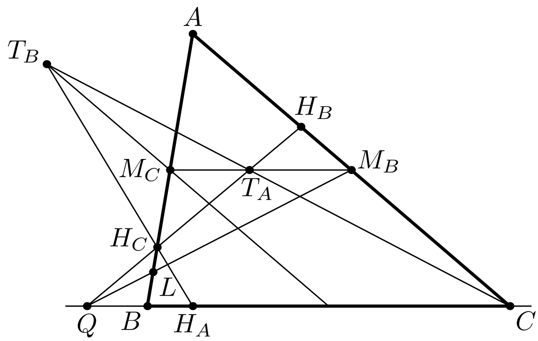

Fig. 5.

Proof. We need some additional notation, which will be used only in this proof. Denote by Q the intersection point of BC and H_BH_C and by L the intersection point of QM_B and AB (see Fig. 5).

Let us apply Desargues' theorem for the triangles H_AH_CQ and the one formed by the lines M_CM_A, M_CM_B, M_BC (this triangle has one vertex at infinity). Then the following statements are equivalent: M_CH_C, M_BQ and the line parallel to CM_B passing through H_A are concurrent and the intersection points of QH_C and M_BM_C, H_AH_C and M_AM_C, QH_A and M_BC are collinear. Notice that H_AH_C meets M_AM_C at T_B, QH_C meets M_CM_B at T_A and QH_A meets M_BC at C. So in order to prove that T_B, T_A, C are collinear we will prove the first statement obtained by Desargues' theorem. It is sufficient to prove that $LH_A \parallel AC$. By Menelaus' theorem

$$\frac{BQ}{QC} \cdot \frac{CM_B}{M_BA} \cdot \frac{AL}{LB} = 1.$$

Then

$$\frac{BL}{AL} = \frac{BQ}{QC}.$$

It is a well-known fact that

$$\frac{BQ}{QC} = \frac{BH_A}{H_AC}.$$

Hence

$$\frac{BL}{LA} = \frac{BH_A}{H_AC},$$

and $LH_A \parallel AC$.

□

Remark 2. *A similar fact to Lemma 3 will hold if one replace the points H_A, H_B, H_C by some points A_1, B_1, C_1 which lie on the respective sides of ABC and AA_1, BB_1, CC_1 are concurrent.*

The next fact describes other properties of T_A, T_B, T_C.

Lemma 4. *The circumcircles of the triangles $T_BT_AH_C$, $T_BT_CH_A$ and $T_CT_AH_B$ have a common point P.*

Proof of Lemma 4. The statement follows from Miquel's theorem applied to the triangle $H_AH_BH_C$ and the points T_C, T_A, T_B (see Fig. 6). □

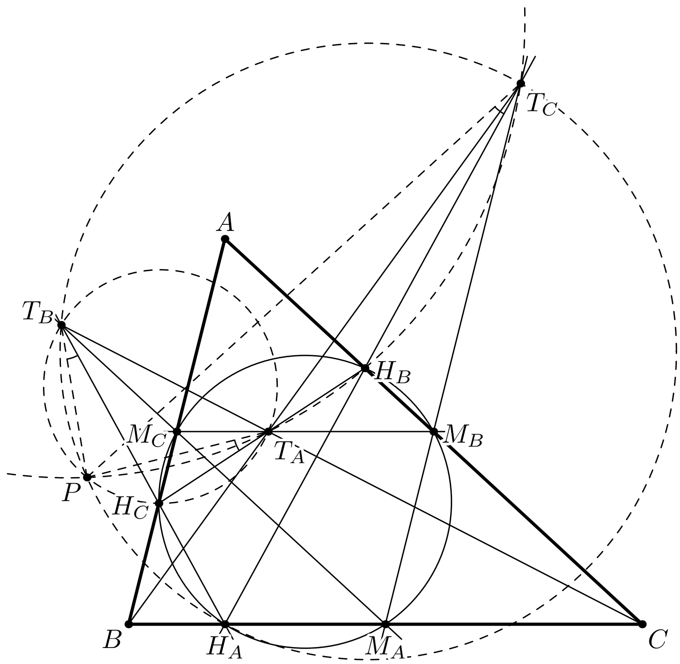

Fig. 6.

