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ON BROCARD'S POINTS IN POLYGONS

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Abstract. In this note we present a synthetic proof of the key lemma, defines in the problem of A. A. Zaslavsky.

For any given convex quadriaterial $ABCD$ there exists a unique point P such that $\angle PAB = \angle PBC = \angle PCD$. Let us call this point the *Brocard point (Br(ABCD))*, and respective angle — *Brocard angle* ($\phi(ABCD)$) of the broken line $ABCD$. You can read the proof of this fact in the beginning of the article by Dimitar Belev about the Brocard points in a convex quadrilateral [1].

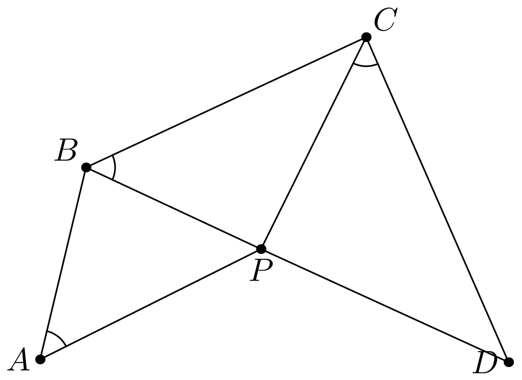

Fig. 1.

In the first volume of JCGeometry [2] A. A. Zaslavsky defines the open problem mentioning that $\phi(ABCD) = \phi(DCBA)$ (namely there are such points P and Q, that $\angle PAB = \angle PBC = \angle PCD = \angle QBA = \angle QCB = \angle QDC = \phi$, moreover $OP = OQ$ and $\angle POQ = 2\phi$) iff $ABCD$ is cyclic, where O is the circumcenter of $ABCD$.

Synthetic proof of these conditions is provided below.

Proof. 1) We have to prove that if $\phi(ABCD) = \phi(DCBA)$, then $ABCD$ is cyclic. Let P and Q be $Br(ABCD)$ and $Br(DCBA)$ respectively. Let us denote angles $\phi(ABCD)$ and $\phi(DCBA)$ by ϕ. We also denote $E = AP \cap BQ$, $G = BP \cap CQ$, and $F = CP \cap DQ$. Using $\angle PAB = \angle PBC = \angle PCD$ and $\angle QBA = \angle QCB = \angle QDC$ we obtain $\angle QEP = \angle AEB = \pi - 2\phi = \angle BGC = \angle GCP$. From this it follows that quadriaterial $QEGP$ is cyclic. Denote its circumcircle by ω. Similarly we can prove that F also belongs to ω.

It remains to prove that the intersection of perpendicular bisectors of sides AB, BC, CD lies on ω. Let us consider the perpendicular bisector of AB. It intersects ω in point K, besides point E lies on it because $\angle EAB = \phi = \angle EBA$.

Then we have $\angle QEK = \frac{\pi}{2} - \phi = \angle PEK$. Therefore point K is the middle of arc PQ and all perpendicular bisectors pass through the middle of arc PQ of ω.

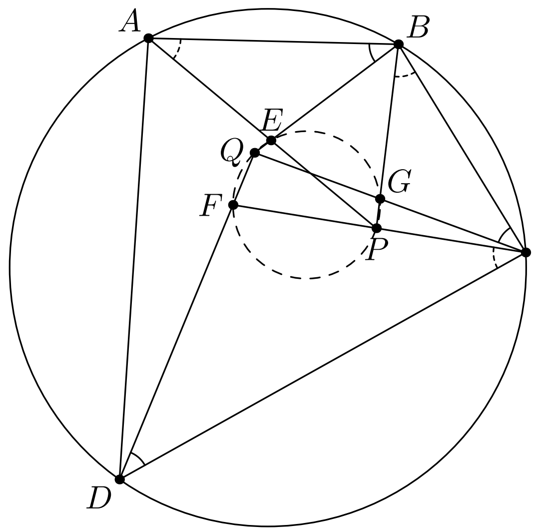

Fig. 2.

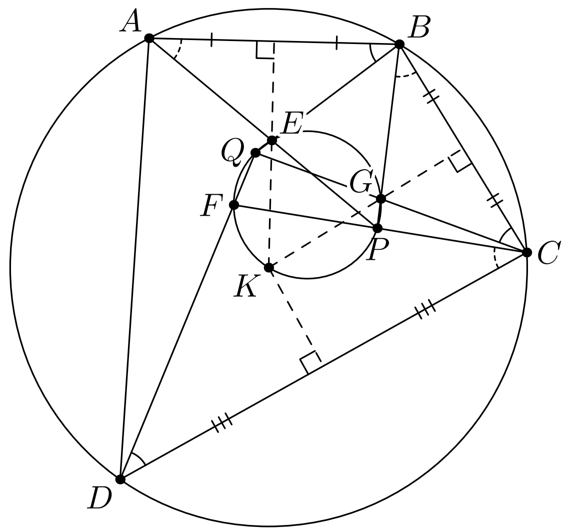

Fig. 3.

2) Given that quariateral $ABCD$ is cyclic, denote point $Br(ABCD)$ by point P and angle $\phi(ABCD)$ by ϕ. There exists point Q that $\angle QBA = \angle QCB = \phi$ and point D' on the line CD that $\angle QD'C = \phi$. Then from the poit 1) we get that quadriaterial $ABCD'$ is inscribed. This implies that points D and D' are the same and $\angle QDC = \phi$. $\square$

Remark. From this it also follows that point K is circumcenter of $ABCD$ and $KP = KQ$, $\angle PKQ = 2\phi$.

It turns out that the above statement may be also used with Brocard polygons.

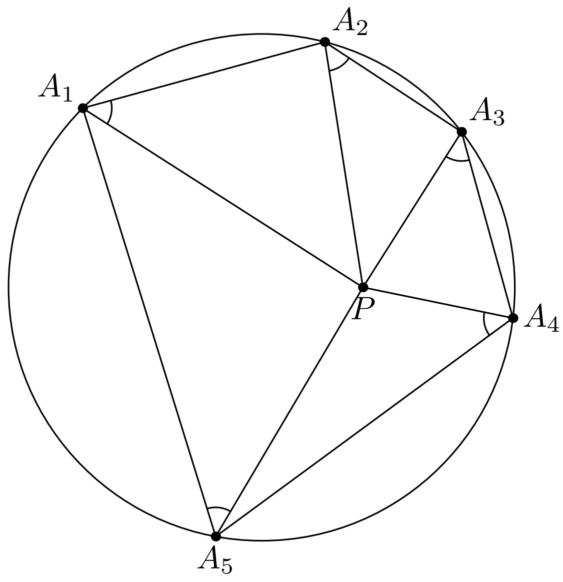

Fig. 4.

Recall that the polygon $A_1A_2 \ldots A_n$ is called Brocard ones if it is cyclic and there exists a unique point P such that $\angle PA_1A_2 = \angle PA_2A_3 = \ldots = \angle PA_nA_1 = \phi$.

Let us prove that a polygon $A_1A_2\dots A_n$ is the Brocard one if and only if there exists unique point P such that $\angle PA_1A_2 = \angle PA_2A_3 = \dots = \angle PA_nA_1 = \phi$ and a unique point Q such that $\angle QA_2A_1 = \angle QA_3A_2 = \dots = \angle PA_1A_n = \phi$.

Notice that we give generalization not for every inscribed polygon but only for Brocard ones.

Proof. **1)** Assume that the polygon $A_1A_2\dots A_n$ is Brocard. Since quadriaterial $A_1A_2A_3A_4$ is cyclic, it follows that there exists unique point Q_1 such that $\angle Q_1A_2A_1 = \angle Q_1A_3A_2 = \angle Q_1A_4A_3 = \phi$ and $OP = OQ_1, \angle POQ_1 = 2\phi$. Since quadriaterial $A_2A_3A_4A_5$ is cyclic, it follows that there exists point Q_2 such that $\angle Q_2A_3A_2 = \angle Q_2A_4A_3 = \angle Q_2A_5A_4 = \phi$ and $OP = OQ_2, \angle POQ_2 = 2\phi$. Obviously points Q_1 and Q_2 are the same. Therefore we obtain point Q such that $\angle QA_2A_1 = \angle QA_3A_2 = \dots = \angle QA_1A_n = \phi$.

2) Given polygon $A_1A_2\dots A_n$ and points P and Q such that $\angle PA_1A_2 = \angle PA_2A_3 = \dots = \angle PA_nA_1 = \phi = \angle QA_2A_1 = \angle QA_3A_2 = \dots = \angle QA_1A_n$. Clearly that quadriaterial $A_iA_{i+1}A_{i+2}A_{i+3}$ is cyclic for all I's. It follows that polygon $A_1A_2\dots A_n$ is cyclic. □

References

[1] D. Belev. Some properties of the brocard points of a cyclic quadrilateral. *Journal of classical geometry*:1–10, 2, 2013.

[2] A. A. Zaslavsky. Brocard's points in quadrilateral. *Journal of classical geometry*:72, 1, 2012.

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ON SOME PROPERTIES OF CONFOCAL CONICS

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Abstract. We prove two theorems concerning confocal conics. The first one is related to bodies invisible from one point. In particular, this theorem is a generalization of Galperin–Plakhov's theorem. The second one is related to billiards bounded by confocal conics and is used to construct bodies invisible from two points. All the proofs are synthetic.

1. Introduction

The results reported here come from the study of invisibility generated by mirror reflections (see [3, 5, 6, 7, 8]). The construction of a body invisible from a fixed point [5, 6] is based on the following geometric statement concerning confocal conics.

The Galperin–Plakhov Theorem (See [4])**.** *Consider two different points F_1 and F_2 in the plane and take an ellipse and a hyperbola with foci at F_1 and F_2. We consider only the branch of the hyperbola associated with F_2 (we shall call it the right branch). Let P and Q be the points of intersection of the ellipse with the right branch of the hyperbola. Consider a ray starting at F_1 and intersecting the right branch of the hyperbola. Denote by X, A the intersection points of this ray with the ellipse and with the branch of the hyperbola. Suppose the focus F_2 lies on the line PQ. Then PQ is the bisector of the angle $\angle AF_2X$ (see Fig. 1).*

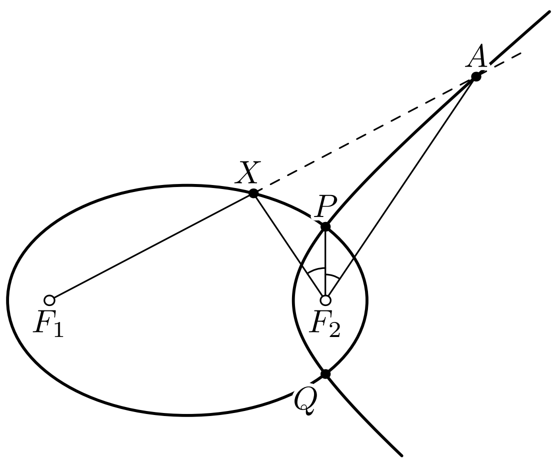

Fig. 1.

In Section 2 we formulate Theorem 1 which is a generalization of the Galperin-Plakhov theorem. In turn, the construction of a body invisible from two points leads to another statements referred to here as Theorem 2. Also Theorem 2 is related to billiards associated with confocal conics (see [9, Chapter 4]).

The paper is organized as follows. In Section 2 we formulate Theorems 1 and 2. In Section 3 we prove and generalize Theorem 1. In Section 4 we prove and generalize Theorem 2.

2. Main results

Theorem 1. *Let confocal ellipse and hyperbola are given. Consider an arbitrary point on the line passing through the intersection points of the ellipse and the right branch of the hyperbola. Draw two tangent lines to the ellipse and to the hyperbola. Then the line through the tangent points passes through a focus (see Fig. 2).*

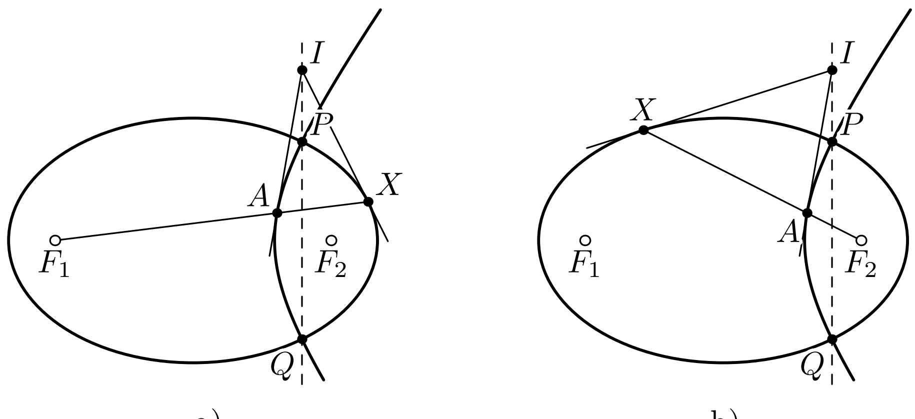

Fig. 2.

In Section 3 we prove Theorem 1 and show how the Galperin–Plakhov Theorem follows from Theorem 1. The following corollary is a limiting case of Theorem 1 in which the focus F_2 converges to infinity.

