ELLIPSE GENERATION RELATED TO ORTHOPOLES

PARIS PAMFILOS

Abstract. In this article we study the generation of an ellipse related to two intersecting circles. The resulting configuration has strong ties to triangle geometry and by means of orthopoles establishes also a relation with cardioids and deltoids.

1. Ellipse generation

Our basic configuration consists of two circles α and β, intersecting at two different points C and D. In this we consider a variable line ε through point C, intersecting the circles, correspondingly, at A and B. Then draw the tangents at these points, intersecting at point E and take the symmetric Z of E w.r.t. to the line AB (See Figure 1). Next theorem describes the locus γ of point Z as the line ε is turning about the point C. The two tangents η and λ to the circles, correspondingly, β and α, at C, are special positions of ε and play, together with their intersections H and L with the circles, an important role in this context. Another important element is line ν, the orthogonal at D to the common chord CD of the two circles. This line intersects circles α and β, correspondingly, at points M and N.

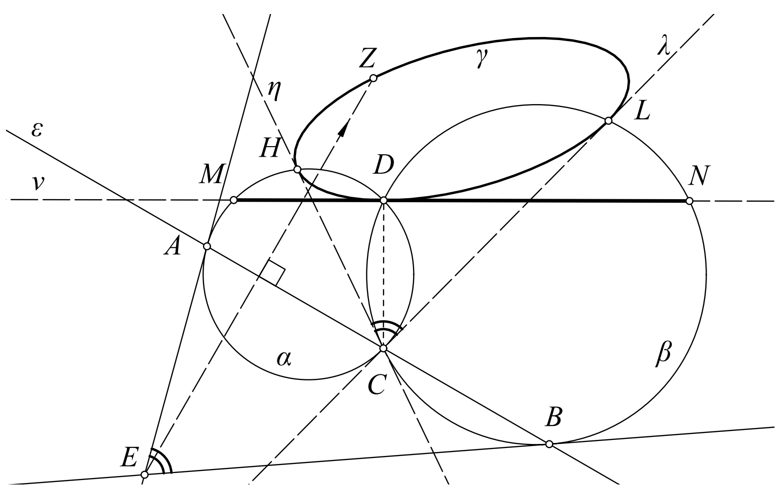

Fig. 1. Ellipse from two intersecting circles

Theorem 1. *As the line AB varies turning about C, the corresponding point Z describes an ellipse γ. This ellipse is also tangent to lines $\{\eta, \lambda, \nu\}$, respectively at the points $\{H, L, D\}$. The ellipse degenerates to the segment MN, when the two circles are orthogonal.*

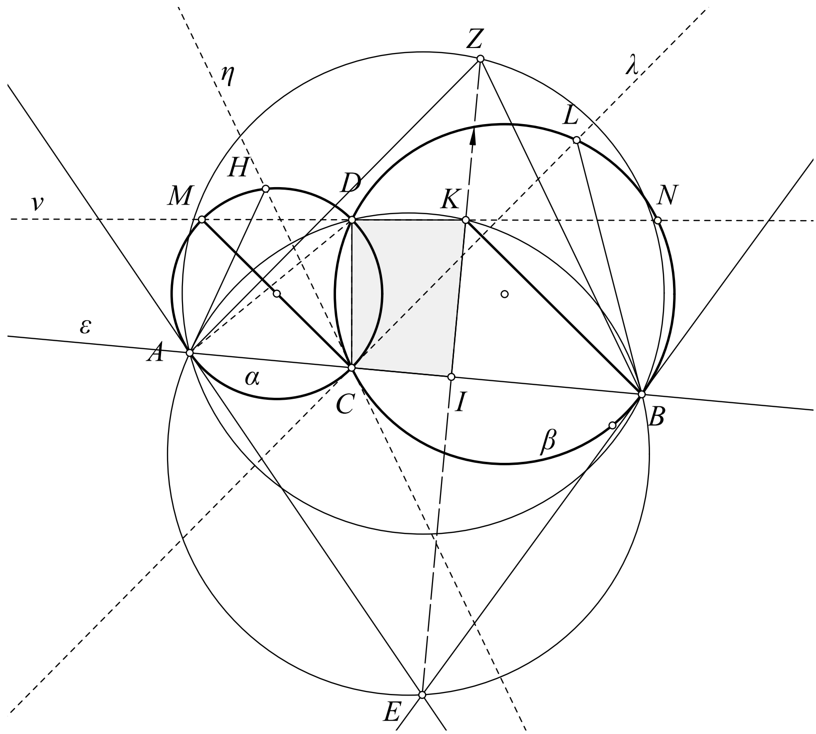

Fig. 2. First properties of the basic configuration

The synthetic proof of the theorem relies on two lemmas, which use the, variable with E, circumcirlce (ABE) of triangle ABE, as well as the intersection points K and I of lines, correspondingly, MN and AB with EZ. (See Figure 2).

Lemma 1. *With the definitions made above, the following are valid properties:*

(1) *The quadrilateral $CDKI$ is a cyclic one and cirlce (ABE) passes through D and K.*
(2) *Lines CM and BK are parallel.*
(3) *Lines AZ and CL are parallel and orthogonal to BK. Analogously, lines CN and AK are parallel, and lines BZ, CH are parallel and orthogonal to the two previous parallels.*
(4) *Lines BL and AH are tangent to the circle (ABE).*
(5) *Triangles HAC and CBL are similar.*

Proof. The first part of statement (1) is obvious, since the opposite lying angles $\widehat{CIK}$ and $\widehat{CDK}$ are right. The second part of (1) follows from the fact that quadrilaterals $ADKE$ and $BKDE$ are cyclic. The reasoning for the two quadrilaterals is the same. For instance, in the case of $ADKE$, this follows from the equality of angles

$$\widehat{MDA} = \frac{\pi}{2} - \widehat{ADC} = \frac{\pi}{2} - \widehat{CAE} = \widehat{AEI}.$$

Statement (2) follows by observing that

$$\widehat{ACM} = \widehat{ADM} = \widehat{ABK},$$

since the quadrilateral $ABKD$ is cyclic.

Statement (3) follows by observing that angle $\widehat{ZAB}$, by symmetry, equals angle $\widehat{BAE}$, which is equal to angle $\widehat{BCL}$, since both angles are formed by the tangents at the extremities of the chord AC of circle α. The orthogonality of CM to CL is obvious, since CM is a diameter of α and CL is tangent at its extremity C.

To prove (4) consider the angle $\widehat{BAE}$, which by symmetry w.r. to AB equals angle $\widehat{BAZ}$, which by the parallelity of AZ to CL equals angle $\widehat{LCB}$. This, by the tangent chord property, for circle (CBL), equals the angle of lines LB and EB, thereby proving the claim.

Property (5) follows from the previous one, showing that the two triangles have, respectively, equal angles at A and B. But also $\widehat{CAH} = \widehat{LCB}$ by the tangent-chord property for the chord AC in circle (HAC). □

In the next lemma, besides the variable circle (ABE) we consider the circle (ABZ) and its other than B intersection point G with circle $\beta = (CBL)$. In addition, point J is the intersection of lines BZ and HD and point Q is the intersection of line CZ with the circle $\alpha = (HAC)$. Finally we consider also the intersection point F of the variable lines AB and ZL, and the analogous intersection point F' of lines AB and ZH. Next lemma shows that F and F' (later not drawn in the figure) move, respectively, on two fixed lines (See Figure 3).

