ON BROCARD'S POINTS IN POLYGONS

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Abstract. In this note we present a synthetic proof of the key lemma, defines in the problem of A. A. Zaslavsky.

For any given convex quadriaterial $ABCD$ there exists a unique point P such that $\angle PAB = \angle PBC = \angle PCD$. Let us call this point the *Brocard point (Br(ABCD))*, and respective angle — *Brocard angle* ($\phi(ABCD)$) of the broken line $ABCD$. You can read the proof of this fact in the beginning of the article by Dimitar Belev about the Brocard points in a convex quadrilateral [1].

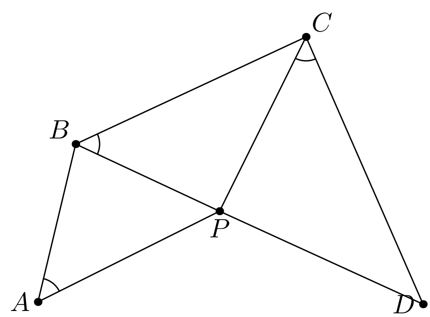

Fig. 1.

In the first volume of JCGeometry [2] A. A. Zaslavsky defines the open problem mentioning that $\phi(ABCD) = \phi(DCBA)$ (namely there are such points P and Q, that $\angle PAB = \angle PBC = \angle PCD = \angle QBA = \angle QCB = \angle QDC = \phi$, moreover $OP = OQ$ and $\angle POQ = 2\phi$) iff $ABCD$ is cyclic, where O is the circumcenter of $ABCD$.

Synthetic proof of these conditions is provided below.

Proof. 1) We have to prove that if $\phi(ABCD) = \phi(DCBA)$, then $ABCD$ is cyclic. Let P and Q be $Br(ABCD)$ and $Br(DCBA)$ respectively. Let us denote angles $\phi(ABCD)$ and $\phi(DCBA)$ by ϕ. We also denote $E = AP \cap BQ$, $G = BP \cap CQ$, and $F = CP \cap DQ$. Using $\angle PAB = \angle PBC = \angle PCD$ and $\angle QBA = \angle QCB = \angle QDC$ we obtain $\angle QEP = \angle AEB = \pi - 2\phi = \angle BGC = \angle GCP$. From this it follows that quadriaterial $QEGP$ is cyclic. Denote its circumcircle by ω. Similarly we can prove that F also belongs to ω.

It remains to prove that the intersection of perpendicular bisectors of sides AB, BC, CD lies on ω. Let us consider the perpendicular bisector of AB. It intersects ω in point K, besides point E lies on it because $\angle EAB = \phi = \angle EBA$.

Then we have $\angle QEK = \frac{\pi}{2} - \phi = \angle PEK$. Therefore point K is the middle of arc PQ and all perpendicular bisectors pass through the middle of arc PQ of ω.

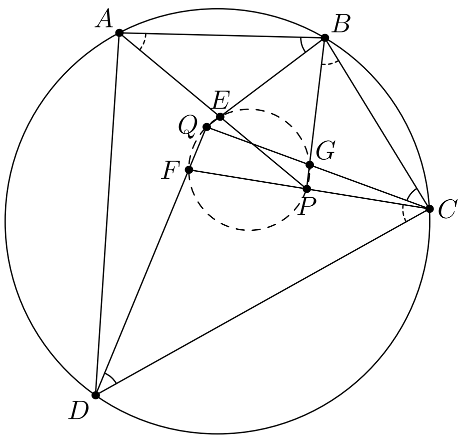

Fig. 2.

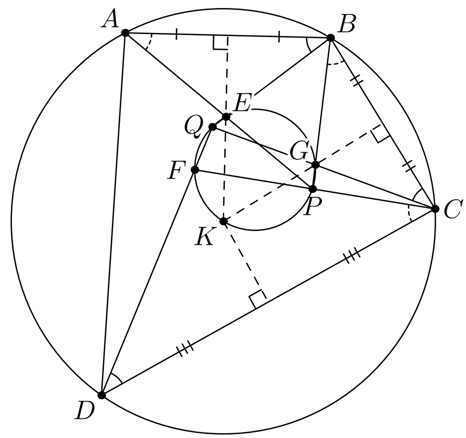

Fig. 3.