Corollary. *Let two intersecting parabolas with common focus and axis of symmetry are given. Coider an arbitrary point on the line passing through the intersection points of the parabolas. Draw two tangent lines to the parabolas. Then the line through the tangent points either passes through the focus or parallel to the axis of symmetry (see Fig. 3).*

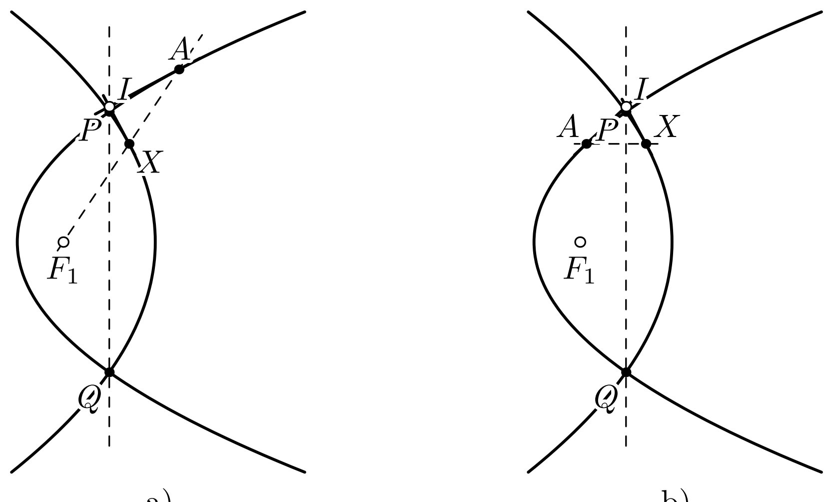

Fig. 3.

The point a) of the following theorem was formulated by A. Yu. Plakhov.

Theorem 2. *Let two confocal ellipses $\mathcal{E}_1$ and $\mathcal{E}_2$ with foci F_1 and F_2 are given. Let a ray with the origin at F_1 intersects $\mathcal{E}_1$ and $\mathcal{E}_2$ at A and B, respectively. Let a ray with the origin at F_2 intersects $\mathcal{E}_1$ and $\mathcal{E}_2$ at C and D, respectively. Suppose the points B and C lie on a branch $\mathcal{H}_1$ of the hyperbola with the foci at F_1 and F_2. Then a) the points A and D lie on a branch $\mathcal{H}_2$ of the hyperbola with the foci at F_1 and F_2 (see Fig. 4)*

b) Consider a ray starting at F_1 intersecting the branch $\mathcal{H}_1$ at P_1. Consider the ray F_2P_1 intersecting the ellipse $\mathcal{E}_2$ at P_2. Analogously, we define the points P_3, P_4, P_5. Then $P_5 = P_1$ (see Fig. 5).

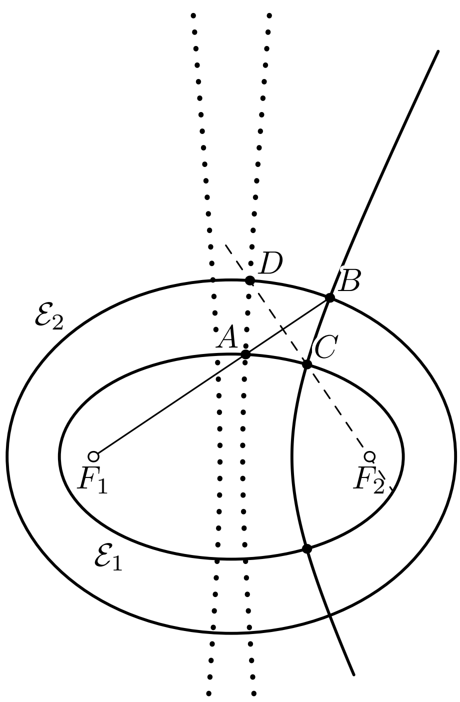

Fig. 4.

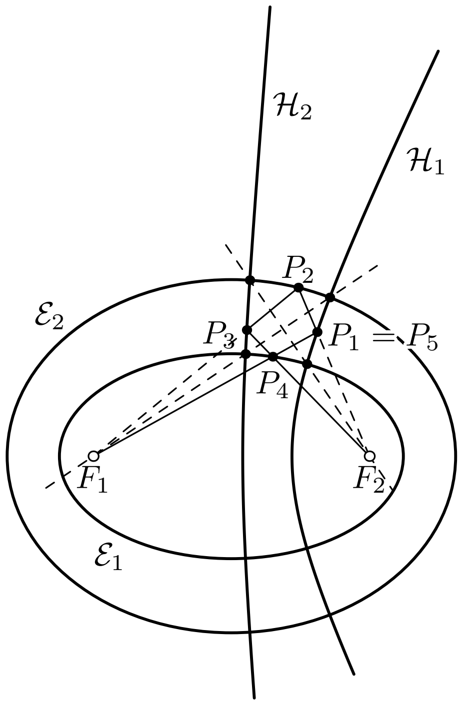

Fig. 5.

In Section 4 we prove Theorem 2 and formulate a generalization of Theorem 2 (b) for an arbitrary number of confocal conics.

3. Proof of Theorem 1

Proof. Denote by P and Q the intersection points the given ellipse $\mathcal{E}$ and the right branch of the hyperbola $\mathcal{H}$. Denote by I the given point on the line PQ and let points X and A be the given tangent points (see Fig. 2). Consider the polar transformation with respect to a circle with the center at F_1. It is well-known that the polar image of each conic with the focus at F_1 is a circle; see [2, the proof of Theorem 3.5]. So the images of $\mathcal{E}$ and $\mathcal{H}$ are the circles $\mathcal{E}'$ and $\mathcal{H}'$, respectively (see Fig. 6).

Note that the focus F_1 is mapped to the line of infinity; the focus F_2 is mapped to the radical axis of the circles $\mathcal{E}'$ and $\mathcal{H}'$; the line PQ is mapped to the intersection point of the common tangent lines to the circles $\mathcal{E}'$ and $\mathcal{H}'$. It easy to see that this intersection point is the internal homothetic center of the circles $\mathcal{E}'$ and $\mathcal{H}'$. Denote by l the image of the point I. Obviously, the line l is passing through the internal homothetic center of the circles $\mathcal{E}'$ and $\mathcal{H}'$; the points X and A are mapped to tangent lines x and a of the circles. Denote by M the intersection point of the lines x and a. It is evident that the point M lies either on the radical axis or on the line of infinity. So the line AX passes through a focus. □

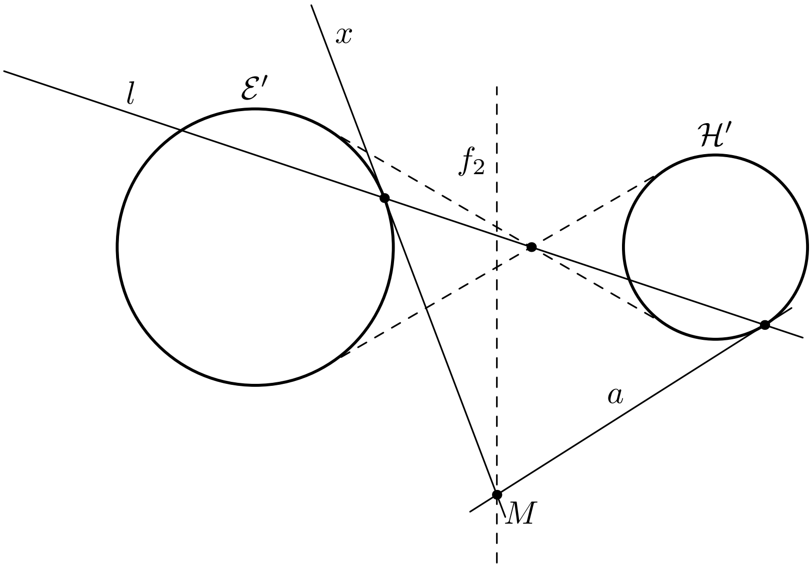

Fig. 6.

Let us show that Theorem 1 is a generalization of the Galperin–Plakhov theorem. We use notations from the proof of Theorem 1. Denote by Y the point of intersection of the ray F_2A with the ellipse $\mathcal{E}$. Denote by B the point of intersection of F_1Y with F_2X (see Fig. 7).

In the sequel we use the following well-known lemma (see [1, problem 11.10]).

Lemma 1. *The quadrangle $AYBX$ is circumscribed and the point B lies on the right branch of the hyperbola $\mathcal{H}$.*

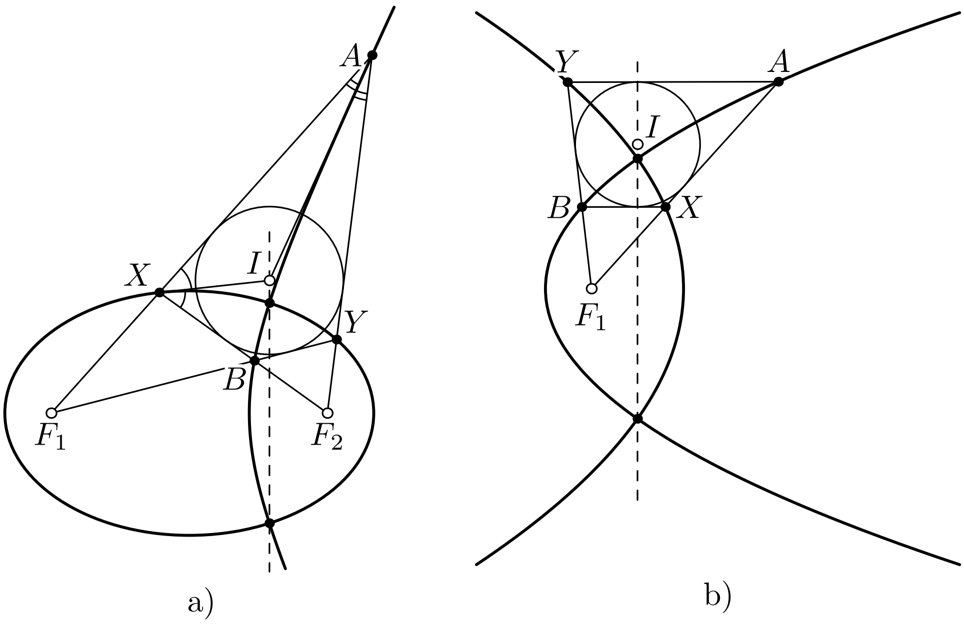

Fig. 7.

From the *optical properties of conics* the point I is the center of the circle inscribed in $AYBX$. Suppose the line PQ passes through the focus F_2. Then PQ is the bissector of the angle AF_2X. So the Galperin-Plakhov theorem is a corollary of Theorem 1.

Let us formulate another related result. Denote by K and L the points of intersection the line XY with F_1F_2 and the line AB with F_1F_2, respectively (see Fig. 8).

Proposition 1. *The lines XY and AB are altitudes in the triangle $\triangle IKL$.*

Proof. Note that the polar line of K with respect to the ellipse $\mathcal{E}$ is the line PQ. It is well-known that AB is a polar line of K with respect to the circle inscribed in $AYBX$. So IK is orthogonal to AB. Analogously IL is orthogonal to XY. □

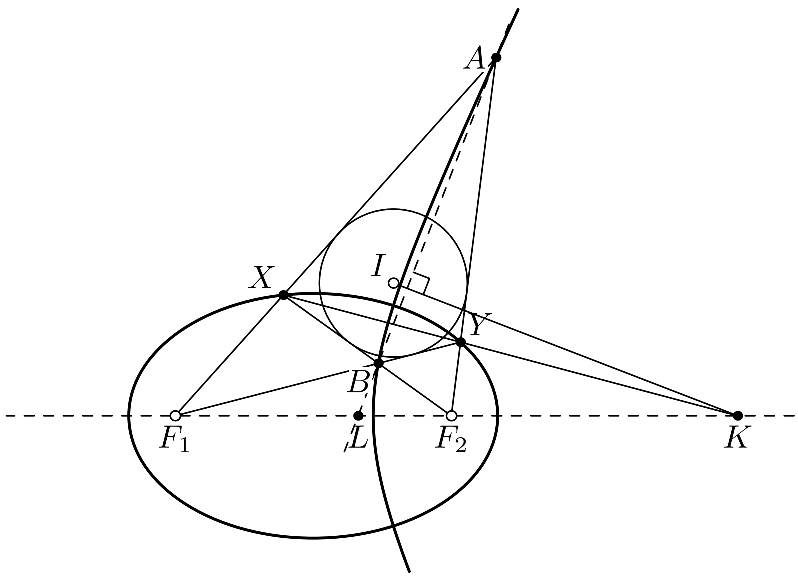

Fig. 8.

Remark. We want to emphasize that two polar lines of the point I with respect to $\mathcal{E}$ and $\mathcal{H}$ intersect on the line PQ, because PQ, XY and AB are altitudes in $\triangle IKL$.

Let us give you Projective generalization of Theorem 1. It is well-known that confocal conics (with foci F_1 and F_2) form a dual pencil of conics (see [2]). Actually, each confocal conic is tangent to 2 pairs of fixed conjugate imaginary lines. The lines of each pair pass through the corresponding focus (F_1 or F_2).

Consider quadrilateral $ABCD$ with opposite sides intersecting at the points F_1, F_2. Consider the dual pencil of consics inscribed in quadrilateral $ABCD$. Consider any two conics from the pencil. Consider a ray from F_1 intersecting both conics and denote by X and Y the intersection points of the ray with conics. Let I be the intersection of tangent lines at X and Y to the conics. Apply a projective trnsformation to the construction of Theorem 1. Then we get the following proposition.

Proposition 2. *a) There exists a straight line passing through I and two intersection points of the conics (see Fig. 9).*

b)the polar lines of I with respect to the conics intersect at the straight line.