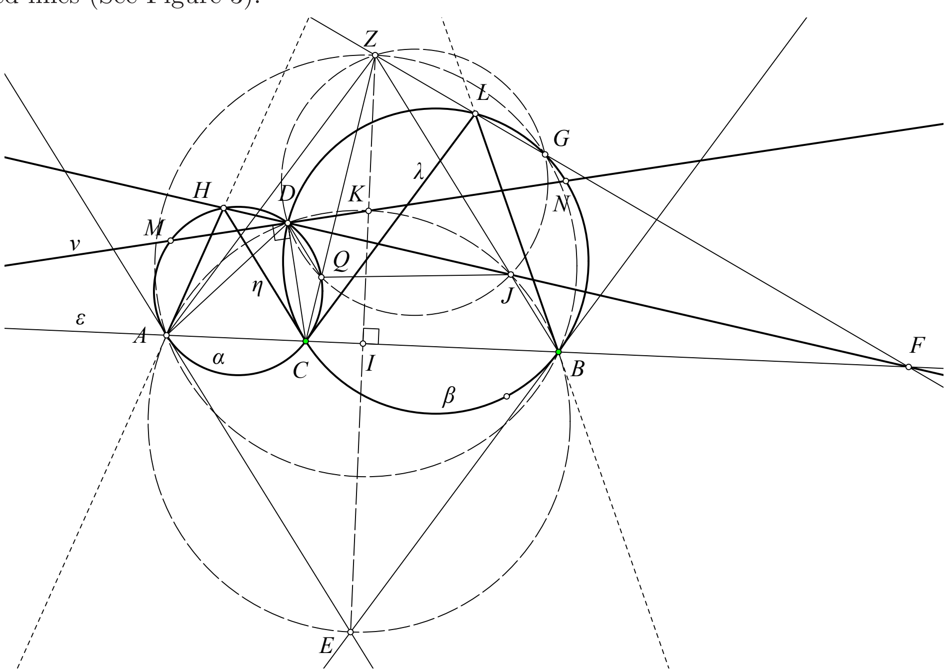

Fig. 3. Coincidences in the basic configuration

Lemma 2. *Under the definitions made above the following are valid properties:*

1. *Point G lies also on the variable line ZL.*
2. *The circle (ABE) passes through point J.*
3. *Circle (ZDJ) passes through Q.*
4. *Circle (ZDJ) passes through G.*
5. *Point F moves on the fixed line HD. Analogously point F' moves on line LD.*

Proof. Property (1) is valid, because, by hypothesis, the two quadrangles $ZABG$ and $LCBG$ are cyclic. Hence, by the parallelity of AZ, CL, their angles at G are the same, which proves the stated collinearity.

Property (2) is valid because angle $\widehat{HDA}$ equals angle $\widehat{HCA}$ and by parallelity of HC to BZ, last angle equals $\widehat{JBA}$, hence quadrangle $JBAD$ is cyclic.

Property (3) is valid because angle $\widehat{DJZ}$, by the cyclic $ABJD$, equals angle $\widehat{DAC}$, which, by the cyclic $DACQ$, equals angle $\widehat{DQZ}$.

Property (4) follows from the fact that angle $\widehat{LGD}$ equals angle $\widehat{LBD}$, which equals angle $\widehat{DAB}$. The last equality following from the fact that BL, as well as, AH are tangent to the circle (ABE), by (4) of the previous lemma. By the tangent-chord property angle $\widehat{DAC}$ is equal to angle $\widehat{DCL}$, thereby proving the claim.

Property (5) follows from the previous claims, which show that lines ZL, HD and AB are, correspondingly, the radical axes of the pairs of circles $(ZDG, ZAB), (ZDG, ABE)$ and (ZAB, ABE), hence point F is the radical center of these three circles. The proof for point F' is similar. □

Proof of the theorem. That the curve traced by Z is a conic, follows from the theorem of Maclaurin (see the appendix below). In fact, by lemma 2(5), the variable triangle ZFF' has its vertices F, F', correspondingly, on two fixed lines $\eta = HD$ and $\eta' = LD$. Besides, all of its sides pass through fixed points: FF' passes through C, ZF passes through L and ZF' through H. By Maclaurin's theorem, the free vertex (Z) (See Figure 4) describes a conic. The fact that the conic is a bounded one i.e. an ellipse, results from the property of the point Z, describing the conic, to be the reflection of point E on line AB. Below (theorem 2) it will be seen that point E describes a bounded curve, while AB is turning about the fixed point C. Hence the reflection Z of E on AB is also bounded.

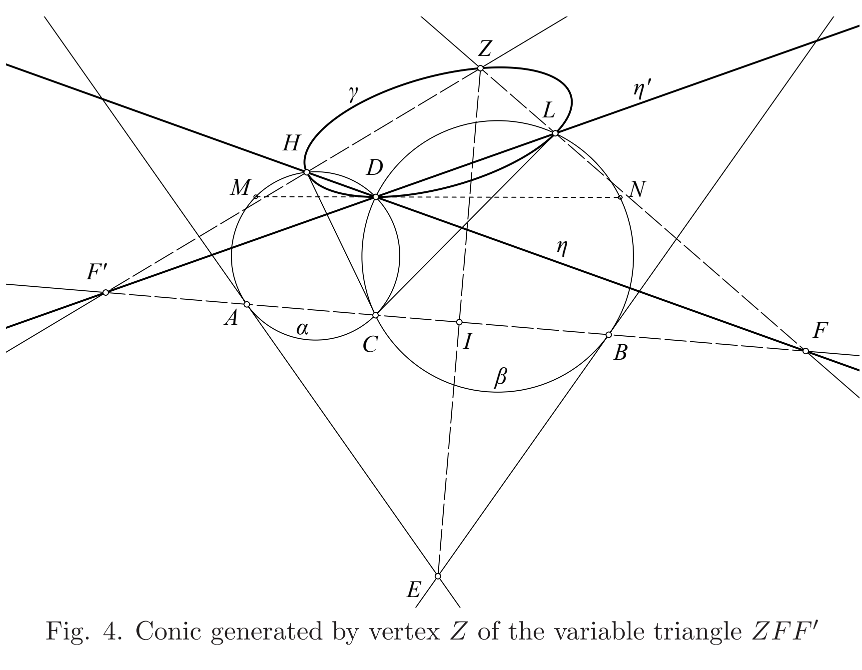

Fig. 4. Conic generated by vertex Z of the variable triangle ZFF'

As discussed in the appendix, the fact that H is on η, implies that CH is tangent to the conic. Analogously CL is also tangent to the conic. The fact that the conic is also tangent to line MN at D (See Figure 4) follows from the convergence of line DZ to line MN, as point E converges to D.

The claim on the degeneracy of the ellipse, when the two circles $\{\alpha, \beta\}$ are orthogonal, follows from the fact that point L (resp. H) becomes identical to N (resp. M), when the circles $\{\alpha, \beta\}$ become orthogonal. In this case triangle CHL becomes right-angled at C. □

2. The cardioid

Here and the sequel we use the notation introduced in the first section. A definition of the cardioid is given in the appendix, where are also discussed some of its known properties, which are relevant for our investigation. For a short synthetic account of the geometry of cardioids we refer to Akopyan's article [3].

Theorem 2. *Given two circles α and β, intersecting at two different points C and D, the intersection point E of their tangents at the extremities of a variable chord AB through C describes a cardioid with its cusp at D.*

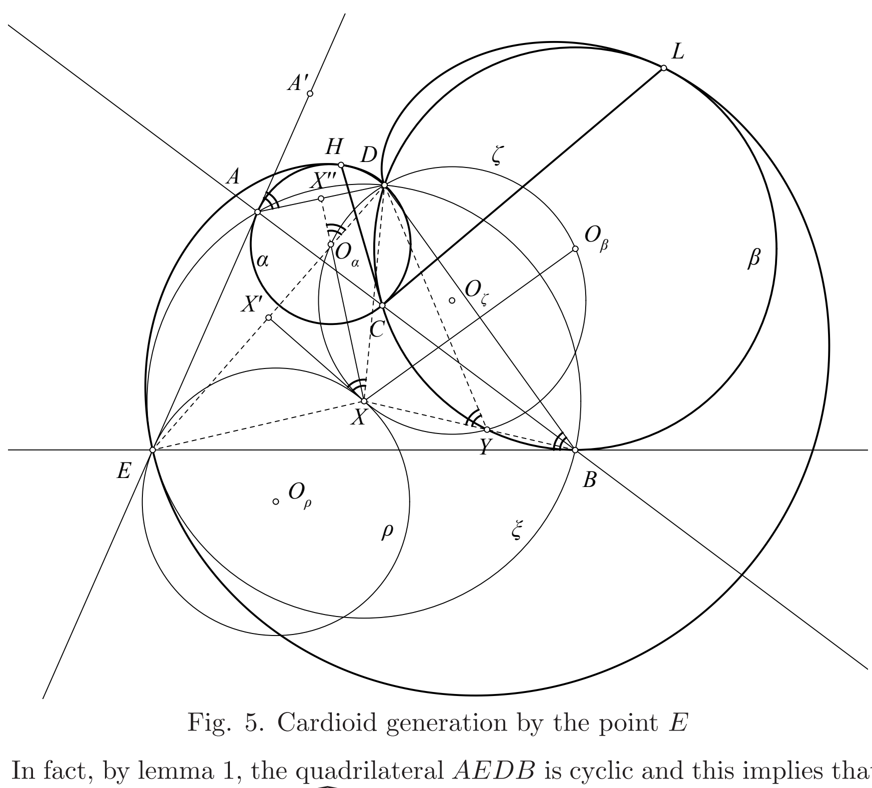

Fig. 5. Cardioid generation by the point E

Proof. In fact, by lemma 1, the quadrilateral $AEDB$ is cyclic and this implies that angle $\widehat{AEB}$ is constant, even equal to $\widehat{HCL}$ (See Figure 5). The result follows then as a special case of a known theorem of Butchart (see Appendix). We proceed though here to a short proof of our special case, since some of its ingredients are important for the subsequent discussion. We prove that the locus of E can be identified with a cardioid in its usual geometric definition. In conformance with this definition, we show that the locus of E

coincides with the locus of a fixed point E of a circle ρ, which is rolling on the fixed equal circle ζ, starting from the point D, which coincides with the cusp of the cardioid.

