

ON BROCARD'S POINTS IN POLYGONS

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ABSTRACT. In this note we present a synthetic proof of the key lemma, defines in the problem of A. A. Zaslavsky.

For any given convex quadriateral $ABCD$ there exists a unique point P such that $\angle PAB = \angle PBC = \angle PCD$. Let us call this point the *Brocard point* ($Br(ABCD)$), and respective angle — *Brocard angle* ($\phi(ABCD)$) of the broken line $ABCD$. You can read the proof of this fact in the beginning of the article by Dimitar Belev about the Brocard points in a convex quadrilateral [1].

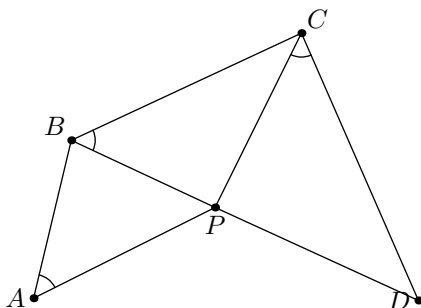


Fig. 1.

In the first volume of JCGeometry [2] A. A. Zaslavsky defines the open problem mentioning that $\phi(ABCD) = \phi(DCBA)$ (namely there are such points P and Q , that $\angle PAB = \angle PBC = \angle PCD = \angle QBA = \angle QCB = \angle QDC = \phi$, moreover $OP = OQ$ and $\angle POQ = 2\phi$) iff $ABCD$ is cyclic, where O is the circumcenter of $ABCD$.

Synthetic proof of these conditions is provided below.

Proof. 1) We have to prove that if $\phi(ABCD) = \phi(DCBA)$, then $ABCD$ is cyclic. Let P and Q be $Br(ABCD)$ and $Br(DCBA)$ respectively. Let us denote angles $\phi(ABCD)$ and $\phi(DCBA)$ by ϕ . We also denote $E = AP \cap BQ$, $G = BP \cap CQ$, and $F = CP \cap DQ$. Using $\angle PAB = \angle PBC = \angle PCD$ and $\angle QBA = \angle QCB = \angle QDC$ we obtain $\angle QEP = \angle AEB = \pi - 2\phi = \angle BGC = \angle GCP$. From this it follows that quadriateral $QEGP$ is cyclic. Denote its circumcircle by ω . Similarly we can prove that F also belongs to ω .

It remains to prove that the intersection of perpendicular bisectors of sides AB , BC , CD lies on ω . Let us consider the perpendicular bisector of AB . It intersects ω in point K , besides point E lies on it because $\angle EAB = \phi = \angle EBA$.

Then we have $\angle QEK = \frac{\pi}{2} - \phi = \angle PEK$. Therefore point K is the middle of arc PQ and all perpendicular bisectors pass through the middle of arc PQ of ω .

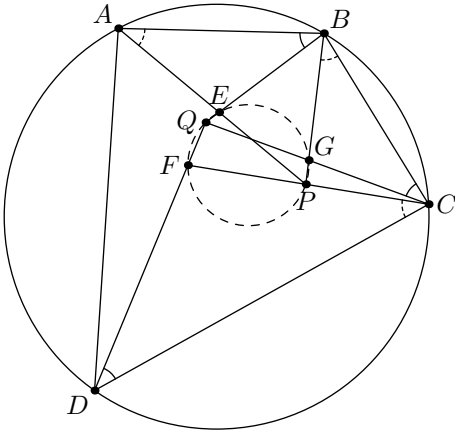


Fig. 2.

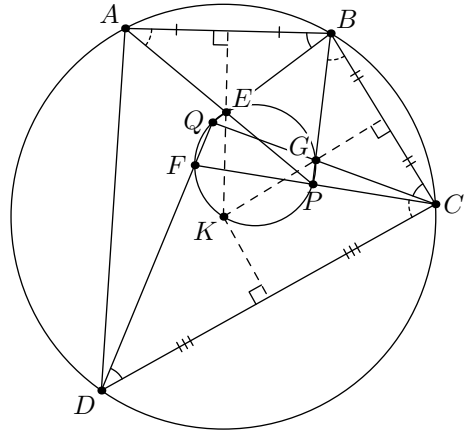


Fig. 3.