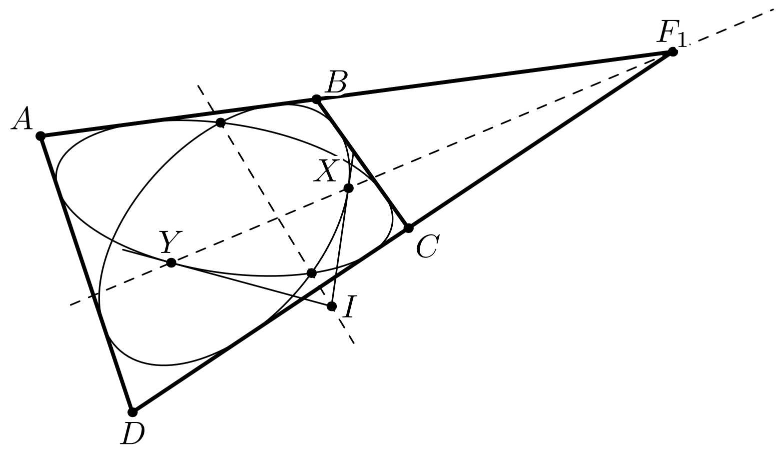

Fig. 9.

4. Proof of Theorem 2

Proof. a) Denote by X the intersection point of F_1D and F_2B; denote by Y the intersection point of F_1B and F_2D; denote by Z the intersection point of F_1C and F_2A; denote by M the intersection point of F_1X and F_2A and denote by N the intersection point of F_1C and F_2X (see Fig. 10).

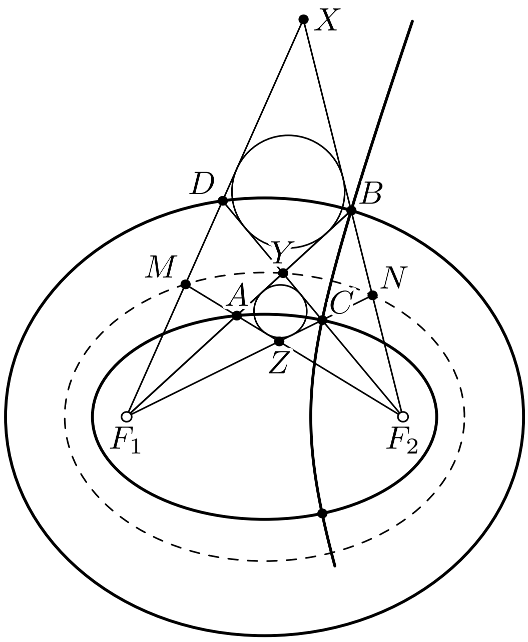

Fig. 10.

The points B and C lie on the same branch $\mathcal{H}_1$. So from Lemma 1 it follows that quadrilateral $CYBN$ is circumscribed and there exists the ellipse $\mathcal{E}$ with foci at F_1 and F_2 passing through the points Y and N. Analogously $YAZC$, $XDYB$ are circumscribed. Hence, the points Y and Z lie on the same branch of a hyperbola with foci F_1 and F_2 and the points X and Y lie on the same branch of a hyperbola with foci F_1 and F_2. So there exists a branch of a hyperbola $\mathcal{H}$ with foci at F_1 and F_2 passing through the points X, Y and Z. So the quadrilateral $XMZN$ is circumscribed. In view of the aforesaid, points M, Y, N lie on the ellipse $\mathcal{E}$. Thus the points A and D lie on the same branch of a hyperbola with foci at F_1, F_2.

b) denote by P_4' the intersecrion point of the ray F_1P_1 with the ellipse $\mathcal{E}_1$. We need to prove that $P_4 = P_4'$. So it is sufficiant to prove that the points P_3, P_4', and F_2 are collinear.

Denote by ϕ_4 and ϕ_2 the hyperbolas with foci at F_1 and F_2 passing through P_4' and P_2, respectivelly. Denote by ϕ_1 and ϕ_3 the ellipses with foci at F_1 and F_2 passing through P_1 and P_3, respectivelly. From Theorem 2 (a) it follows that the intersection point of ϕ_4 with ϕ_1 lies on the straigt line F_2D. Analogously, the intersection point of ϕ_1 with ϕ_2 lies on the line F_1B and the intersection point ϕ_2 with ϕ_3 lies on the line F_2D. From Theorem 2 (a) for conics ϕ_1, ϕ_2, ϕ_3, ϕ_4 it follows that the intersection point of ϕ_3 with ϕ_4 lies on the line F_1B. Finally, from Theorem 2 (a) for conics $\mathcal{E}_1$, ϕ_3, ϕ_4, $\mathcal{H}_2$ it follows that points P_3, P_4', F_2 are collinear. □

Remark. Let $F_1P_4+P_4F_2 = a_1, F_1P_2+P_2F_2 = a_2, F_1P_1-P_1F_2 = b_1, F_1P_3-P_3F_2 = b_2$. Notice that $a_2 + b_1 - b_2 - a_1 = P_4P_1 + P_1P_2 + P_2P_3 + P_3P_4$. So the perimeter of the quadrangle $P_1P_2P_3P_4$ is fixed.

Now we formulate a generalization of Theorem 2 (b) for an arbitrary number of confocal conics. Consider two different points F_1 and F_2 in the plane and take $n = 2k+1$ conics $\varphi_1, \varphi_2, \ldots, \varphi_n$ with foci at F_1 and F_2. Take a point P_1 on φ_1 and consider the ray F_1P_1. The ray reflects from the conic φ_1 at the point P_1; then the line containing the reflected ray passes through F_2. Denote by P_2 the intercection point of the reflected ray and φ_2. Analogously we define the points $P_3, ..., P_n$. The last ray P_nF_2 passes through point F_2. Denote by P_0 the intersection point of the line F_1P_1 with the line P_nF_2

Proposition 3. *The point P_0 lies on the fixed conic with foci at F_1 and F_2.*

Proof. The proof for the basis of induction $n = 3$ is analogous to the proof of Theorem 2 (b). Suppose that the statement of the Proposition is true for $n = 2k - 1$. Let $n = 2k + 1$. By the basis of induction for every 3 consecutive conics $\varphi_{i-1}, \varphi_i, \varphi_{i+1}$ there exists the conic ϕ_i such that the construction associated with the conics $\varphi_1, \ldots, \varphi_{i-1}, \varphi_i, \varphi_{i+1}, \ldots, \varphi_n$ may be reduced to the construction associated with the conics $\varphi_1, \ldots, \phi_i, \ldots, \varphi_n$. So, inductive step is proved. □

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References

[1] A. V. Akopyan. *Geometry in figures.* Createspace, 2011.

[2] A. V. Akopyan and A. A. Zaslavsky. *Geometry of conics.* Vol. 26 of *Mathematical World.* American Mathematical Society, Providence, RI, 2007. ISBN: 978-0-8218-4323-9.

[3] A. Aleksenko and A. Plakhov. Bodies of zero resistance and bodies invisible in one direction. *Nonlinearity*, 22(6):1247–1258, Apr. 2009.

[4] G. A. Gal'perin and A. Y. Plakhov. A geometric problem that leads to the billiard law of reflection. *Matematicheskoe prosveshenie. Tret'ya seriya,* 15:176–181, 2011.

[5] A. Plakhov. *Exterior Billiards: Systems with Impacts Outside Bounded Domains.* Springer, 2012.

[6] A. Plakhov and V. Roshchina. Bodies invisible from one point. *Arxiv preprint arxiv:1112.6167*, 2011.

[7] A. Plakhov and V. Roshchina. Bodies with mirror surface invisible from two points. *Nonlinearity*, 27(6):1193–1203, May 2014.
[8] A. Plakhov and V. Roshchina. Invisibility in billiards. *Nonlinearity*, 24(3):847–854, Feb. 2011.
[9] S. Tabachnikov. *Geometry and billiards*. Vol. 30. American Mathematical Soc., 2005.

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ELLIPSE GENERATION RELATED TO ORTHOPOLES

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Abstract. In this article we study the generation of an ellipse related to two intersecting circles. The resulting configuration has strong ties to triangle geometry and by means of orthopoles establishes also a relation with cardioids and deltoids.

1. Ellipse generation

Our basic configuration consists of two circles α and β, intersecting at two different points C and D. In this we consider a variable line ε through point C, intersecting the circles, correspondingly, at A and B. Then draw the tangents at these points, intersecting at point E and take the symmetric Z of E w.r. to the line AB (See Figure 1). Next theorem describes the locus γ of point Z as the line ε is turning about the point C. The two tangents η and λ to the circles, correspondingly, β and α, at C, are special positions of ε and play, together with their intersections H and L with the circles, an important role in this context. Another important element is line ν, the orthogonal at D to the common chord CD of the two circles. This line intersects circles α and β, correspondingly, at points M and N.

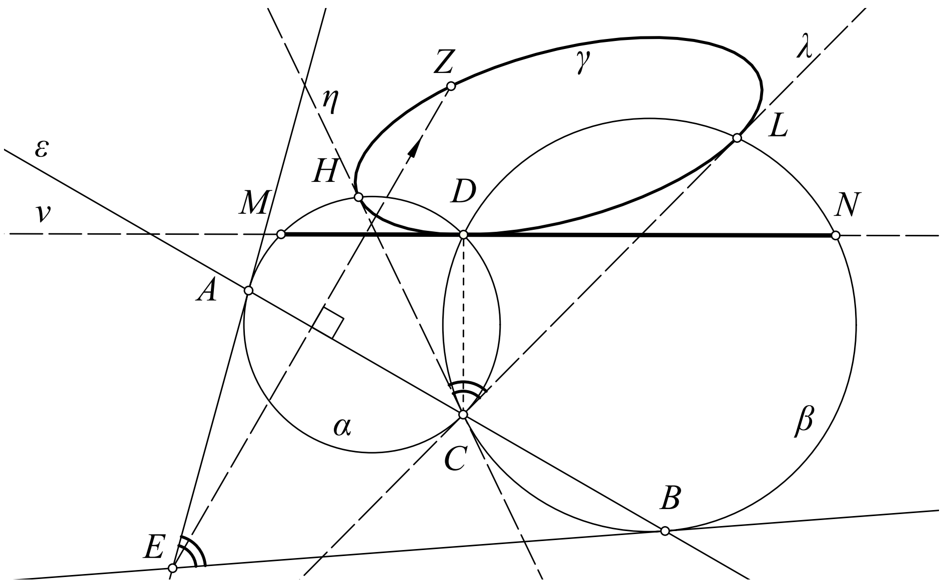

Fig. 1. Ellipse from two intersecting circles

Theorem 1. *As the line AB varies turning about C, the corresponding point Z describes an ellipse γ. This ellipse is also tangent to lines $\{\eta, \lambda, \nu\}$, respectively at the points $\{H, L, D\}$. The ellipse degenerates to the segment MN, when the two circles are orthogonal.*

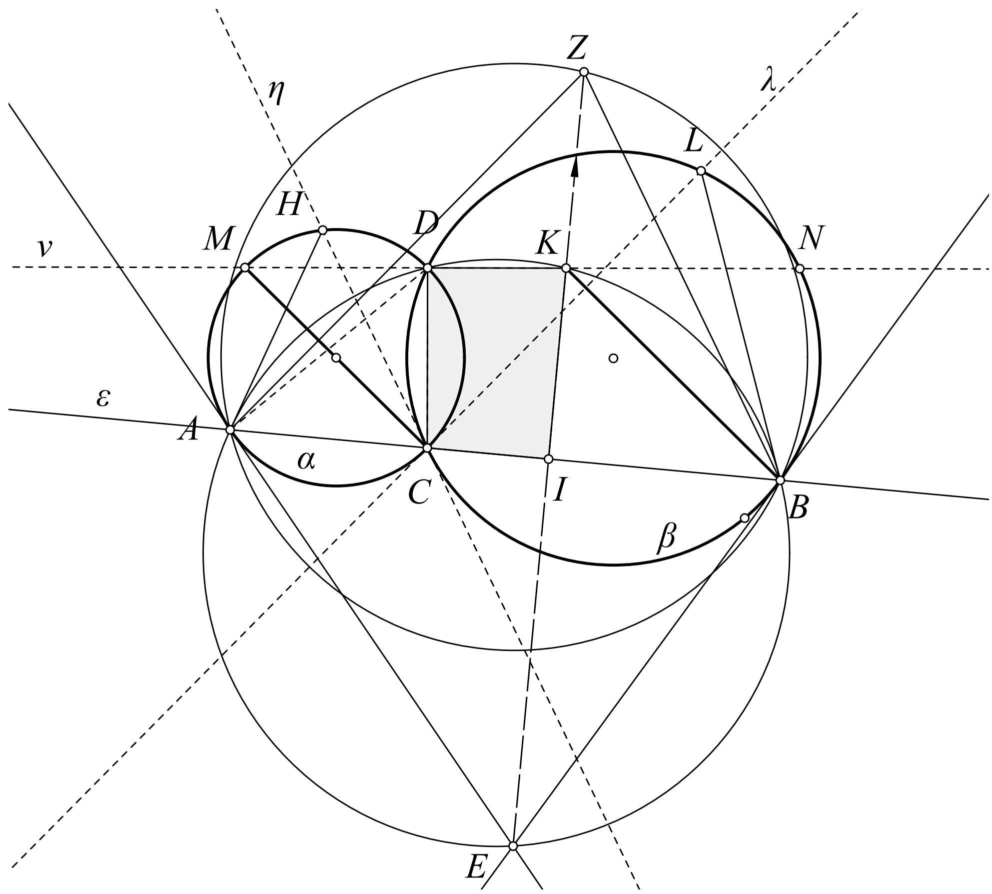

Fig. 2. First properties of the basic configuration

The synthetic proof of the theorem relies on two lemmas, which use the, variable with E, circumcirlce (ABE) of triangle ABE, as well as the intersection points K and I of lines, correspondingly, MN and AB with EZ. (See Figure 2).