For the proof, notice first that the circumcenter X of the variable triangle ABE describes a circle ζ passing through the centers O_α, O_β of α, β and the point D. This follows from the definition of X, which is the intersection of medial lines of segments DA and DB, which always pass correspondingly through O_α, O_β and build an angle $\widehat{O_\alpha X O_\beta}$ complementary to $\widehat{ADB}$, which is constant and equal to angle $\widehat{O_\alpha D O_\beta}$.

A second remark is that the intersection point Y of circle ζ and line XB is also on the circle β. In fact, if X', X'' are correspondingly the middles of segments DE, DA, then, from the cyclic quadrilateral $YXO_\alpha D$, follows that angle $\widehat{XYD}$ equals angle $\widehat{X''O_\alpha D}$. This, by the tangent-chord property, equals angle $\widehat{A'AD}$. This in turn, by the cyclic $DAEB$ is equal to $\widehat{EBD}$, which by the tangent-chord property is equal to angle $\widehat{BLD}$. This shows that the quadrilateral $YBLD$ is cyclic and proves the claim.

By the definition of X, angle $\widehat{X'XD}$ is also equal to $\widehat{EBD}$. It follows that $X'X$ is tangent at X to circle ζ. Thus, taking the symmetric O_ρ of O_ζ w.r. to X we define circle ρ with center at O_ρ, which is equal to ζ, tangent to it at X and passing through E. It follows that the locus point E defines on circle ρ an arc XE, which is equal to the arc XD of the fixed circle ζ. This completes the proof of the rolling property of ρ on ζ and proves the theorem. □

Remark 1. As is pointed out in the appendix, the cardioid is symmetric w.r. to the line DO_ζ and up to congruence completely determined by the circle ζ. Later we will see that circle ζ is congruent to the Euler circle of a triangle circumscribing the ellipse γ.

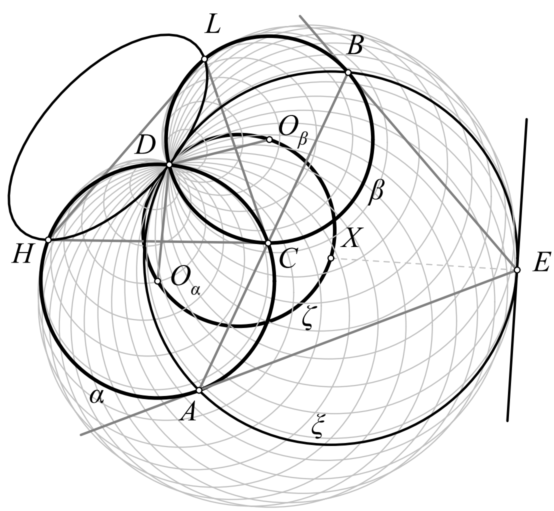

Fig. 6. Cardioid as envelope of circles

Remark 2. Notice that the cardioid can be also generated by fixing a point D on a circle ζ and considering all circles ζ with radii $r_\zeta = |DX|$, where their center X is another arbitrary point of ζ. The envelope of the family of all these circles (See Figure 6) is a cardioid. The two circles α, β of our configuration are two particular circle-members of such a family. The circumcircle ξ of the variable triangle ABE belongs also to this family. By its property to be envelope of all these circles, follows that the cardioid is tangent to the circles α, β and ξ. Referring back, to figure 5, and the fact, shown there that $X'X$ is the medial line of DE, follows that the points of tangency of these circles with the cardioid are respectively points H, L, and E.

3. A circumscribing triangle

In this section we focus on the triangle MNV, whose sides are shown to be tangent to the conic γ. Besides line MN, already met in the previous paragraphs, we consider here the lines MV and NV, which are the tangents, correspondingly, at M and N of circles α and β (See Figure 7). The context here is that of *triangle centers* and the *inscribed conics* in triangles, which have these centers as *perspectors*. A general reference for these concepts is the book of Akopyan and Zaslavsky on the geometry of conics [2, p. 105] and Yiu on the geometry of the triangle [31]. Of particular importance for our subject is the concept of the *orthopole* and the orthopolar conic, defined and discussed in short in the appendix.

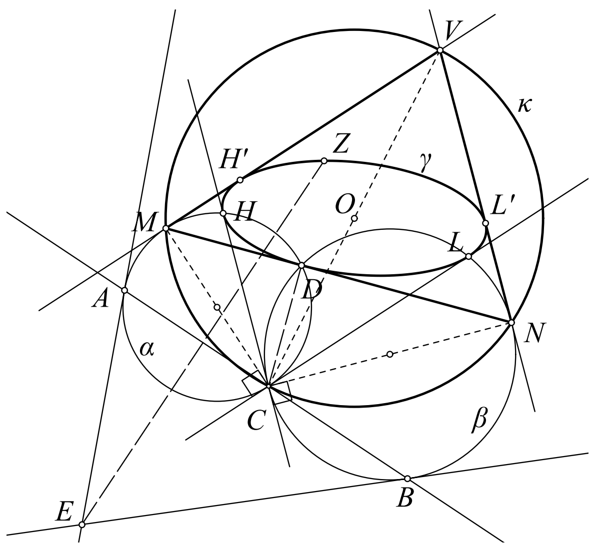

Fig. 7. The circumscribed triangle MNV

Theorem 3. *The ellipse γ is an inscribed conic in the triangle MNV. The center of the ellipse coincides with the cicrumcenter O of the triangle and this identifies it with the inconic with perspector X_{69}. This is the ellipse generated by the orthopoles of all lines passing through the De Longchamps point X_{20} of the triangle MNV.*

Proof. The proof on the triangle MNV circumscribing the ellipse follows from lemma 1(3), asserting that lines $\{AZ, BZ\}$ (See Figure 7) are always parallel, respectively, to $\{CL, CH\}$. Thus, the ellipse is contained in the strip consisting of CL and its parallel MH' from M. It follows that the ellipse has exactly one common point H' with this parallel, which, consequently, is a tangent to the ellipse. Analogously, the ellipse is tangent to the line NL', which is parallel to CH from N. Note that the reasoning shows also that points H' and L' are, respectively, symmetric of L and H w.r. to the center O of the conic.

The proof on X_{69} follows by identifying the ellipse with the orthopolar conic of the De Longchamps point X_{20} of the triangle MNV, which is the ellipse tangent to the triangle and having its center at the circumcenter of the triangle. This ellipse is the locus of points which are orthopoles of the lines passing through X_{20} (see appendix).

The fact, in turn, that the circumcenter O of the triangle MNV is the center of the ellipse follows by observing that quadrangle $CMVN$ is cyclic, its angles at M and N beign right, and the ellipse is inscribed in the parallelogram defined by the lines CH, CL, VN, and VM. This implies that the center of the ellipse is the intersection point of the diagonals of the parallelogram, one of which (CV) is a diameter of the circumcircle κ. $\square$

4. A bitangent triangle

Of particular interest for our discussion is also the *bitangent* to γ triangle CHL, i.e. a triangle that has only two of its sides (CH, CL) tangent to γ, correspondingly, at points H and L. Of interest also is the circumcircle δ of triangle CHL. In the formulation of the next theorem enter the concepts of the *symmedians* and the *Brocard points* of a triangle, for which we refer to the aforementioned book of Yiu. The following theorem expresses the main properties of this configuration (See Figure 8).