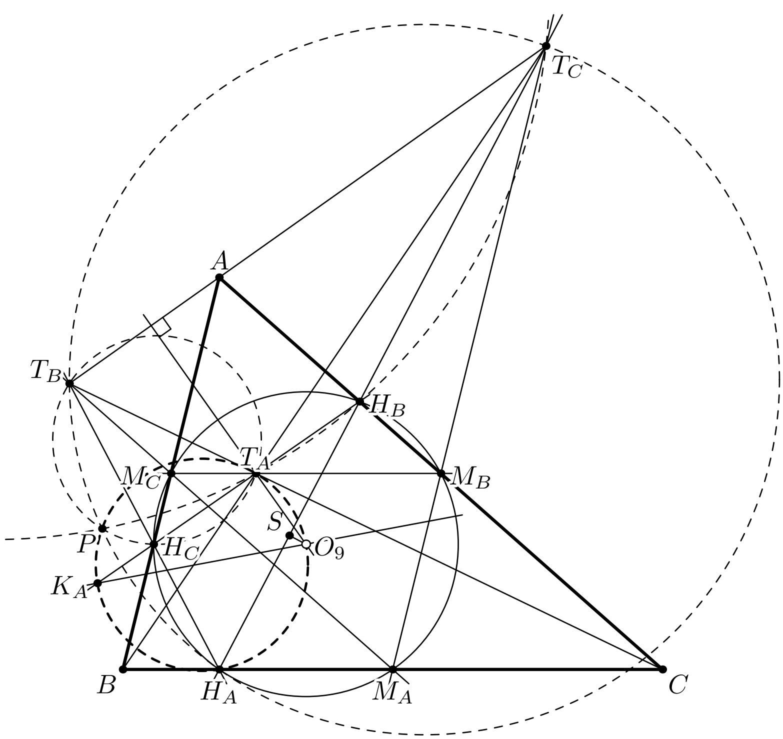

Fig. 7.

Now we claim that the point P lies on the circumcircle of the triangle $K_AH_AO_9$. This fact combined with that for $K_BH_BO_9$ and $K_CH_CO_9$ is equivalent to Theorem 2.

Proof of Theorem 2. In order to prove that the circumcircle of $O_9K_AH_A$ passes through P, we will show that $\angle(T_AP, PH_A) = \angle(T_AO_9, O_9H_A)$ (See Fig. 7).

Firstly, we will prove that $O_9T_A \perp T_BT_C$. Let H_BM_C and H_CM_B intersect at U. It is a well-known fact that AU is the polar line of T_A with respect to the nine-point circle. Applying Pascal's theorem on the hexagon $M_BM_AM_CH_BH_AH_C$ we obtain that T_B, T_C, U are collinear. Hence T_BT_C is the polar line of T_A, consequently, $O_9T_A \perp T_BT_C$. Let S be the foot of the perpendicular from O_9 to H_AH_B. Then $\angle(T_AO_9, O_9S) = \angle(T_BT_C, T_CS)$. Also we have $\angle(SO_9, O_9H_A) = \angle(H_BH_C, H_CH_A)$. So

$$\begin{aligned}
&\angle(T_AO_9, O_9H_A) = \angle(T_AO_9, O_9S) + \angle(SO_9, O_9H_A) = \\
&= \angle(T_BT_C, T_CS) + \angle(H_BH_C, H_CH_A) = \\
&= \angle(T_BT_C, T_CH_A) + \angle(T_AH_C, H_CT_B) = \\
&= \angle(T_BP, PH_A) + \angle(T_AP, PT_B) = \angle(T_AP, PH_A)
\end{aligned}$$

and we are done. □

Theorem 3. *T_BT_C, M_BM_C and K_AH_A are concurrent.*

Proposition 3. *K_AH_A is tangent to the circumcircle of $T_AT_BH_A$ at H_A.*

Proof. We have

$$\begin{aligned}
&\angle(K_AH_A, H_AP) = \angle(K_AT_A, T_AP) = \\
&= \angle(H_CT_A, T_AP) = \angle(H_CT_B, T_BH_C) = \angle(H_AT_B, T_BP).
\end{aligned}$$