Lemma 1. *With the definitions made above, the following are valid properties:*

(1) *The quadrilateral $CDKI$ is a cyclic one and cirlce (ABE) passes through D and K.*
(2) *Lines CM and BK are parallel.*
(3) *Lines AZ and CL are parallel and orthogonal to BK. Analogously, lines CN and AK are parallel, and lines BZ, CH are parallel and orthogonal to the two previous parallels.*
(4) *Lines BL and AH are tangent to the circle (ABE).*
(5) *Triangles HAC and CBL are similar.*

Proof. The first part of statement (1) is obvious, since the opposite lying angles $\widehat{CIK}$ and $\widehat{CDK}$ are right. The second part of (1) follows from the fact that quadrilaterals $ADKE$ and $BKDE$ are cyclic. The reasoning for the two quadrilaterals is the same. For instance, in the case of $ADKE$, this follows from the equality of angles

$$\widehat{MDA} = \frac{\pi}{2} - \widehat{ADC} = \frac{\pi}{2} - \widehat{CAE} = \widehat{AEI}.$$

Statement (2) follows by observing that

$$\widehat{ACM} = \widehat{ADM} = \widehat{ABK},$$

since the quadrilateral $ABKD$ is cyclic.

Statement (3) follows by observing that angle $\widehat{ZAB}$, by symmetry, equals angle $\widehat{BAE}$, which is equal to angle $\widehat{BCL}$, since both angles are formed by the tangents at the extremities of the chord AC of circle α. The orthogonality of CM to CL is obvious, since CM is a diameter of α and CL is tangent at its extremity C.

To prove (4) consider the angle $\widehat{BAE}$, which by symmetry w.r. to AB equals angle $\widehat{BAZ}$, which by the parallelity of AZ to CL equals angle $\widehat{LCB}$. This, by the tangent chord property, for circle (CBL), equals the angle of lines LB and EB, thereby proving the claim.

Property (5) follows from the previous one, showing that the two triangles have, respectively, equal angles at A and B. But also $\widehat{CAH} = \widehat{LCB}$ by the tangent-chord property for the chord AC in circle (HAC). □

In the next lemma, besides the variable circle (ABE) we consider the circle (ABZ) and its other than B intersection point G with circle $\beta = (CBL)$. In addition, point J is the intersection of lines BZ and HD and point Q is the intersection of line CZ with the circle $\alpha = (HAC)$. Finally we consider also the intersection point F of the variable lines AB and ZL, and the analogous intersection point F' of lines AB and ZH. Next lemma shows that F and F' (later not drawn in the figure) move, respectively, on two fixed lines (See Figure 3).

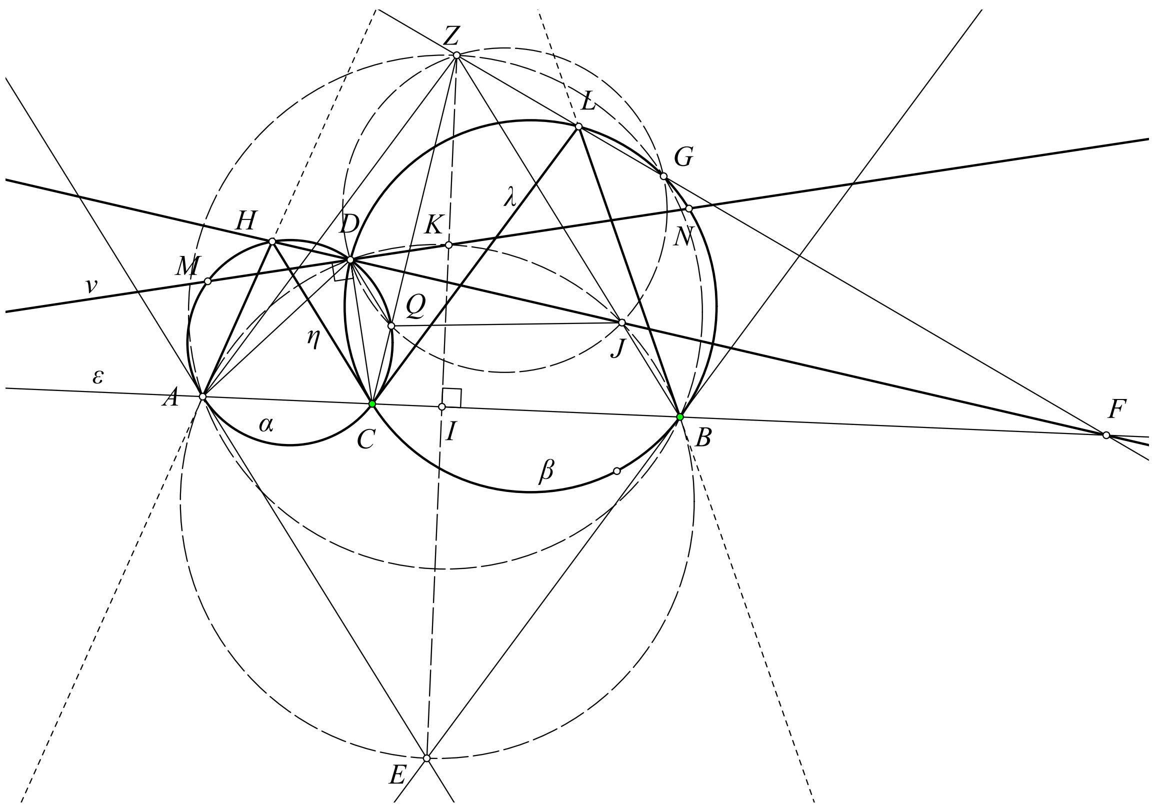

Fig. 3. Coincidences in the basic configuration

Lemma 2. *Under the definitions made above the following are valid properties:*

(1) *Point G lies also on the variable line ZL.*
(2) *The circle (ABE) passes through point J.*
(3) *Circle (ZDJ) passes through Q.*
(4) *Circle (ZDJ) passes through G.*
(5) *Point F moves on the fixed line HD. Analogously point F' moves on line LD.*

Proof. Property (1) is valid, because, by hypothesis, the two quadrangles $ZABG$ and $LCBG$ are cyclic. Hence, by the parallelity of AZ, CL, their angles at G are the same, which proves the stated collinearity.

Property (2) is valid because angle $\widehat{HDA}$ equals angle $\widehat{HCA}$ and by parallelity of HC to BZ, last angle equals $\widehat{JBA}$, hence quadrangle $JBAD$ is cyclic.

Property (3) is valid because angle $\widehat{DJZ}$, by the cyclic $ABJD$, equals angle $\widehat{DAC}$, which, by the cyclic $DACQ$, equals angle $\widehat{DQZ}$.

Property (4) follows from the fact that angle $\widehat{LGD}$ equals angle $\widehat{LBD}$, which equals angle $\widehat{DAB}$. The last equality following from the fact that BL, as well as, AH are tangent to the circle (ABE), by (4) of the previous lemma. By the tangent-chord property angle $\widehat{DAC}$ is equal to angle $\widehat{DCL}$, thereby proving the claim.

Property (5) follows from the previous claims, which show that lines ZL, HD and AB are, correspondingly, the radical axes of the pairs of circles $(ZDG, ZAB), (ZDG, ABE)$ and (ZAB, ABE), hence point F is the radical center of these three circles. The proof for point F' is similar. □

Proof of the theorem. That the curve traced by Z is a conic, follows from the theorem of Maclaurin (see the appendix below). In fact, by lemma 2(5), the variable triangle ZFF' has its vertices F, F', correspondingly, on two fixed lines $\eta = HD$ and $\eta' = LD$. Besides, all of its sides pass through fixed points: FF' passes through C, ZF passes through L and ZF' through H. By Maclaurin's theorem, the free vertex (Z) (See Figure 4) describes a conic. The fact that the conic is a bounded one i.e. an ellipse, results from the property of the point Z, describing the conic, to be the reflection of point E on line AB. Below (theorem 2) it will be seen that point E describes a bounded curve, while AB is turning about the fixed point C. Hence the reflection Z of E on AB is also bounded.

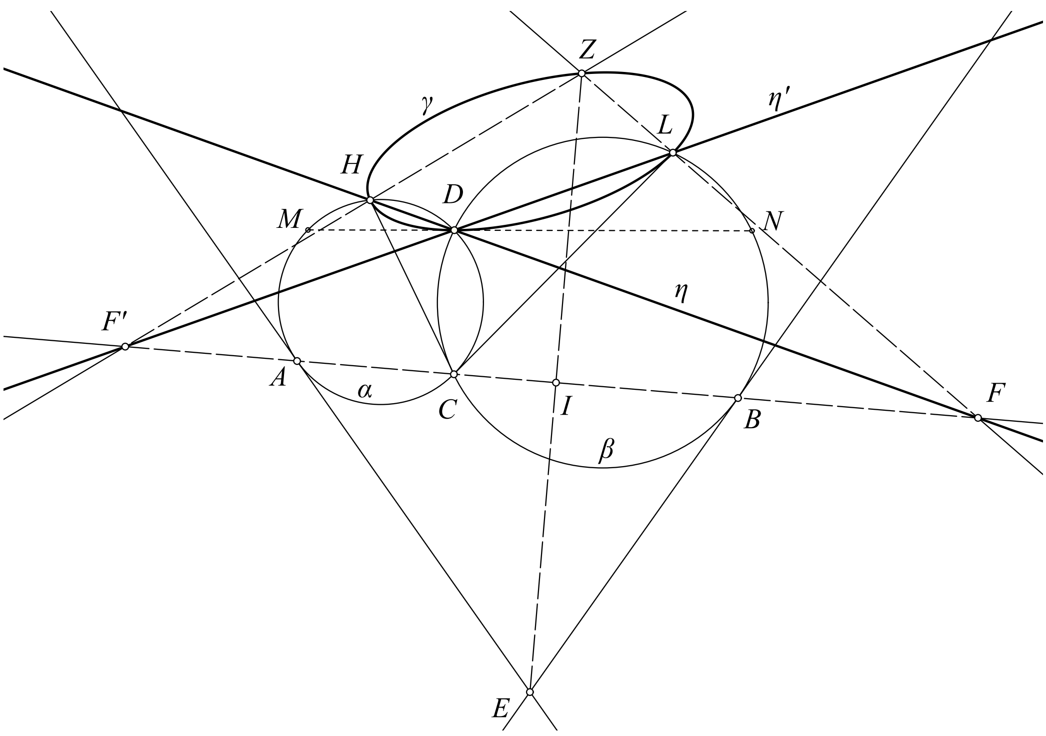

Fig. 4. Conic generated by vertex Z of the variable triangle ZFF'

As discussed in the appendix, the fact that H is on η, implies that CH is tangent to the conic. Analogously CL is also tangent to the conic. The fact that the conic is also tangent to line MN at D (See Figure 4) follows from the convergence of line DZ to line MN, as point E converges to D.

The claim on the degeneracy of the ellipse, when the two circles $\{\alpha, \beta\}$ are orthogonal, follows from the fact that point L (resp. H) becomes identical to N (resp. M), when the circles $\{\alpha, \beta\}$ become orthogonal. In this case triangle CHL becomes right-angled at C. □

2. The cardioid

Here and the sequel we use the notation introduced in the first section. A definition of the cardioid is given in the appendix, where are also discussed some of its known properties, which are relevant for our investigation. For a short synthetic account of the geometry of cardioids we refer to Akopyan's article [3].

Theorem 2. *Given two circles α and β, intersecting at two different points C and D, the intersection point E of their tangents at the extremities of a variable chord AB through C describes a cardioid with its cusp at D.*

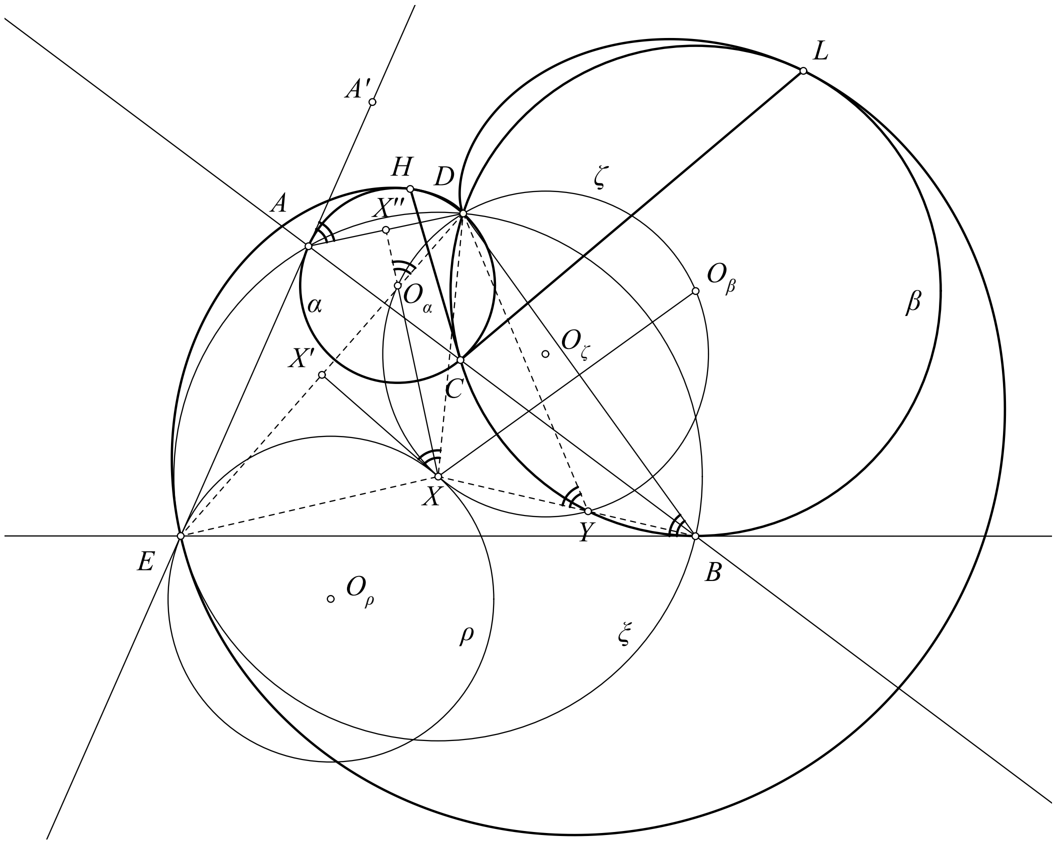

Fig. 5. Cardioid generation by the point E

Proof. In fact, by lemma 1, the quadrilateral $AEDB$ is cyclic and this implies that angle $\widehat{AEB}$ is constant, even equal to $\widehat{HCL}$ (See Figure 5). The result follows then as a special case of a known theorem of Butchart (see Appendix). We proceed though here to a short proof of our special case, since some of its ingredients are important for the subsequent discussion. We prove that the locus of E can be identified with a cardioid in its usual geometric definition. In conformance with this definition, we show that the locus of E

coincides with the locus of a fixed point E of a circle ρ, which is rolling on the fixed equal circle ζ, starting from the point D, which coincides with the cusp of the cardioid.