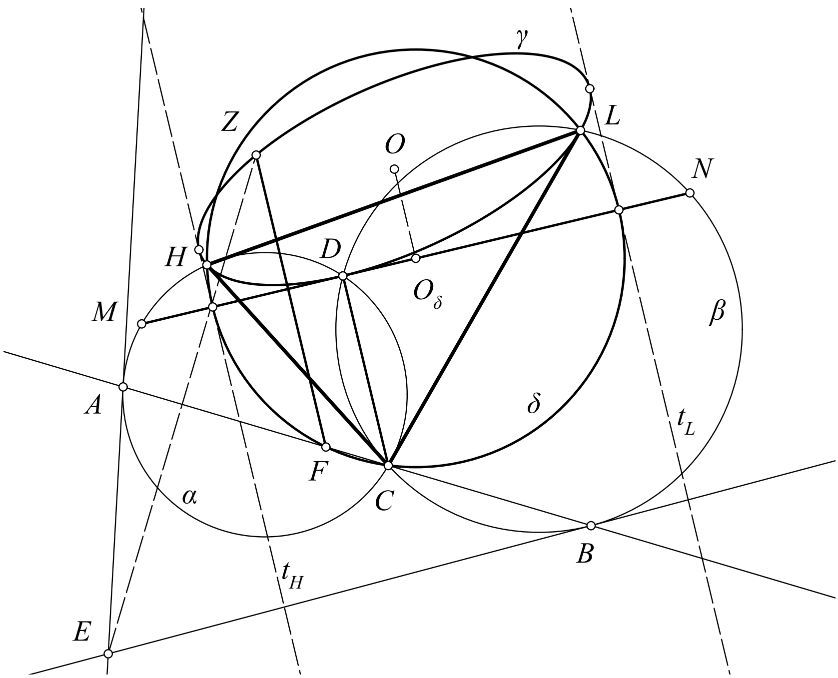

Fig. 8. The bitangent to γ triangle CHL

Theorem 4. *Line CD is a symmedian for the triangle CHL and point D coincides with a vertex of its second Brocard triangle. The circumcenter O_δ is the middle of the segment MN. If point F is the intersection of δ with the variable line AB, then line FZ is parallel to the fixed line CD. The circle δ and the ellipse γ have two common parallel tangents t_H and t_L, which are orthogonal to line MN.*

The proof of the theorem relies on the following two lemmas. In these we consider the second intersection point U of the circle $\delta = (CHL)$ with the line AB and the triangle $EA'B'$ with sides the lines EA, EB and the tangent t_U to δ at U, points A', B' being, correspondingly, the intersections of this tangent with lines EA and EB (See Figure 9).

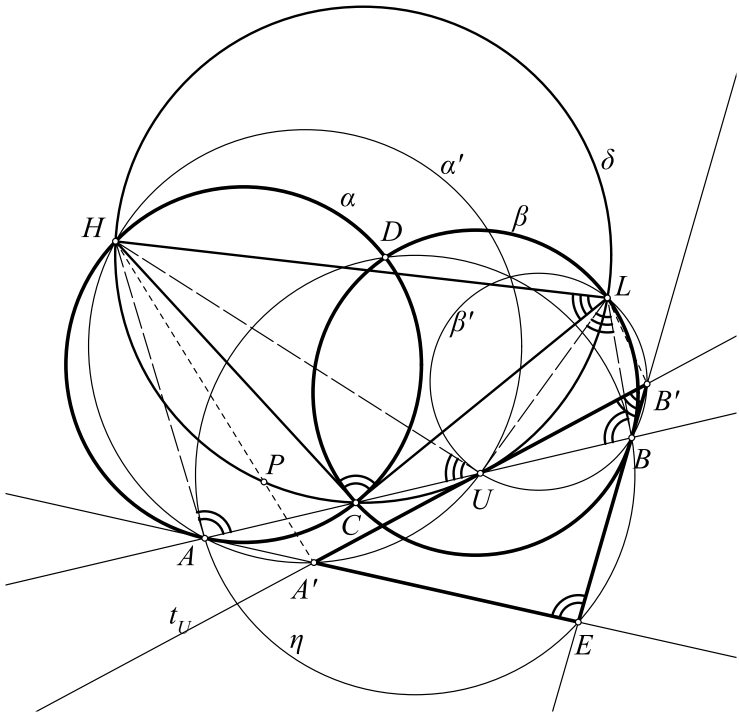

Fig. 9. The similar triangles: fixed CHL and variable $A'EB'$

Lemma 3. *Under the definitions made above, the following are valid properties:*

1. *Quadrangles $UBB'L$ and $UA'AH$ are cyclic.*
2. *Triangle $A'EB'$ is similar to HCL.*
3. *Triangles $HA'U, UB'L$ and HUL are similar.*
4. *Point U is the middle of $A'B'$.*

Proof. To prove (1), write angle $\widehat{UB'B} = \widehat{UBE} - \widehat{BUB'}$. By the tangent-chord theorem these angles are equal: $\widehat{UBE} = \widehat{CLB}$, $\widehat{BUB'} = \widehat{CLU}$. It follows that $\widehat{ULB} = \widehat{UB'B}$ and this proves the first claim for quadrilateral $UBB'L$. Analogous is the proof for the quadrilateral $UA'AH$.

To prove (2), notice that $\widehat{CLB} = \widehat{HLU}$, which implies $\widehat{ULB} = \widehat{HLC}$. This follows from the equalities of angles: $\widehat{CLB} = \widehat{CBE} = \widehat{ACH} = \widehat{HLU}$. Analogously follows that $\widehat{EA'B'} = \widehat{CHL}$ and thereby the proof of the claim.

To prove (3) use property (4) of theorem 1. By this, lines AH and BL are tangent to the circle $\eta = (ABE)$ (See Figure 9). Using the previously proven properties this implies that $\widehat{HA'U} = \widehat{HAU} = \widehat{UBL} = \widehat{UB'L}$. Similarly, $\widehat{UHA'} = \widehat{UAA'} = \widehat{LBB'} = \widehat{LUB'}$. By the tangent chord property for cirlce δ follows that $\widehat{UHL} = \widehat{LUB'}$. Analogously also $\widehat{HLU} = \widehat{HUA'}$, thereby completing the proof of this claim.

The proof of (4) follows by observing that HU is the bisector of angle $\widehat{A'HL}$, hence divides the corresponding arc of circle δ in two equal parts and $UP = UL$, where P the second intersection of line HA' with circle δ. This implies that triangles UPA' and ULB' are congruent and completes the proof of this claim. $\square$

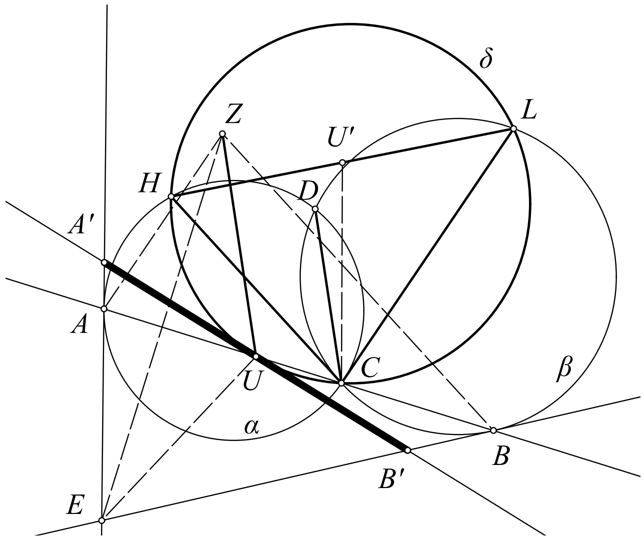

Fig. 10. The direction of UZ

Lemma 4. *The variable line UZ, where U is the second intersection point of AB with circle δ, is always parallel to line CD.*

Proof. For the proof we use the result of the previous lemma, by which, the variable triangle $A'EB'$ remains similar to the fixed one HCL. We use also the well known result identifying point D with a vertex of the second Brocard triangle of the triangle HCL ([12, p.283]). By this result, the symmedian point of triangle CHL is on line CD, which then is isogonal w.r. respect to this triangle to its median CU' (See Figure 10). To prove the lemma we show that lines UZ and CD are equal inclined to side CH of the triangle CHL. In fact, by (3) of lemma 1, line CH is parallel to BZ, and, by symmetry, $\widehat{UZB} = \widehat{UEB}$. Last angle, by the similarity of triangles $A'EB'$ and HCL and because U is the middle of $A'B'$, equals $\widehat{U'CL}$. This angle, by the noticed above isogonality of CD and CU', equals angle $\widehat{HCD}$, thereby completing the proof of the lemma. □

The proof of theorem 4 follows immediately from the preceding lemma.

5. An Euler circle

In this section we turn back to triangle MNV, discussed in section 3, and its Euler cirlce θ, combining results obtained in the previous paragraphs. The main property of this circle is its equality with the circle ζ, introduced in section 2, and carrying the circumcenters of all triangles EAB. This equality is realized by a point-symmetry with respect to the center O_δ of the circle δ, introduced in section 4. The main properties in this context, are easy to derive consequences of the previous discussion and are listed in the following theorem. Some additional elements, involved in this theorem, are the orthocenter H' of triangle MNV and the feet V', M' and V' of the altitudes from the corresponding vertices of the triangle (See Figure 11).