2) Given that quariateral $ABCD$ is cyclic, denote point $Br(ABCD)$ by point P and angle $\phi(ABCD)$ by ϕ. There exists point Q that $\angle QBA = \angle QCB = \phi$ and point D' on the line CD that $\angle QD'C = \phi$. Then from the poit 1) we get that quadriaterial $ABCD'$ is inscribed. This implies that points D and D' are the same and $\angle QDC = \phi$. □

Remark. From this it also follows that point K is circumcenter of $ABCD$ and $KP = KQ$, $\angle PKQ = 2\phi$.

It turns out that the above statement may be also used with Brocard polygons.

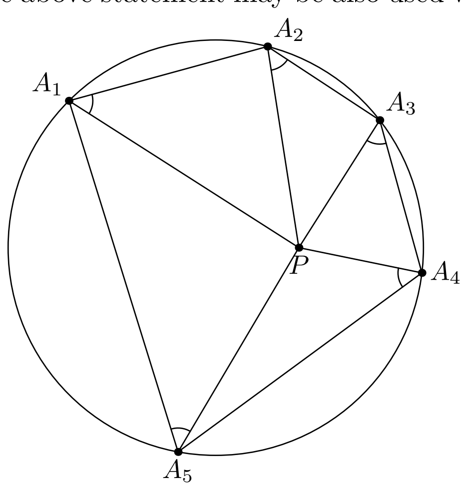

Fig. 4.

Recall that the polygon $A_1A_2 \dots A_n$ is called Brocard ones if it is cyclic and there exists a unique point P such that $\angle PA_1A_2 = \angle PA_2A_3 = \dots = \angle PA_nA_1 = \phi$.

Let us prove that a polygon $A_1A_2\dots A_n$ is the Brocard one if and only if there exists unique point P such that $\angle PA_1A_2 = \angle PA_2A_3 = \dots = \angle PA_nA_1 = \phi$ and a unique point Q such that $\angle QA_2A_1 = \angle QA_3A_2 = \dots = \angle PA_1A_n = \phi$.

Notice that we give generalization not for every inscribed polygon but only for Brocard ones.

Proof. **1)** Assume that the polygon $A_1A_2\dots A_n$ is Brocard. Since quadriaterial $A_1A_2A_3A_4$ is cyclic, it follows that there exists unique point Q_1 such that $\angle Q_1A_2A_1 = \angle Q_1A_3A_2 = \angle Q_1A_4A_3 = \phi$ and $OP = OQ_1, \angle POQ_1 = 2\phi$. Since quadriaterial $A_2A_3A_4A_5$ is cyclic, it follows that there exists point Q_2 such that $\angle Q_2A_3A_2 = \angle Q_2A_4A_3 = \angle Q_2A_5A_4 = \phi$ and $OP = OQ_2, \angle POQ_2 = 2\phi$. Obviously points Q_1 and Q_2 are the same. Therefore we obtain point Q such that $\angle QA_2A_1 = \angle QA_3A_2 = \dots = \angle QA_1A_n = \phi$.

2) Given polygon $A_1A_2\dots A_n$ and points P and Q such that $\angle PA_1A_2 = \angle PA_2A_3 = \dots = \angle PA_nA_1 = \phi = \angle QA_2A_1 = \angle QA_3A_2 = \dots = \angle QA_1A_n$. Clearly that quadriaterial $A_iA_{i+1}A_{i+2}A_{i+3}$ is cyclic for all I's. It follows that polygon $A_1A_2\dots A_n$ is cyclic. □

References

[1] D. Belev. Some properties of the brocard points of a cyclic quadrilateral. *Journal of classical geometry*:1–10, 2, 2013.

[2] A. A. Zaslavsky. Brocard's points in quadrilateral. *Journal of classical geometry*:72, 1, 2012.

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