For the proof, notice first that the circumcenter X of the variable triangle ABE describes a circle ζ passing through the centers O_α, O_β of α, β and the point D. This follows from the definition of X, which is the intersection of medial lines of segments DA and DB, which always pass correspondingly through O_α, O_β and build an angle $\widehat{O_\alpha X O_\beta}$ complementary to $\widehat{ADB}$, which is constant and equal to angle $\widehat{O_\alpha D O_\beta}$.

A second remark is that the intersection point Y of circle ζ and line XB is also on the circle β. In fact, if X', X'' are correspondingly the middles of segments DE, DA, then, from the cyclic quadrilateral $YXO_\alpha D$, follows that angle $\widehat{XYD}$ equals angle $\widehat{X''O_\alpha D}$. This, by the tangent-chord property, equals angle $\widehat{A'AD}$. This in turn, by the cyclic $DAEB$ is equal to $\widehat{EBD}$, which by the tangent-chord property is equal to angle $\widehat{BLD}$. This shows that the quadrilateral $YBLD$ is cyclic and proves the claim.

By the definition of X, angle $\widehat{X'XD}$ is also equal to $\widehat{EBD}$. It follows that $X'X$ is tangent at X to circle ζ. Thus, taking the symmetric O_ρ of O_ζ w.r. to X we define circle ρ with center at O_ρ, which is equal to ζ, tangent to it at X and passing through E. It follows that the locus point E defines on circle ρ an arc XE, which is equal to the arc XD of the fixed circle ζ. This completes the proof of the rolling property of ρ on ζ and proves the theorem. $\square$

Remark 1. As is pointed out in the appendix, the cardioid is symmetric w.r. to the line DO_ζ and up to congruence completely determined by the circle ζ. Later we will see that circle ζ is congruent to the Euler circle of a triangle circumscribing the ellipse γ.

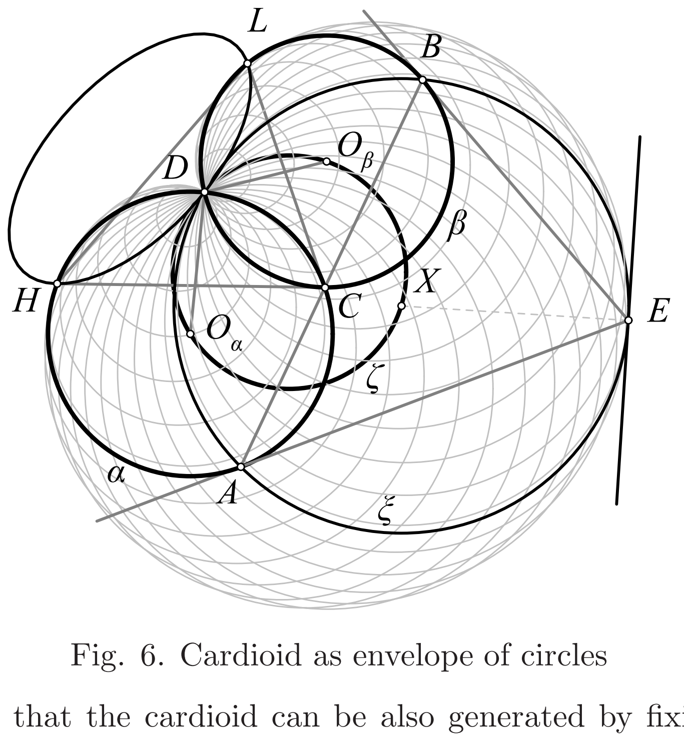

Fig. 6. Cardioid as envelope of circles

Remark 2. Notice that the cardioid can be also generated by fixing a point D on a circle ζ and considering all circles ζ with radii $r_\zeta = |DX|$, where their center X is another arbitrary point of ζ. The envelope of the family of all these circles (See Figure 6) is a cardioid. The two circles α, β of our configuration are two particular circle-members of such a family. The circumcircle ξ of the variable triangle ABE belongs also to this family. By its property to be envelope of all these circles, follows that the cardioid is tangent to the circles α, β and ξ. Referring back, to figure 5, and the fact, shown there that $X'X$ is the medial line of DE, follows that the points of tangency of these circles with the cardioid are respectively points H, L, and E.

3. A circumscribing triangle

In this section we focus on the triangle MNV, whose sides are shown to be tangent to the conic γ. Besides line MN, already met in the previous paragraphs, we consider here the lines MV and NV, which are the tangents, correspondingly, at M and N of circles α and β (See Figure 7). The context here is that of *triangle centers* and the *inscribed conics* in triangles, which have these centers as *perspectors.* A general reference for these concepts is the book of Akopyan and Zaslavsky on the geometry of conics [2, p. 105] and Yiu on the geometry of the triangle [31]. Of particular importance for our subject is the concept of the *orthopole* and the orthopolar conic, defined and discussed in short in the appendix.

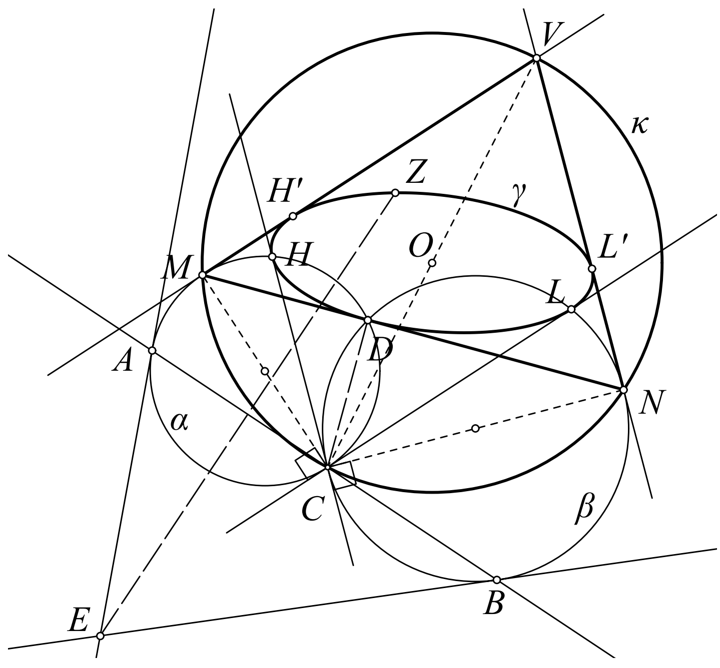

Fig. 7. The circumscribed triangle MNV

Theorem 3. *The ellipse γ is an inscribed conic in the triangle MNV. The center of the ellipse coincides with the cicrumcenter O of the triangle and this identifies it with the inconic with perspector X_{69}. This is the ellipse generated by the orthopoles of all lines passing through the De Longchamps point X_{20} of the triangle MNV.*

Proof. The proof on the triangle MNV circumscribing the ellipse follows from lemma 1(3), asserting that lines $\{AZ, BZ\}$ (See Figure 7) are always parallel, respectively, to $\{CL, CH\}$. Thus, the ellipse is contained in the strip consisting of CL and its parallel MH' from M. It follows that the ellipse has exactly one common point H' with this parallel, which, consequently, is a tangent to the ellipse. Analogously, the ellipse is tangent to the line NL', which is parallel to CH from N. Note that the reasoning shows also that points H' and L' are, respectively, symmetric of L and H w.r. to the center O of the conic.

The proof on X_{69} follows by identifying the ellipse with the orthopolar conic of the De Longchamps point X_{20} of the triangle MNV, which is the ellipse tangent to the triangle and having its center at the circumcenter of the triangle. This ellipse is the locus of points which are orthopoles of the lines passing through X_{20} (see appendix).

The fact, in turn, that the circumcenter O of the triangle MNV is the center of the ellipse follows by observing that quadrangle $CMVN$ is cyclic, its angles at M and N beign right, and the ellipse is inscribed in the parallelogram defined by the lines CH, CL, VN, and VM. This implies that the center of the ellipse is the intersection point of the diagonals of the parallelogram, one of which (CV) is a diameter of the circumcircle κ. □

4. A bitangent triangle

Of particular interest for our discussion is also the *bitangent* to γ triangle CHL, i.e. a triangle that has only two of its sides (CH, CL) tangent to γ, correspondingly, at points H and L. Of interest also is the circumcircle δ of triangle CHL. In the formulation of the next theorem enter the concepts of the *symmedians* and the *Brocard points* of a triangle, for which we refer to the aforementioned book of Yiu. The following theorem expresses the main properties of this configuration (See Figure 8).

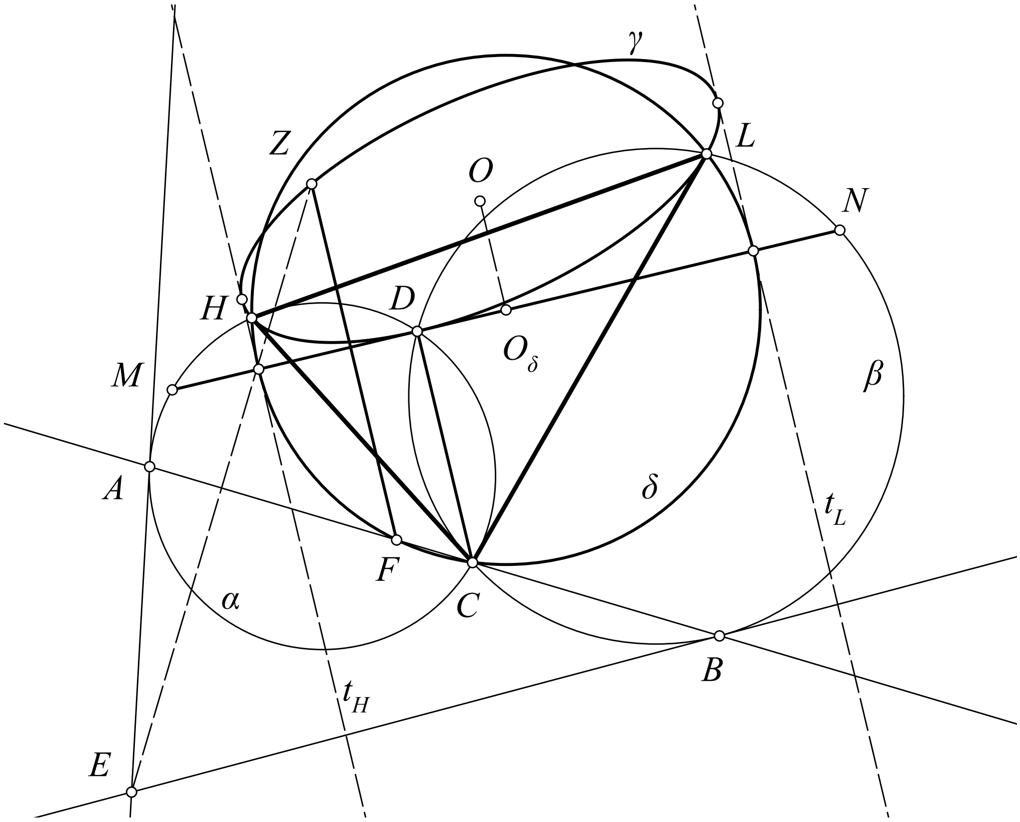

Fig. 8. The bitangent to γ triangle CHL

Theorem 4. *Line CD is a symmedian for the triangle CHL and point D coincides with a vertex of its second Brocard triangle. The circumcenter O_δ is the middle of the segment MN. If point F is the intersection of δ with the variable line AB, then line FZ is parallel to the fixed line CD. The circle δ and the ellipse γ have two common parallel tangents t_H and t_L, which are orthogonal to line MN.*

The proof of the theorem relies on the following two lemmas. In these we consider the second intersection point U of the circle $\delta = (CHL)$ with the line AB and the triangle $EA'B'$ with sides the lines EA, EB and the tangent t_U to δ at U, points A', B' being, correspondingly, the intersections of this tangent with lines EA and EB (See Figure 9).