Theorem 5. *Under the definitions made above the following are valid properties:*

1. *Point C is the symmetric of the orthocenter H' of triangle MNV w.r. to the middle O_δ of its side MN.*

(2) *The circumcircle δ of triangle CHL passes through H' and points $\{L, H\}$ are on the altitudes of triangle MNV, at distance from the vertex equal to the distance of H' from the corresponding altitude-foot.*
(3) *The circle $\zeta = (DO_\alpha O_\beta)$ is the symmetric of the Euler circle θ of MNV w.r. to the middle O_δ of MN.*

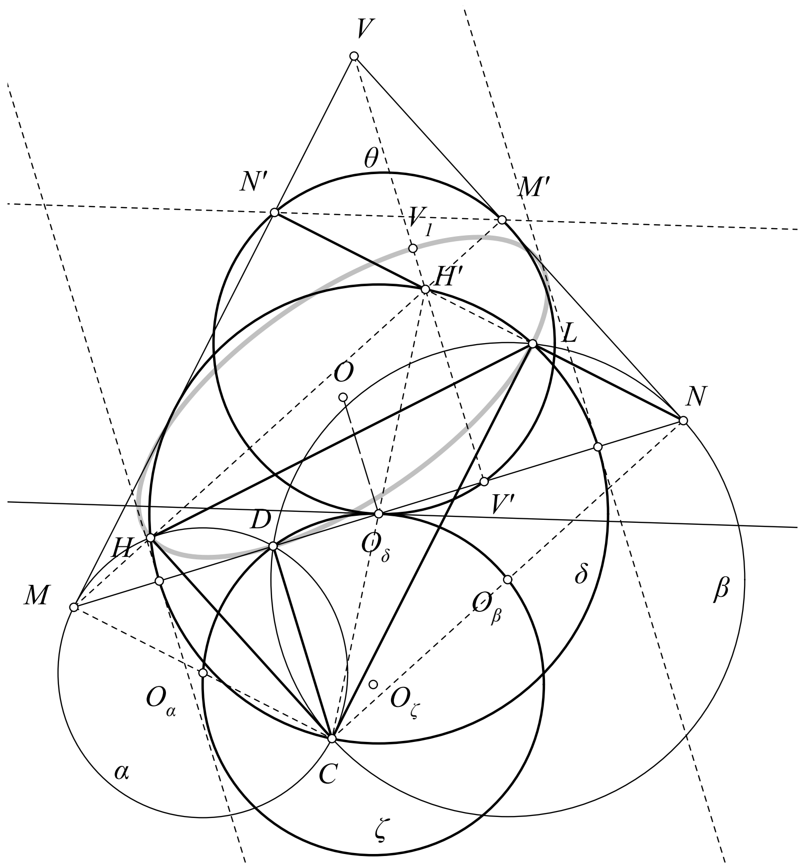

Fig. 11. ζ symmetric to the Euler circle θ w.r. to O_δ

Proof. Properties (1) and (2) follow from the presence of the parallelogram p_1, having for sides the lines $\{CH, HL, NV, VM\}$. As proved in theorem 3, the circumcenter O of triangle MNV is the middle of the diagonal VC. Besides, lines CD and VV' are both orthogonal to MN. Thus, the middle O of VC projects on the middle of DV' and this shows that $|DO_\delta| = |O_\delta V'|$. Since CL is orthogonal to NN' and CH is orthogonal to MM', it follows that they meet on the altitude VV' at a point H', which is diametral to C w.r. to circle δ.

Property (3) follows by considering the symmetrics of points O_α, O_β w.r. to O_δ. Because $CNH'M$ is a parallelogram and O_δ is its center, these two points coincide respectively with the middles of $H'N$ and $H'M$, which are points of the Euler circle θ of MNV. This shows the stated symmetry about O_δ. □

There are several properties that follow from the previous theorems, and their proofs are easy exercises. Thus, for example, segment DC is equal to $H'V'$, which is also equal to VV_1, where V_1 is the intersection of DO with the altitute VV' (See Figure 11). It is also easily seen that the ellipse γ passes through V_1, that line DH is parallel to ON, that CV is orthogonal to $N'M'$, which is parallel to the common tangent at O_δ of the circles ζ and θ. Perhaps, we should notice aslo that the ellipse γ passes through six easily constructible points. The first triad of these points consists of the symmetrics of the feet of the altitudes w.r. to the middles of the respective sides, like the point D, which is symmetric to V' w.r. to O_δ. The other triad of points are on the altitudes, like V_1 in distance $|VV_1| = |H'V'| = |DC|$.

6. Inverse construction

Next theorem deals with the inverse procedure, succeeding to reproduce a given ellipse γ by the procedure described above, applying it at a point $D \in \gamma$.

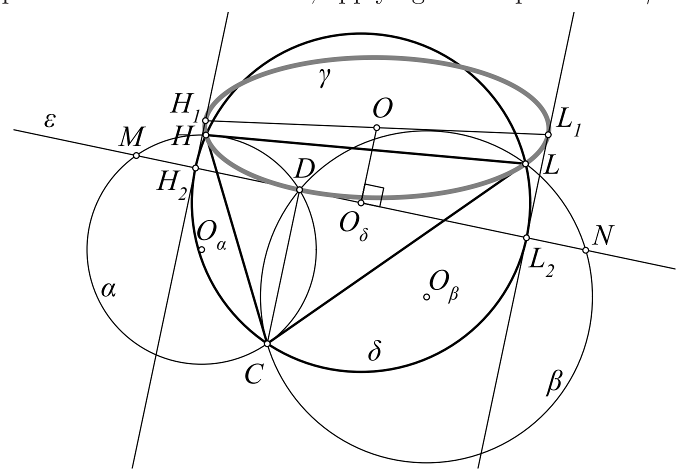

Fig. 12. Constructing two circles α, β at every point of an ellipse

Theorem 6. *Given an ellipse γ, for every point $D \in \gamma$ there are two uniquely defined circles $\{\alpha, \beta\}$, which generate the ellipse by the preceding method.*

Proof. This is a direct consequence of theorem 4. In fact, consider the tangent ε of the conic at an arbitrary point D and project its center O to point O_δ on it (See Figure 12). Then consider the conjugate to the direction of OO_δ, diameter H_1L_1 of the ellipse. The circle δ is defined by its center at O_δ and diameter equal to the length of the projection H_2L_2 of H_1L_1 on line ε. Let C be the intersection point of δ on the normal of the ellipse at D, lying on the other side than O. Draw from C the tangents CH, CL to the ellipse. There are defined two circles $\alpha = (CDH)$, $\beta = (CDL)$. In view of theorem 1, the two intersecting circles define, by the procedure studied above, the ellipse γ. □

A question that comes up from the previous property is, about the behavior of the configurations that result from the various places of point D on the ellipse. To this answers the following statement.

Theorem 7. *For all places of D on γ the segment OC is of constant length. Thus, all resulting triangles MNV that circumscribe the ellipse γ are also inscribed in the same circle κ, which has the radius $r_\kappa = |OC| = a + b$, where $\{a, b\}$ are the axes of the ellipse and has the same center with it. Consequently the cardioids for the various places of D are congruent to each other.*

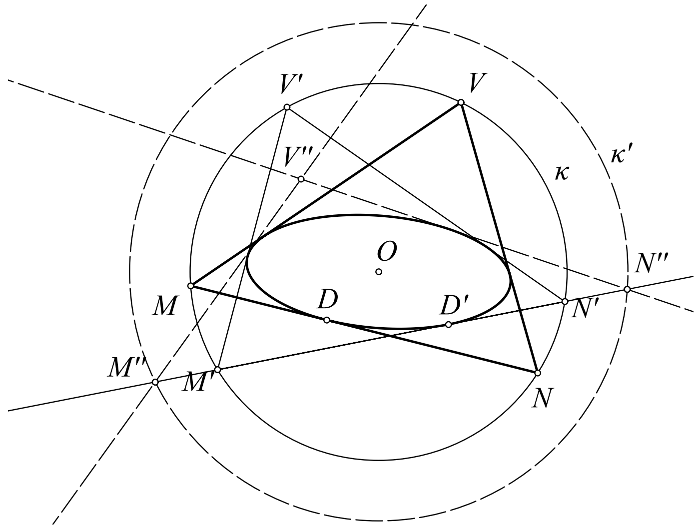

Fig. 13. Applying Poncelet's porism

Proof. The proof of this could be given by a calculation, using, for example, known formulas for the tangents from a point, the chords of contact points to the ellipse etc. ([26, pp.221-233]). The property, though, lends itself for a synthetic proof. In fact, consider an ellipse γ, a point $D \in \gamma$, the corresponding triangle MNV, constructed by the procedure, and its circumcircle κ (See Figure 13). By Poncelet's porism ([2, p.93], [7, p.203 (II)]), for every other tangent $M'N'$, at another point D', we would obtain a triangle $M'N'V'$ which is also circumscribed in γ and inscribed in κ. Now, if there were a triangle $M''N''V''$ circumscribed to γ and inscribed in another concentric circle κ', of radius, say, greater than that of κ, then, by applying again Poncelet's porism, we could construct the triangle so that the lines $M'N'$ and $M''N''$ coincide. Then it is readily seen that the corresponding vertex V'' cannot lie on κ'. This contradiction shows that, by our procedure, the constructed circumscribed about γ triangles MNV have all the same circumcircle κ. The claim about the radius follows then easily by simply considering D to be at a vertex of the ellipse and evaluating $r_\kappa = |OC| = |CD|+|L_1L_2| = |DL_2|+|L_2L_1| = a+b$ (See Figure 14). The result for the cardioids is an immediate consequence, of their generation by the rolling of a circle on another circle, both being congruent to the Euler circle of triangle MNV. $\square$