2) Given that quadriateral $ABCD$ is cyclic, denote point $Br(ABCD)$ by point P and angle $\phi(ABCD)$ by ϕ . There exists point Q that $\angle QBA = \angle QCB = \phi$ and point D' on the line CD that $\angle QD'C = \phi$. Then from the point 1) we get that quadriateral $ABCD'$ is inscribed. This implies that points D and D' are the same and $\angle QDC = \phi$. \square

Remark. From this it also follows that point K is circumcenter of $ABCD$ and $KP = KQ$, $\angle PKQ = 2\phi$.

It turns out that the above statement may be also used with Brocard polygons.

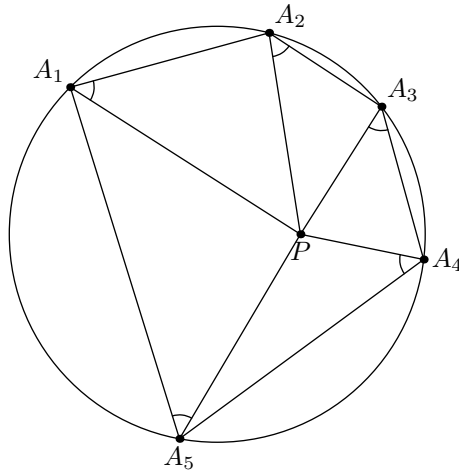


Fig. 4.

Recall that the polygon $A_1A_2 \dots A_n$ is called Brocard ones if it is cyclic and there exists a unique point P such that $\angle PA_1A_2 = \angle PA_2A_3 = \dots = \angle PA_nA_1 = \phi$.

Let us prove that a polygon $A_1A_2 \dots A_n$ is the Brocard one if and only if there exists unique point P such that $\angle PA_1A_2 = \angle PA_2A_3 = \dots = \angle PA_nA_1 = \phi$ and a unique point Q such that $\angle QA_2A_1 = \angle QA_3A_2 = \dots = \angle QA_1A_n = \phi$.

Notice that we give generalization not for every inscribed polygon but only for Brocard ones.

Proof. 1) Assume that the polygon $A_1A_2 \dots A_n$ is Brocard. Since quadriateral $A_1A_2A_3A_4$ is cyclic, it follows that there exists unique point Q_1 such that $\angle Q_1A_2A_1 = \angle Q_1A_3A_2 = \angle Q_1A_4A_3 = \phi$ and $OP = OQ_1, \angle POQ_1 = 2\phi$. Since quadriateral $A_2A_3A_4A_5$ is cyclic, it follows that there exists point Q_2 such that $\angle Q_2A_3A_2 = \angle Q_2A_4A_3 = \angle Q_2A_5A_4 = \phi$ and $OP = OQ_2, \angle POQ_2 = 2\phi$. Obviously points Q_1 and Q_2 are the same. Therefore we obtain point Q such that $\angle QA_2A_1 = \angle QA_3A_2 = \dots = \angle QA_1A_n = \phi$.

2) Given polygon $A_1A_2 \dots A_n$ and points P and Q such that $\angle PA_1A_2 = \angle PA_2A_3 = \dots = \angle PA_nA_1 = \phi = \angle QA_2A_1 = \angle QA_3A_2 = \dots = \angle QA_1A_n$. Clearly that quadriateral $A_iA_{i+1}A_{i+2}A_{i+3}$ is cyclic for all i 's. It follows that polygon $A_1A_2 \dots A_n$ is cyclic. \square

REFERENCES

- [1] D. Belev. Some properties of the brocard points of a cyclic quadrilateral. *Journal of classical geometry*:1–10, 2, 2013.
- [2] A. A. Zaslavsky. Brocard's points in quadrilateral. *Journal of classical geometry*:72, 1, 2012.

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