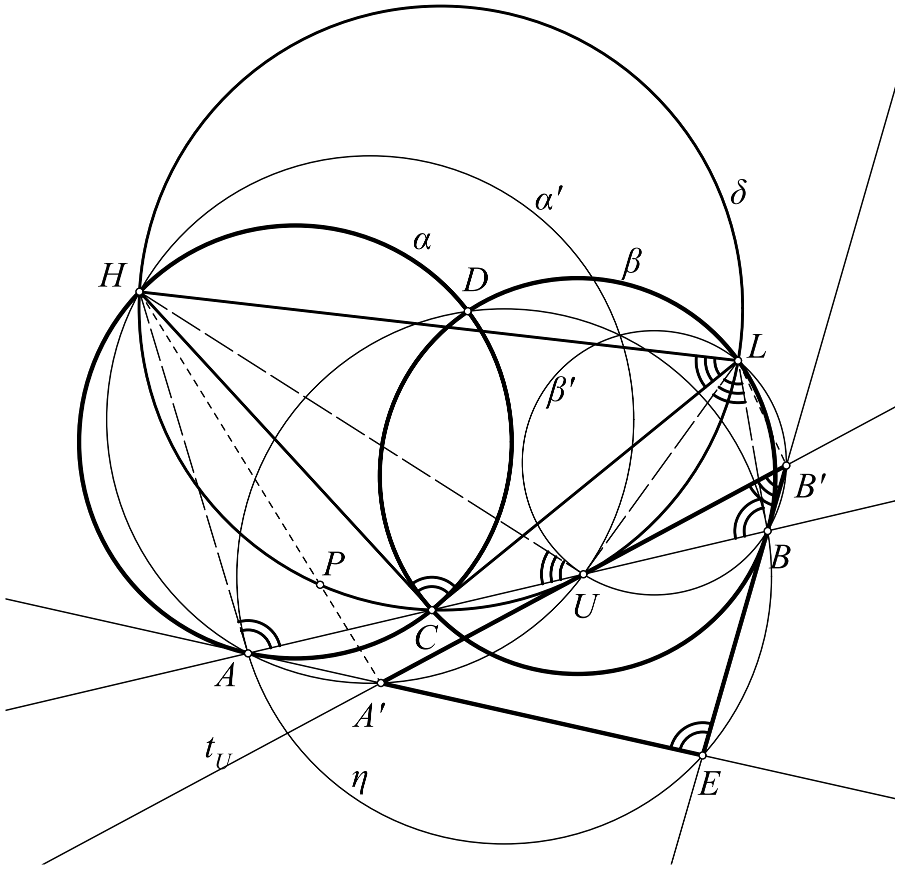

Fig. 9. The similar triangles: fixed CHL and variable $A'EB'$

Lemma 3. *Under the definitions made above, the following are valid properties:*

(1) *Quadrangles $UBB'L$ and $UA'AH$ are cyclic.*
(2) *Triangle $A'EB'$ is similar to HCL.*
(3) *Triangles $HA'U, UB'L$ and HUL are similar.*
(4) *Point U is the middle of $A'B'$.*

Proof. To prove (1), write angle $\widehat{UB'B} = \widehat{UBE} - \widehat{BUB'}$. By the tangent-chord theorem these angles are equal: $\widehat{UBE} = \widehat{CLB}$, $\widehat{BUB'} = \widehat{CLU}$. It follows that $\widehat{ULB} = \widehat{UB'B}$ and this proves the first claim for quadrilateral $UBB'L$. Analogous is the proof for the quadrilateral $UA'AH$.

To prove (2), notice that $\widehat{CLB} = \widehat{HLU}$, which implies $\widehat{ULB} = \widehat{HLC}$. This follows from the equalities of angles: $\widehat{CLB} = \widehat{CBE} = \widehat{ACH} = \widehat{HLU}$. Analogously follows that $\widehat{EA'B'} = \widehat{CHL}$ and thereby the proof of the claim.

To prove (3) use property (4) of theorem 1. By this, lines AH and BL are tangent to the circle $\eta = (ABE)$ (See Figure 9). Using the previously proven properties this implies that $\widehat{HA'U} = \widehat{HAU} = \widehat{UBL} = \widehat{UB'L}$. Similarly, $\widehat{UHA'} = \widehat{UAA'} = \widehat{LBB'} = \widehat{LUB'}$. By the tangent chord property for cirlce δ follows that $\widehat{UHL} = \widehat{LUB'}$. Analogously also $\widehat{HLU} = \widehat{HUA'}$, thereby completing the proof of this claim.

The proof of (4) follows by observing that HU is the bisector of angle $\widehat{A'HL}$, hence divides the corresponding arc of circle δ in two equal parts and $UP = UL$, where P the second intersection of line HA' with circle δ. This implies that triangles UPA' and ULB' are congruent and completes the proof of this claim. $\square$

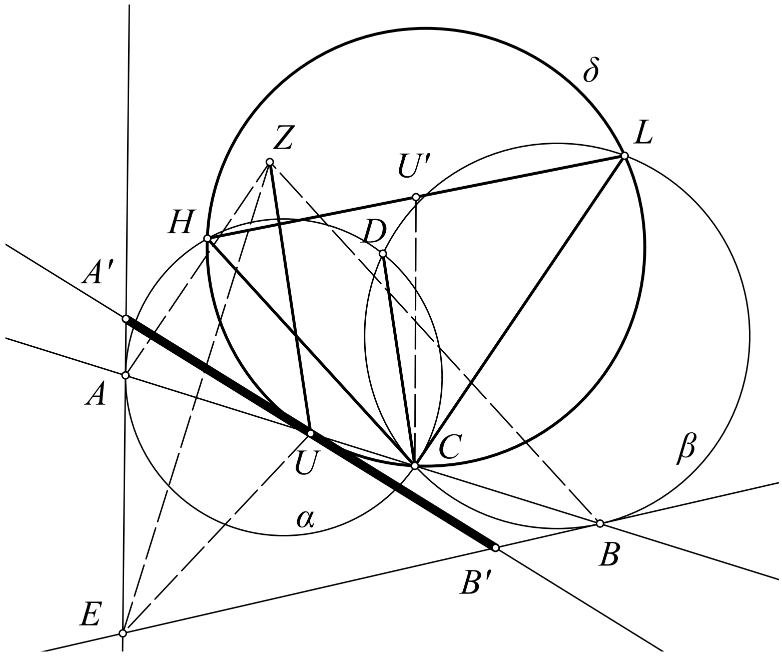

Fig. 10. The direction of UZ

Lemma 4. *The variable line UZ, where U is the second intersection point of AB with circle δ, is always parallel to line CD.*

Proof. For the proof we use the result of the previous lemma, by which, the variable triangle $A'EB'$ remains similar to the fixed one HCL. We use also the well known result identifying point D with a vertex of the second Brocard triangle of the triangle HCL ([12, p.283]). By this result, the symmedian point of triangle CHL is on line CD, which then is isogonal w.r. respect to this triangle to its median CU' (See Figure 10). To prove the lemma we show that lines UZ and CD are equal inclined to side CH of the triangle CHL. In fact, by (3) of lemma 1, line CH is parallel to BZ, and, by symmetry, $\widehat{UZB} = \widehat{UEB}$. Last angle, by the similarity of triangles $A'EB'$ and HCL and because U is the middle of $A'B'$, equals $\widehat{U'CL}$. This angle, by the noticed above isogonality of CD and CU', equals angle $\widehat{HCD}$, thereby completing the proof of the lemma. □

The proof of theorem 4 follows immediately from the preceding lemma.

5. An Euler circle

In this section we turn back to triangle MNV, discussed in section 3, and its Euler cirlce θ, combining results obtained in the previous paragraphs. The main property of this circle is its equality with the circle ζ, introduced in section 2, and carrying the circumcenters of all triangles EAB. This equality is realized by a point-symmetry with respect to the center O_δ of the circle δ, introduced in section 4. The main properties in this context, are easy to derive consequences of the previous discussion and are listed in the following theorem. Some additional elements, involved in this theorem, are the orthocenter H' of triangle MNV and the feet V', M' and V' of the altitudes from the corresponding vertices of the triangle (See Figure 11).

Theorem 5. *Under the definitions made above the following are valid properties:*

(1) *Point C is the symmetric of the orthocenter H' of triangle MNV w.r. to the middle O_δ of its side MN.*

(2) *The circumcircle* δ *of triangle* CHL *passes through* H' *and points* $\{L, H\}$ *are on the altitudes of triangle* MNV, *at distance from the vertex equal to the distance of* H' *from the corresponding altitude-foot.*
(3) *The circle* $\zeta = (DO_\alpha O_\beta)$ *is the symmetric of the Euler circle* θ *of* MNV *w.r. to the middle* O_δ *of* MN.

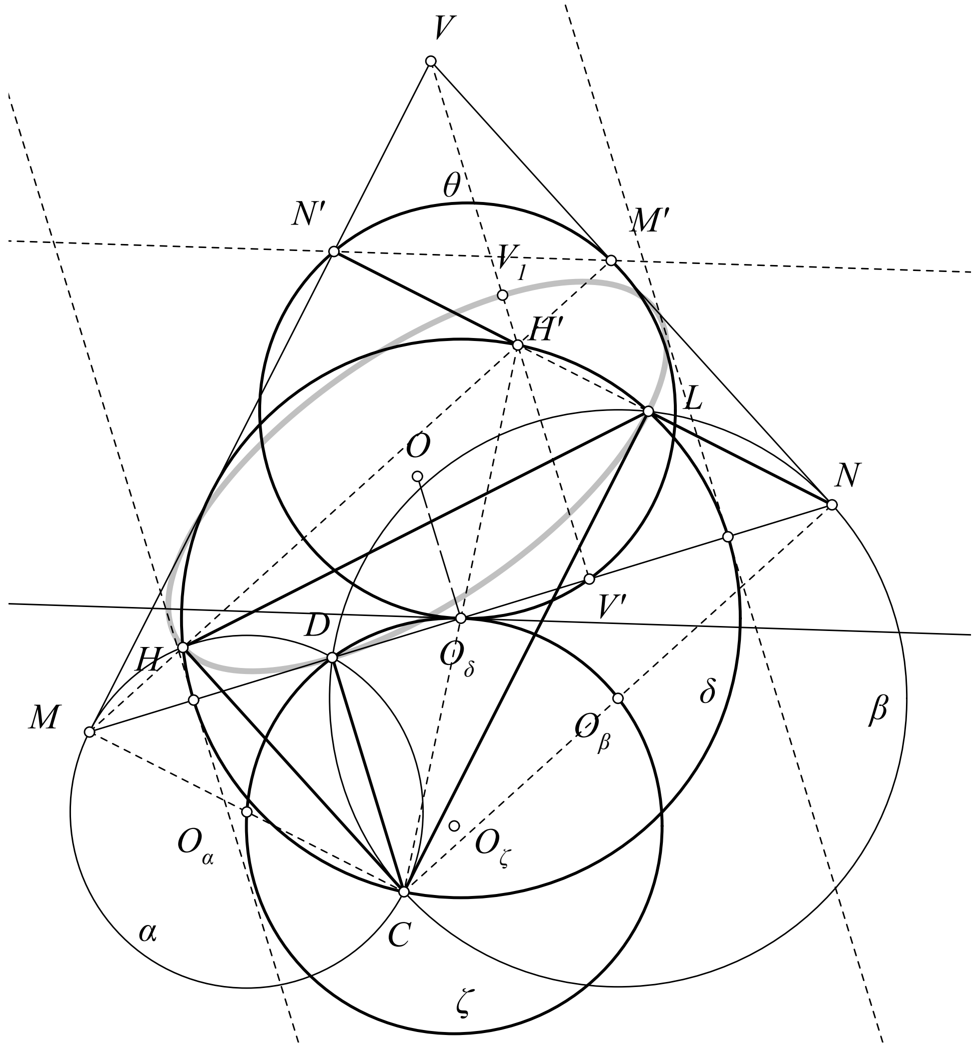

Fig. 11. ζ symmetric to the Euler circle θ w.r. to O_δ

Proof. Properties (1) and (2) follow from the presence of the parallelogram p_1, having for sides the lines $\{CH, HL, NV, VM\}$. As proved in theorem 3, the circumcenter O of triangle MNV is the middle of the diagonal VC. Besides, lines CD and VV' are both orthogonal to MN. Thus, the middle O of VC projects on the middle of DV' and this shows that $|DO_\delta| = |O_\delta V'|$. Since CL is orthogonal to NN' and CH is orthogonal to MM', it follows that they meet on the altitude VV' at a point H', which is diametral to C w.r. to circle δ.

Property (3) follows by considering the symmetrics of points O_α, O_β w.r. to O_δ. Because $CNH'M$ is a parallelogram and O_δ is its center, these two points coincide respectively with the middles of $H'N$ and $H'M$, which are points of the Euler circle θ of MNV. This shows the stated symmetry about O_δ. □

There are several properties that follow from the previous theorems, and their proofs are easy exercises. Thus, for example, segment DC is equal to $H'V'$, which is also equal to VV_1, where V_1 is the intersection of DO with the altitute VV' (See Figure 11). It is also easily seen that the ellipse γ passes through V_1, that line DH is parallel to ON, that CV is orthogonal to $N'M'$, which is parallel to the common tangent at O_δ of the circles ζ and θ. Perhaps, we should notice aslo that the ellipse γ passes through six easily constructible points. The first triad of these points consists of the symmetrics of the feet of the altitudes w.r. to the middles of the respective sides, like the point D, which is symmetric to V' w.r. to O_δ. The other triad of points are on the altitudes, like V_1 in distance $|VV_1| = |H'V'| = |DC|$.