Corollary 8. *For every ellipse γ with center O, axes $\{a,b\}$ and every triangle circumscribed to the ellipse and having O for circumcenter, its vertices lie on the circle $\kappa(O,a+b)$.*

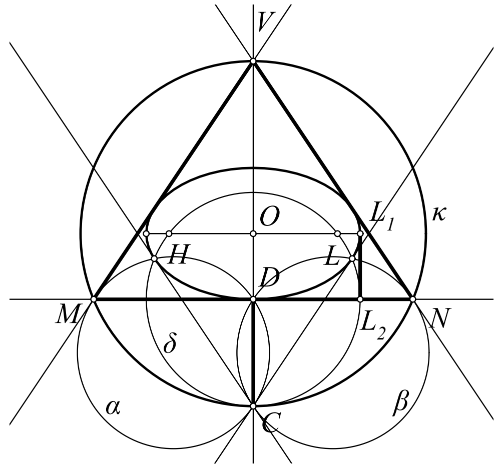

Fig. 14. $r_\kappa = a + b$

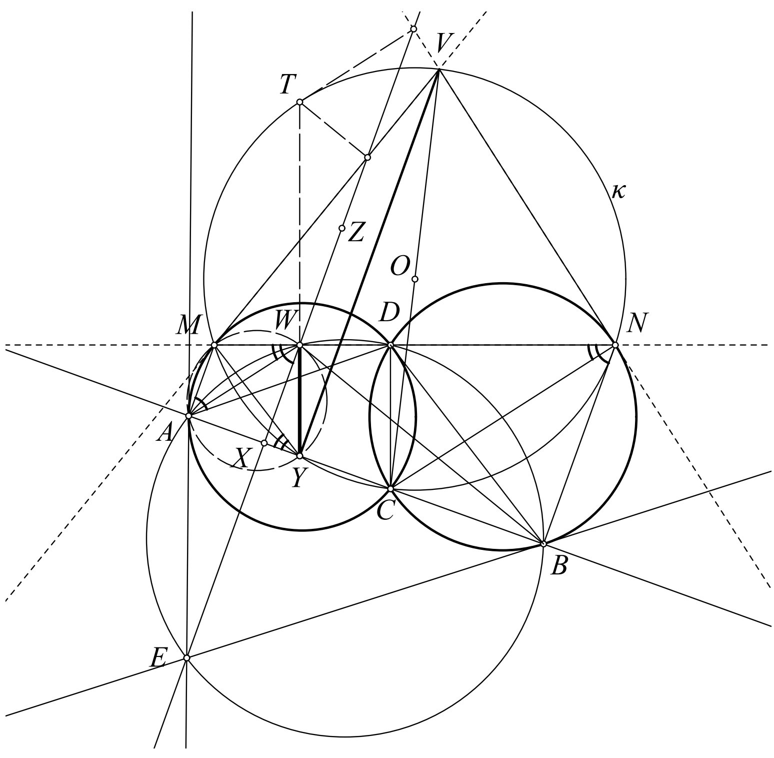

Fig. 15. EZ is a Simson line of triangle MNV

7. Simson lines and deltoid

The construction of the ellipse γ from two intersecting circles $\{\alpha, \beta\}$, by our procedure, leads to the triangle MNV and its associated deltoid, i.e. the envelope of its Simson lines. The key property in this respect is the following (See Figure 15).

Theorem 9. *For all points E, line EZ is a Simson line of the triangle MNV.*

We prove first that the intersection point W of MN and EZ and the other than C intersection point Y of the circumcircle κ of MNV and line AB define a segment WY which is orthogonal to MN. This is so because $WYAM$ is cyclic. This in turn follows from the equality of angles $\widehat{MAW} = \widehat{AWX} = \widehat{ABE} = \widehat{CNB}$, which implies that AW and CN are parallel, since MA, WX, NB are parallel being all orthogonal to AB. From the parallelity of AW to CN follows that $\widehat{MWA} = \widehat{DNC} = \widehat{MYA}$, later because $MYCN$ is cyclic. This shows that $\widehat{MWA} = \widehat{MYA}$ and implies the orthogonality of WY to MN. Let T be the other intersection point of YW with the circle $\kappa = (MNV)$, then, since VC is a diameter of κ, line VY is orthogonal to AB, hence parallel to EZ. By a well known theorem for the orientation of the Simson lines (see theorem 15 in the appendix), the line EZ is the Simson line of the point T on the circle κ. This completes the proof of the theorem.

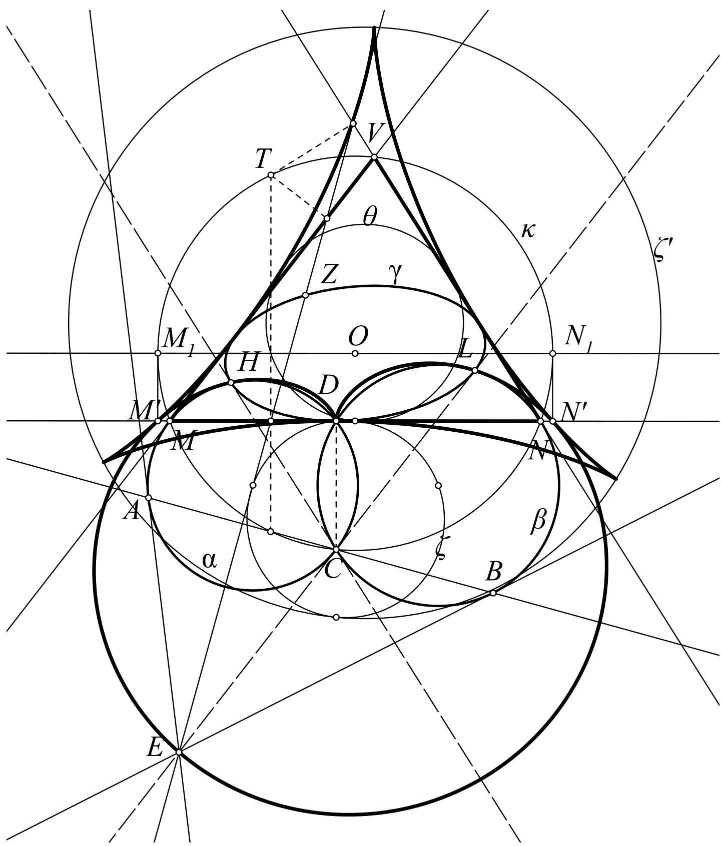

Fig. 16. The deltoid-envelope of lines EZ

Figure 16 shows the deltoid-envelope of the Simson lines of triangle MNV together with the corresponding cardioid-locus of point E. There are various relations visible in this figure, which are consequences of our discussion or can be proved easily. I mention only next property, which can be proved by considering special places of the point E.

Theorem 10. *The deltoid passes through point D and is tangent to the cardioid at two points $\{M', N'\}$ on line MN, which are the projections of the diameter M_1N_1 of the circumcircle κ of MNV, which is parallel to the side MN.*

It is remarkable that both the deltoid ([22, p.72]) and the cardioid ([22, p. 34]) can be generated from a point of a circle congruent to the Euler circle θ of MNV which rolls on another circle. The deltoid is generated by rolling such a circle inside the circle ζ' which is concentring to the Euler circle and has the radius $r = 3r_{\theta}$. The cardioid is generated by rolling circle θ outside its congruent ζ.