□

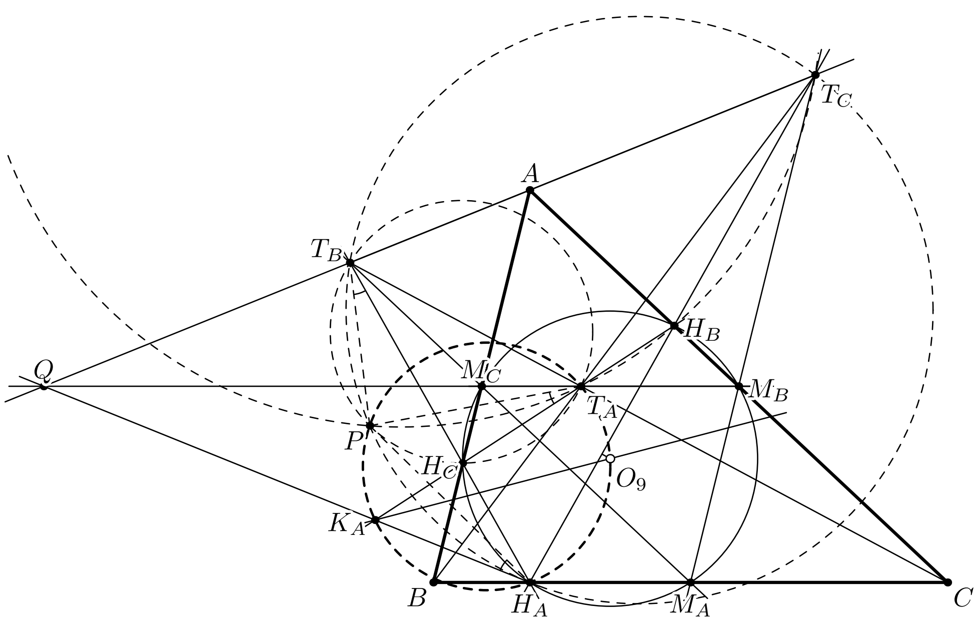

Fig. 8.

Proof of Theorem 3. Let T_BT_C intersect K_AH_A at Q (see Fig. 8). By Proposition 3 QH_A is tangent to the circumcircle of the triangle $T_BH_AT_C$. Also H_AA bisects $\angle T_BH_AT_C$. Hence $AQ = QH_A$ since

$$\angle QAH_A = \angle T_BT_CH_A + \angle AH_AT_C = \angle T_BH_AQ + \angle T_BH_AA = \angle QH_AA.$$

So Q belongs to the perpendicular bisector of AH_A, which is obviously M_AM_B. □

The author is grateful to Vladyslav Vovchenko for a short proof of the Lemma 3 and to Georgiy Shevchenko for careful reading of the manuscript.

References

[1] 53-rd International Mathematical Olympiad, Shortlisted Problems with Solutions. 2012. URL: `https://www.imo-official.org/problems.aspx`.

[2] 54-th International Mathematical Olympiad. 2013. URL: `https://www.imo-official.org/problems.aspx`.

[3] N. K. Agakhanov, I. I. Bogdanov, P. A. Kozhevnikov, O. K. Podlipsky, and D. A. Tereshin. *All-russian mathematical olympiad 1993–2006*. M.:MCCME, 2007.

[4] V. V. Prasolov. *Problems in plane geometry.* Russian. M.:MCCME, 2006.

Taras Shevchenko National University of Kyiv
E-mail address: dkhilko@ukr.net