6. Inverse construction

Next theorem deals with the inverse procedure, succeeding to reproduce a given ellipse γ by the procedure described above, applying it at a point $D \in \gamma$.

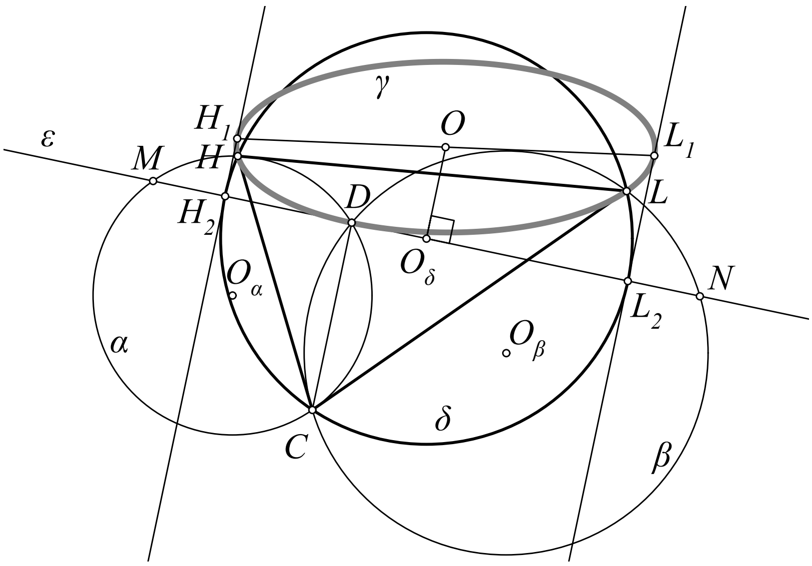

Fig. 12. Constructing two circles α, β at every point of an ellipse

Theorem 6. *Given an ellipse γ, for every point $D \in \gamma$ there are two uniquely defined circles $\{\alpha, \beta\}$, which generate the ellipse by the preceding method.*

Proof. This is a direct consequence of theorem 4. In fact, consider the tangent ε of the conic at an arbitrary point D and project its center O to point O_δ on it (See Figure 12). Then consider the conjugate to the direction of OO_δ, diameter H_1L_1 of the ellipse. The circle δ is defined by its center at O_δ and diameter equal to the length of the projection H_2L_2 of H_1L_1 on line ε. Let C be the intersection point of δ on the normal of the ellipse at D, lying on the other side than O. Draw from C the tangents CH, CL to the ellipse. There are defined two circles $\alpha = (CDH)$, $\beta = (CDL)$. In view of theorem 1, the two intersecting circles define, by the procedure studied above, the ellipse γ. □

A question that comes up from the previous property is, about the behavior of the configurations that result from the various places of point D on the ellipse. To this answers the following statement.

Theorem 7. *For all places of D on γ the segment OC is of constant length. Thus, all resulting triangles MNV that circumscribe the ellipse γ are also inscribed in the same circle κ, which has the radius $r_\kappa = |OC| = a + b$, where $\{a, b\}$ are the axes of the ellipse and has the same center with it. Consequently the cardioids for the various places of D are congruent to each other.*

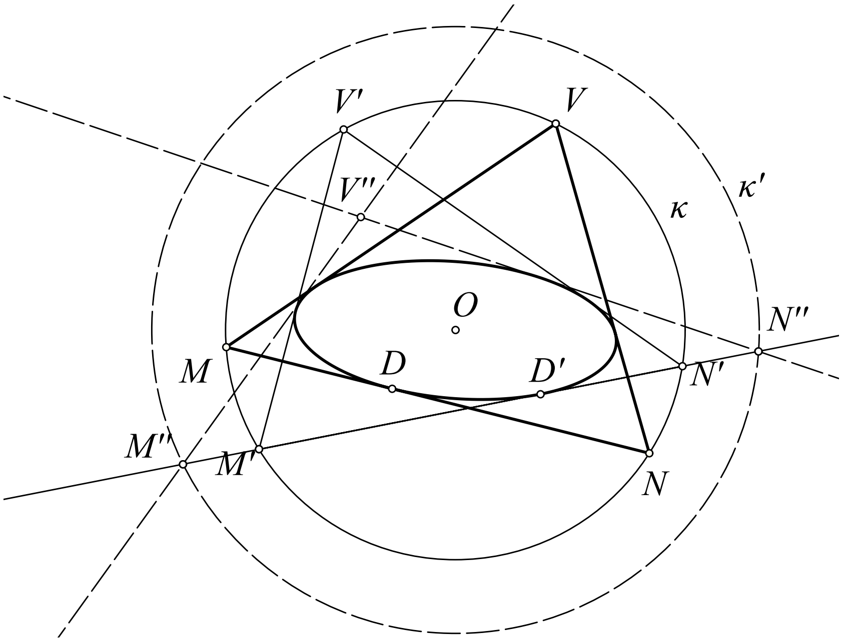

Fig. 13. Applying Poncelet's porism

Proof. The proof of this could be given by a calculation, using, for example, known formulas for the tangents from a point, the chords of contact points to the ellipse etc. ([26, pp.221-233]). The property, though, lends itself for a synthetic proof. In fact, consider an ellipse γ, a point $D \in \gamma$, the corresponding triangle MNV, constructed by the procedure, and its circumcircle κ (See Figure 13). By Poncelet's porism ([2, p.93], [7, p.203 (II)]), for every other tangent $M'N'$, at another point D', we would obtain a triangle $M'N'V'$ which is also circumscribed in γ and inscribed in κ. Now, if there were a triangle $M''N''V''$ circumscribed to γ and inscribed in another concentric circle κ', of radius, say, greater than that of κ, then, by applying again Poncelet's porism, we could construct the triangle so that the lines $M'N'$ and $M''N''$ coincide. Then it is readily seen that the corresponding vertex V'' cannot lie on κ'. This contradiction shows that, by our procedure, the constructed circumscribed about γ triangles MNV have all the same circumcircle κ. The claim about the radius follows then easily by simply considering D to be at a vertex of the ellipse and evaluating $r_\kappa = |OC| = |CD|+|L_1L_2| = |DL_2|+|L_2L_1| = a+b$ (See Figure 14). The result for the cardioids is an immediate consequence, of their generation by the rolling of a circle on another circle, both being congruent to the Euler circle of triangle MNV. □

Corollary 8. *For every ellipse γ with center O, axes $\{a, b\}$ and every triangle circumscribed to the ellipse and having O for circumcenter, its vertices lie on the circle $\kappa(O, a+b)$.*

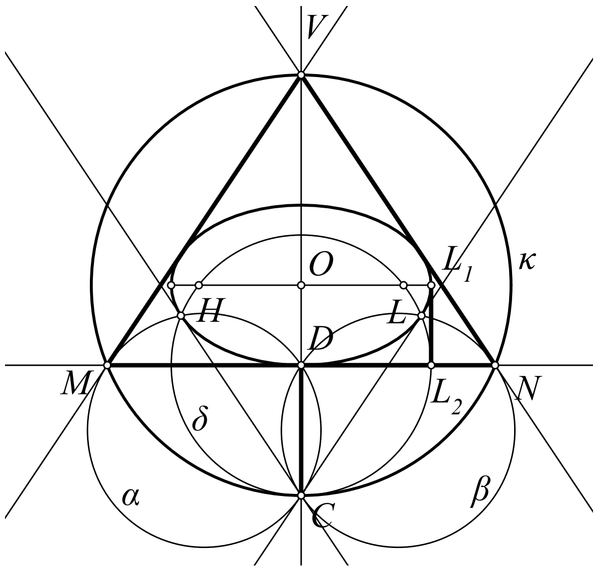

Fig. 14. $r_\kappa = a + b$

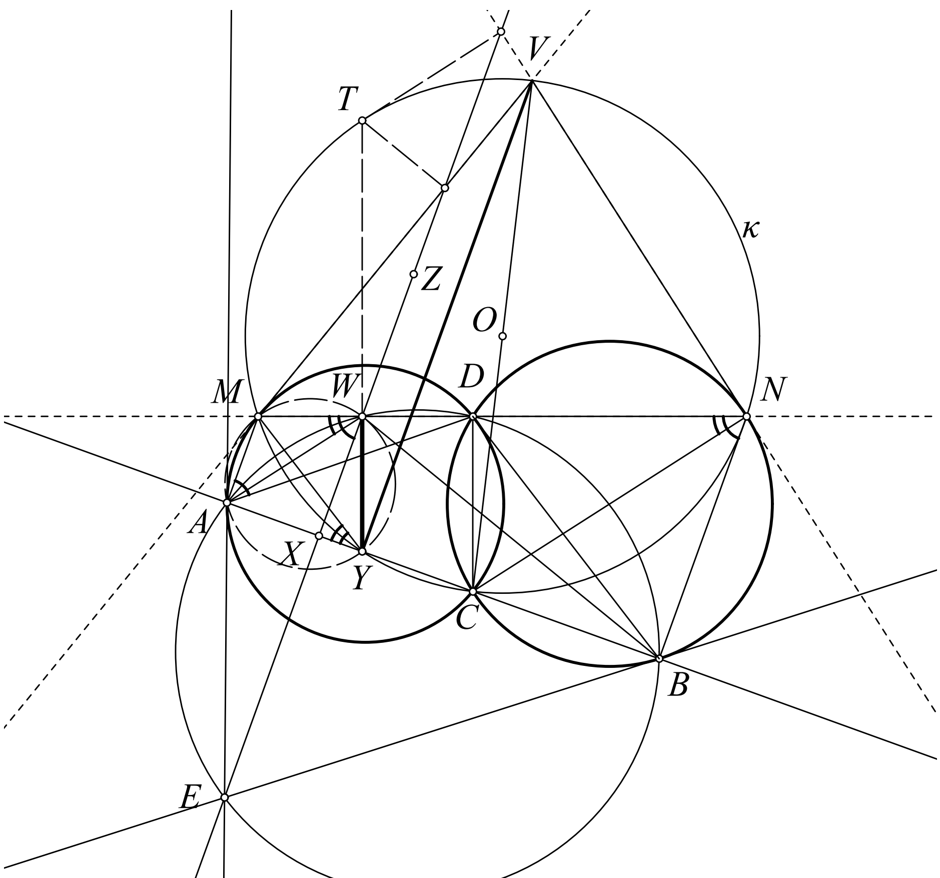

Fig. 15. EZ is a Simson line of triangle MNV

7. Simson lines and deltoid

The construction of the ellipse γ from two intersecting circles $\{\alpha, \beta\}$, by our procedure, leads to the triangle MNV and its associated deltoid, i.e. the envelope of its Simson lines. The key property in this respect is the following (See Figure 15).

Theorem 9. *For all points E, line EZ is a Simson line of the triangle MNV.*

We prove first that the intersection point W of MN and EZ and the other than C intersection point Y of the circumcircle κ of MNV and line AB define a segment WY which is orthogonal to MN. This is so because $WYAM$ is cyclic. This in turn follows from the equality of angles $\widehat{MAW} = \widehat{AWX} = \widehat{ABE} = \widehat{CNB}$, which implies that AW and CN are parallel, since MA, WX, NB are parallel being all orthogonal to AB. From the parallelity of AW to CN follows that $\widehat{MWA} = \widehat{DNC} = \widehat{MYA}$, later because $MYCN$ is cyclic. This shows that $\widehat{MWA} = \widehat{MYA}$ and implies the orthogonality of WY to MN. Let T be the other intersection point of YW with the circle $\kappa = (MNV)$, then, since VC is a diameter of κ, line VY is orthogonal to AB, hence parallel to EZ. By a well known theorem for the orientation of the Simson lines (see theorem 15 in the appendix), the line EZ is the Simson line of the point T on the circle κ. This completes the proof of the theorem.