Remark 3. In the discussion we have tacitly assumed that the two intersecting circles $\{\alpha, \beta\}$, defining our configuration, intersect at an obtuse angle, equivalently the angle $\widehat{HCL} = \widehat{MVN}$ is acute. The case of circles intersecting at an acute angle can be handled using, essentially, the same arguments. Figure 17 shows a configuration for such a case.

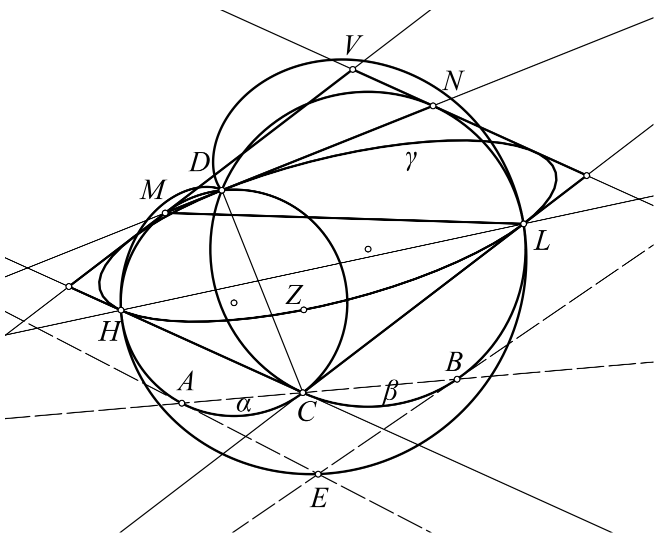

Fig. 17. The case of obtuse angles

8. Appendix

In this section we gather together some well known facts used in the proofs, mainly on Maclaurin's theorem, properties of the Cardioids, Simson lines, and orthopoles.

8.1. **Maclaurin's theorem.** Maclaurin's theorem describes the generation of a conic by the free vertex Z of a variable triangle ZFF', whose other two vertices F, F' are bound to move on two fixed lines η and η'. The precise statement is the following ([6, II, p.13], [11, p.72], [27, p.299]).

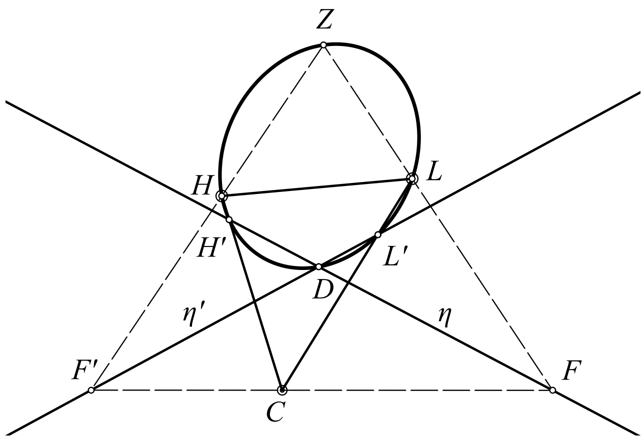

Fig. 18. Maclaurin's theorem

Theorem 11. *Let the variable triangle ZFF' have its two vertices F and F' move, correspondingly on two fixed lines η and η', while its side-lines ZF, ZF' and FF' pass, correspondingly through three fixed points L, H and C. Then its free vertex Z describes a conic. This conic passes through the points H, L and the intersection point D of lines η, η'. The conic passes also through the intersection points H', L', correspondingly, of the line-pairs (η, HC) and (η', CL).*

The theorem, in its previous form, applies for three points $\{C, H, L\}$ and two lines $\{\varepsilon, \varepsilon'\}$ *in general position* (See Figure 18). In the special case, in which point H is contained in line η, points H and H' coincide and the conic is tangent to line CH at H. Analogous is the behavior for line CL if L is contained in η'. This particular case applies to our configuration, impying the tangency of the conic γ to lines CH, CL, correspondingly at H and L.

8.2. **Cardioid.** One way to define the cardioid ([3] [22, p.34], [30, p.89], [23, p.142]), is to consider it as the geometric locus of a fixed point E of a circle ρ rolling on a circle ζ of equal radius, the rolling circle starting to roll at the point D of ζ. The resulting curve has a cusp at D (See Figure 19). The circle ζ could be called the *basic* circle of the cardioid. The cardioid is completely determined by its basic circle and a point D on it, defining its cusp. Among the many properties of this remarkable curve is the fact that the one-sided tangents at D coincide and both contain the center O_ζ of the fixed circle and line DO_ζ is a symmetry axis of the curve. If X is the contact point of the circles ρ and ζ, X' is the middle of DE, and O_ρ is the center of ρ, then, since the arcs XE on ρ and XD on ζ are equal, the quadrilateral $DEO_\rho O_\zeta$ is an equilateral trapezium and XX' is orthogonal to DE, which is parallel to $O_\rho O_\zeta$. Taking the circle σ in symmetric position of ρ w.r. to O_ζ and considering the corresponding point E' of the locus, we realize analogously that $DE'O_\sigma O_\zeta$ is an equilateral trapezium and $EE'O_\sigma O_\rho$ is a parallelogram. This proves the following fundamental property of the cardioid.

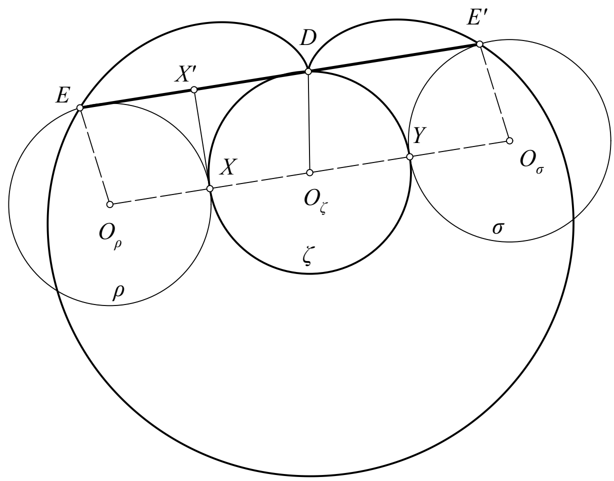

Fig. 19. Cardioid generation by a rolling circle

Theorem 12. *Every line through the cusp D of the cardioid defines a chord EE' of it, which has the constant length $4r_\zeta$, where r_ζ is the radius of the fixed circle O_ζ.*

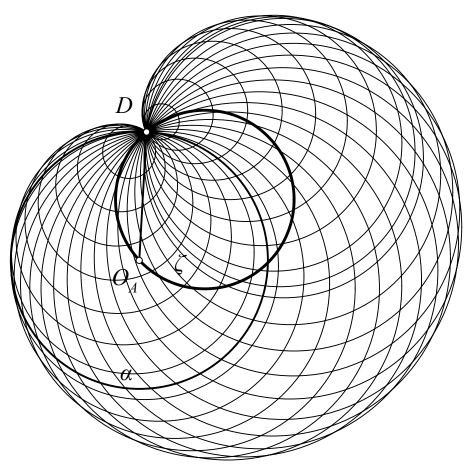

Fig. 20. Cardioid generation as envelope of circles

Adopting the previous definition, we can formulate an alternative definition of the cardioid in the form of the following theorem.

Theorem 13. *Let D be a fixed point on a circle ζ. For each other point O_α on ζ define the circle α with center at O_α and radius the distance $r_\alpha = |DO_\alpha|$. Then the envelope of all these circles α is a cardioid with basic circle ζ and its cusp at D.*

Cardioids have been extensively studied in the past ([22, p.34], [30, p.89], [23, p.142], [24, p.73], [9]). For a short account and references see Archibald's article [5]. The following theorem, due to Butchard ([10]), applies to more general situations than it is the case with our configuration.

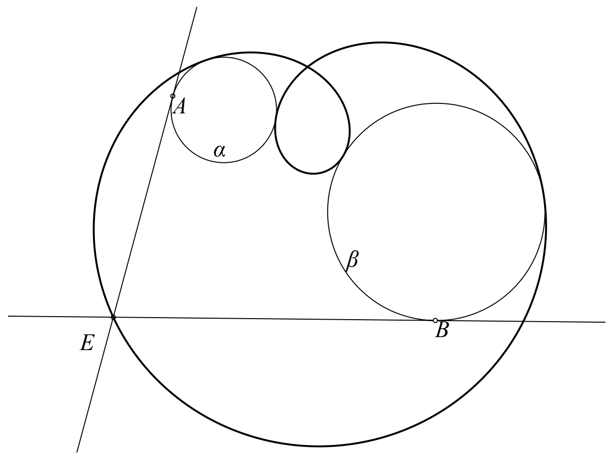

Fig. 21. Cardioid generation by a constant angle $\widehat{AEB}$

Theorem 14. *If a, constant in measure, angle AEB has its legs EA, EB, respectively, tangent to two circles α and β, then its vertex E describes a cardioid.*

8.3. **Simson lines.** The Simson line s_P of a triangle ABC w.r. to a point P on its circumcircle $c = (ABC)$ is the line carrying the projections P_1, P_2, P_3 of P (See Figure 22), respectively, on sides BC, CA and AB ([12, p.140]). A property of Simson lines, used in our discussion, is the one expressed by the following proposition ([12, p.142]).

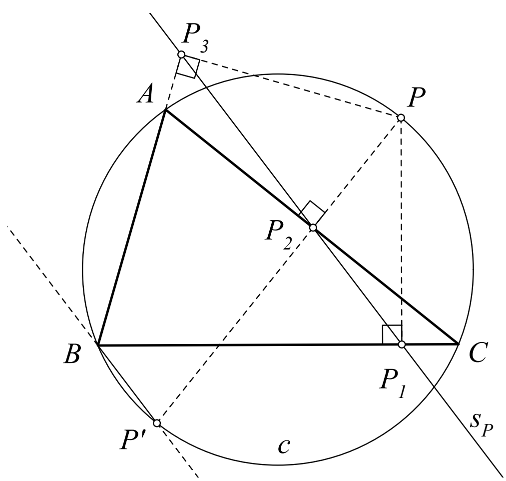

Fig. 22. Simson lines

Theorem 15. *If for a point P on the circumcircle c of triangle ABC and its projection on a side, say P_2 on CA, line PP_2 is extended to cut c at a second point P', then line $P'B$ is parallel to the Simson line s_P of P.*

Another fact used in the discussion is also the following ([1, p.240], [28, p.231], [14, p.101]),([15, p.563],[8, p.224]) (See Figure 23).

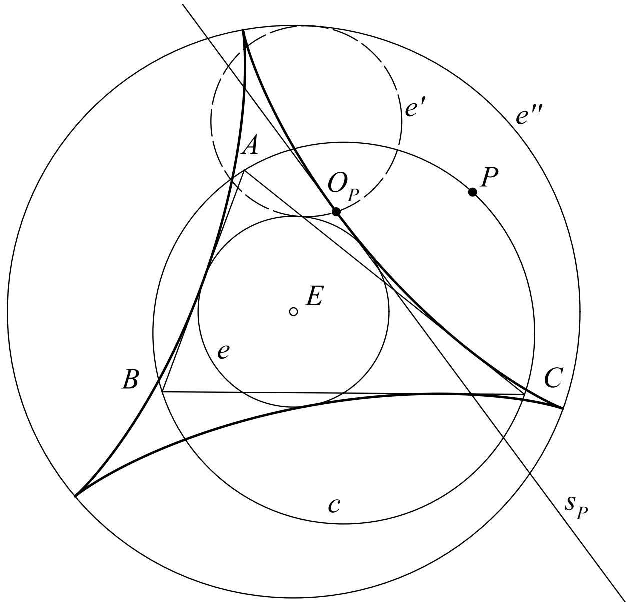

Fig. 23. Deltoid: the envelope of Simson lines

Theorem 16. *The evelope of all Simson lines s_p of triangle ABC is an algebraic curve of degree four, called deltoid. This curve is generated by a point O_p of a circle e' equal to the Euler circle e of the triangle, which roles inside a circle e'' concentric to e and of triple radius.*

8.4. **Orthopoles.** The orthopole of a line ε w.r. to a triangle ABC results by a process of double projection of each vertex of the triangle. Vertex A is projected on line ε to the point A' and point A', in turn, is projected on the opposite side BC, to point A''. Analogously

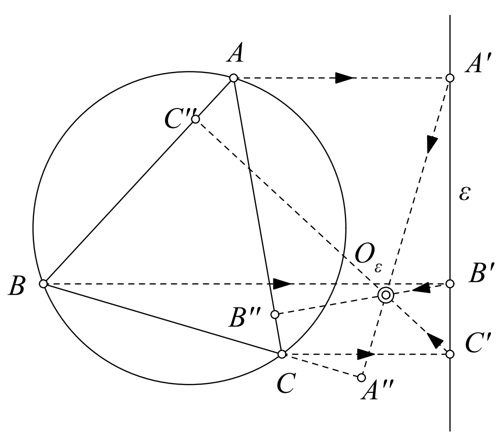

Fig. 24. The orthopole of a line w.r. to a triangle

are defined the points B', B'' and C', C'' (See Figure 24). It is proved that the lines $A'A''$, $B'B''$ and $C'C''$ intersect at a point O_ε which is the orthopole of the line ε w.r. to ABC ([21, p.17], [16, p.49], [19, p.106], [18]). The main properties used in this article are expressed by the following two theorems. The first of them relating the orthopole to Simson lines and the deltoid they envelope (See Figure 25).

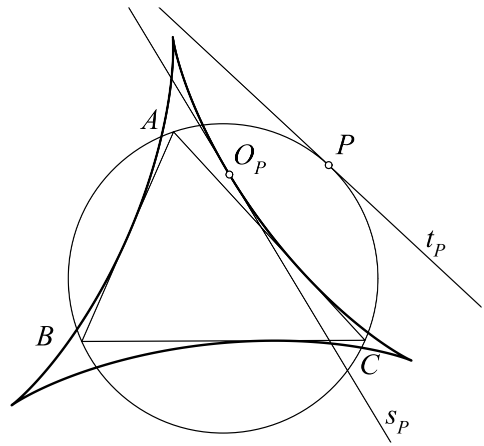

Fig. 25. The deltoid contact point O_P of the Simson line s_P

Theorem 17. *The orthopole O_P of the tangnet t_P of a point P on the circumcircle of triangle ABC is on the Simson line s_P of that point and coincides with its contact point with the deltoid-envelope of all Simson lines of the triangle.*

The second property is concerned with the orthopoles w.r. to a fixed triangle and all the lines through a given point D. Their orthopoles define a conic called the *orthopolar* conic of point D w.r. to the triangle (See Figure 26).

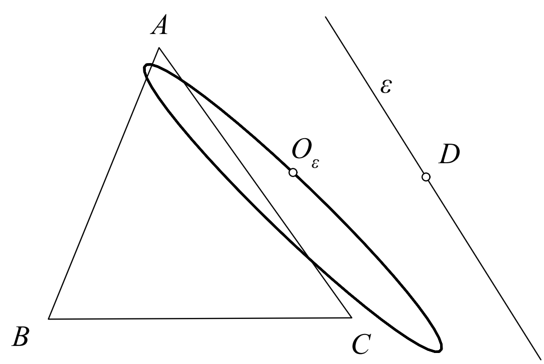

Fig. 26. The orthopolar conic of point D w.r. to the triangle ABC

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Theorem 18. *The orthopoles w.r. to the triangle ABC of all lines through the point D generate a conic.*

Given the triangle ABC, there is a particular orthopolar conic inscribed in the triangle. Next theorem describes how this is done ([4], [13], [19, p.125], [16, p.46], [18]) (See Figure 27).

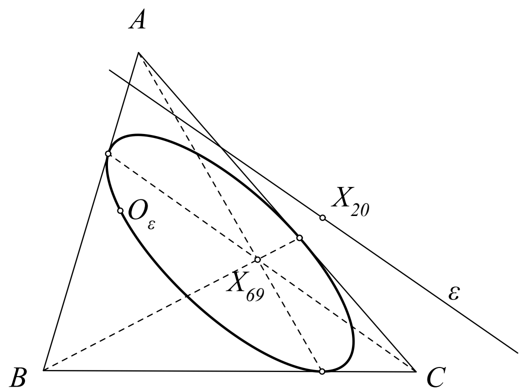

Fig. 27. The orthopolar conic of point X_{20}

Theorem 19. *Given the triangle ABC, the orthopolar conic of its triangle center X_{20} (called the Delongchamps point of the triangle), is a conic inscribed in the triangle, with perspector the triangle center X_{69}. This is an ellipse tangent to the sides of the triangle whose center coincides with the circumcenter of the triangle.*

The standard reference for general triangle centers is Kimberling's encyclopedia of triangle centers [20]. For the particular centers X_{20} and X_{69}, the previously cited facts and many other interesting properties can be found in the articles [17], [25], [29].

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E-mail address: `pamfilos@math.uoc.gr`

UNIVERSITY OF CRETE, GREECE