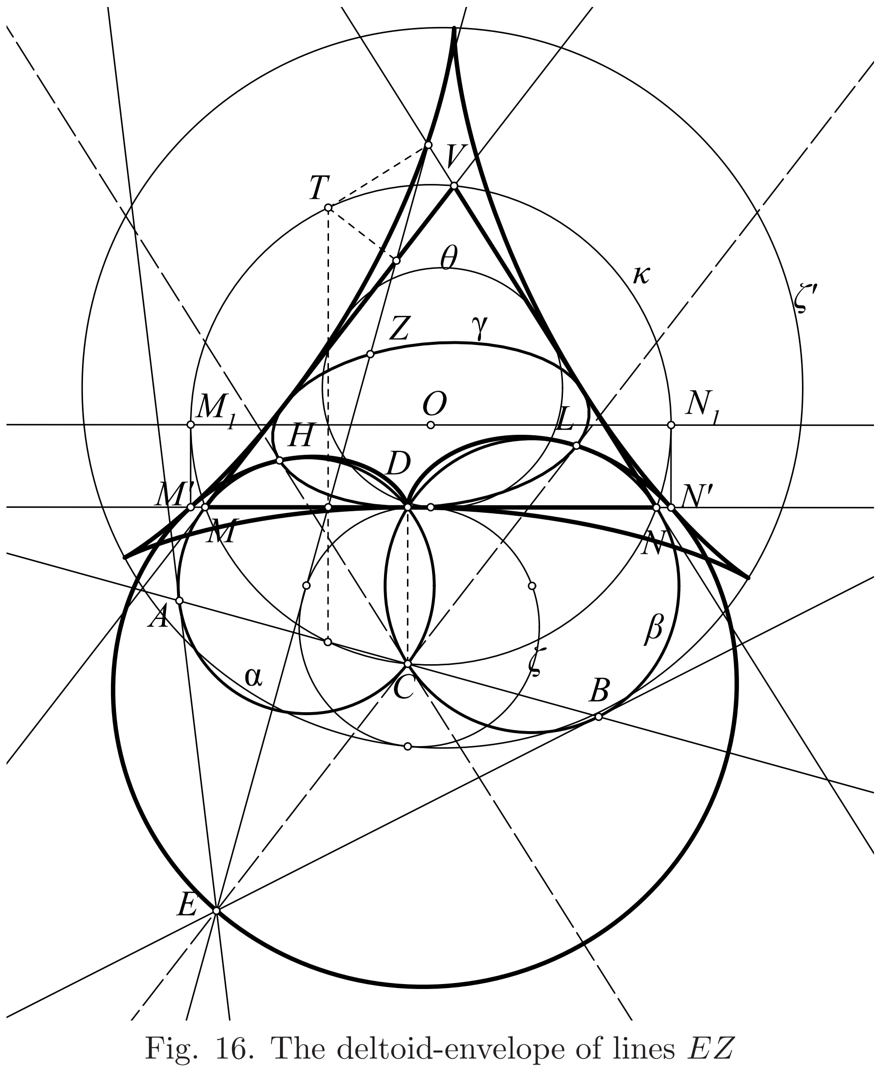

Fig. 16. The deltoid-envelope of lines EZ

Figure 16 shows the deltoid-envelope of the Simson lines of triangle MNV together with the corresponding cardioid-locus of point E. There are various relations visible in this figure, which are consequences of our discussion or can be proved easily. I mention only next property, which can be proved by considering special places of the point E.

Theorem 10. *The deltoid passes through point D and is tangent to the cardioid at two points $\{M', N'\}$ on line MN, which are the projections of the diameter M_1N_1 of the circumcircle κ of MNV, which is parallel to the side MN.*

It is remarkable that both the deltoid ([22, p.72]) and the cardioid ([22, p. 34]) can be generated from a point of a circle congruent to the Euler circle θ of MNV which rolls on another circle. The deltoid is generated by rolling such a circle inside the circle ζ' which is concentring to the Euler circle and has the radius $r = 3r_\theta$. The cardioid is generated by rolling circle θ outside its congruent ζ.

Remark 3. In the discussion we have tacitly assumed that the two intersecting circles $\{\alpha, \beta\}$, defining our configuration, intersect at an obtuse angle, equivalently the angle $\widehat{HCL} = \widehat{MVN}$ is acute. The case of circles intersecting at an acute angle can be handled using, essentially, the same arguments. Figure 17 shows a configuration for such a case.

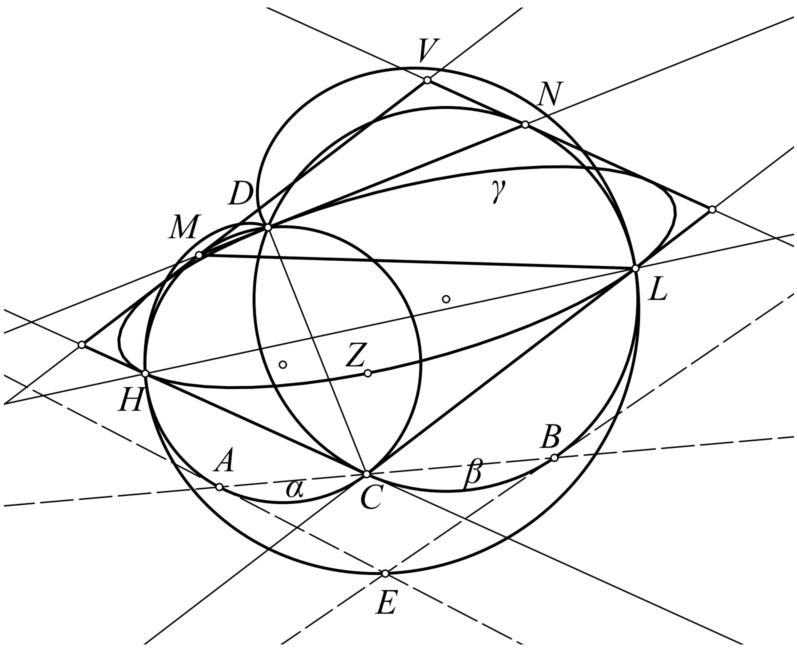

Fig. 17. The case of obtuse angles

8. Appendix

In this section we gather together some well known facts used in the proofs, mainly on Maclaurin's theorem, properties of the Cardioids, Simson lines, and orthopoles.

8.1. **Maclaurin's theorem.** Maclaurin's theorem describes the generation of a conic by the free vertex Z of a variable triangle ZFF', whose other two vertices F, F' are bound to move on two fixed lines η and η'. The precise statement is the following ([6, II, p.13], [11, p.72], [27, p.299]).

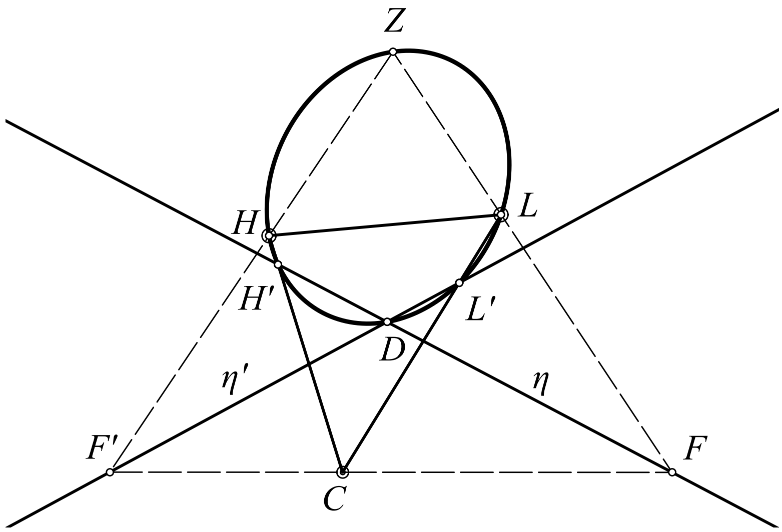

Fig. 18. Maclaurin's theorem

Theorem 11. *Let the variable triangle ZFF' have its two vertices F and F' move, correspondingly on two fixed lines η and η', while its side-lines ZF, ZF' and FF' pass, correspondingly through three fixed points L, H and C. Then its free vertex Z describes a conic. This conic passes through the points H, L and the intersection point D of lines η, η'. The conic passes also through the intersection points H', L', correspondingly, of the line-pairs (η, HC) and (η', CL).*

The theorem, in its previous form, applies for three points $\{C, H, L\}$ and two lines $\{\varepsilon, \varepsilon'\}$ *in general position* (See Figure 18). In the special case, in which point H is contained in line η, points H and H' coincide and the conic is tangent to line CH at H. Analogous is the behavior for line CL if L is contained in η'. This particular case applies to our configuration, impying the tangency of the conic γ to lines CH, CL, correspondingly at H and L.

8.2. **Cardioid.** One way to define the cardioid ([3] [22, p.34], [30, p.89], [23, p.142]), is to consider it as the geometric locus of a fixed point E of a circle ρ rolling on a circle ζ of equal radius, the rolling circle starting to roll at the point D of ζ. The resulting curve has a cusp at D (See Figure 19). The circle ζ could be called the *basic* circle of the cardioid. The cardioid is completely determined by its basic circle and a point D on it, defining its cusp. Among the many properties of this remarkable curve is the fact that the one-sided tangents at D coincide and both contain the center O_ζ of the fixed circle and line DO_ζ is a symmetry axis of the curve. If X is the contact point of the circles ρ and ζ, X' is the middle of DE, and O_ρ is the center of ρ, then, since the arcs XE on ρ and XD on ζ are equal, the quadrilateral $DEO_\rho O_\zeta$ is an equilateral trapezium and XX' is orthogonal to DE, which is parallel to $O_\rho O_\zeta$. Taking the circle σ in symmetric position of ρ w.r. to O_ζ and considering the corresponding point E' of the locus, we realize analogously that $DE'O_\sigma O_\zeta$ is an equilateral trapezium and $EE'O_\sigma O_\rho$ is a parallelogram. This proves the following fundamental property of the cardioid.

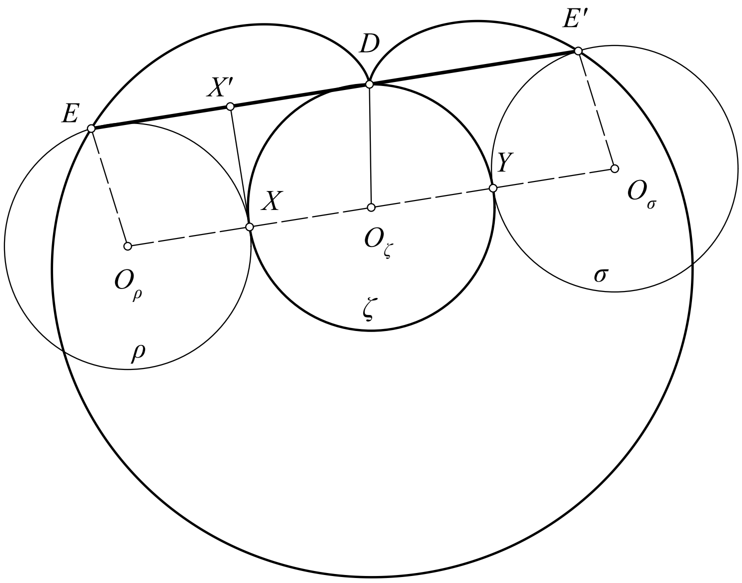

Fig. 19. Cardioid generation by a rolling circle

Theorem 12. *Every line through the cusp D of the cardioid defines a chord EE' of it, which has the constant length $4r_\zeta$, where r_ζ is the radius of the fixed circle O_ζ.*

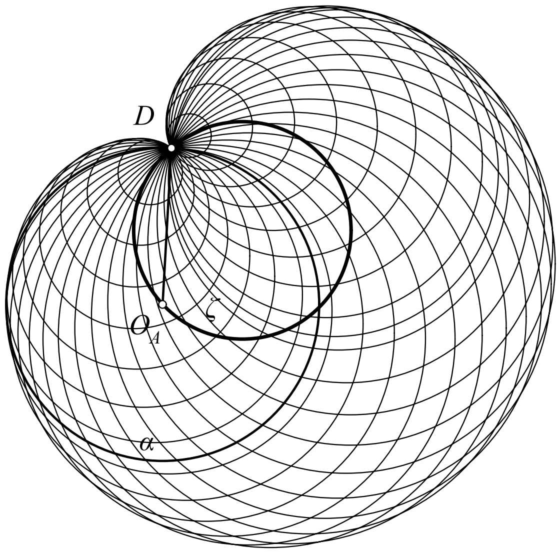

Fig. 20. Cardioid generation as envelope of circles

Adopting the previous definition, we can formulate an alternative definition of the cardioid in the form of the following theorem.

Theorem 13. *Let D be a fixed point on a circle ζ. For each other point O_α on ζ define the circle α with center at O_α and radius the distance $r_\alpha = |DO_\alpha|$. Then the envelope of all these circles α is a cardioid with basic circle ζ and its cusp at D.*

Cardioids have been extensively studied in the past ([22, p.34], [30, p.89], [23, p.142], [24, p.73], [9]). For a short account and references see Archibald's article [5]. The following theorem, due to Butchard ([10]), applies to more general situations than it is the case with our configuration.

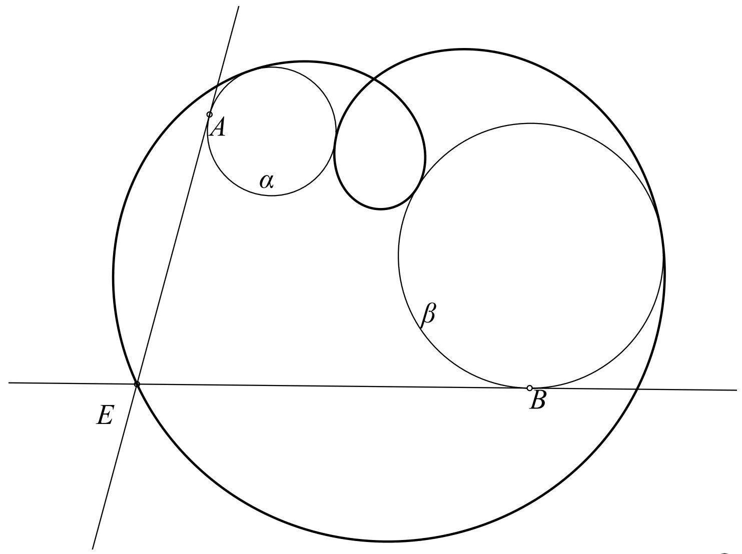

Fig. 21. Cardioid generation by a constant angle $\widehat{AEB}$

Theorem 14. *If a, constant in measure, angle AEB has its legs EA, EB, respectively, tangent to two circles α and β, then its vertex E describes a cardioid.*

8.3. **Simson lines.** The Simson line s_P of a triangle ABC w.r. to a point P on its circumcircle $c = (ABC)$ is the line carrying the projections P_1, P_2, P_3 of P (See Figure 22), respectively, on sides BC, CA and AB ([12, p.140]). A property of Simson lines, used in our discussion, is the one expressed by the following proposition ([12, p.142]).

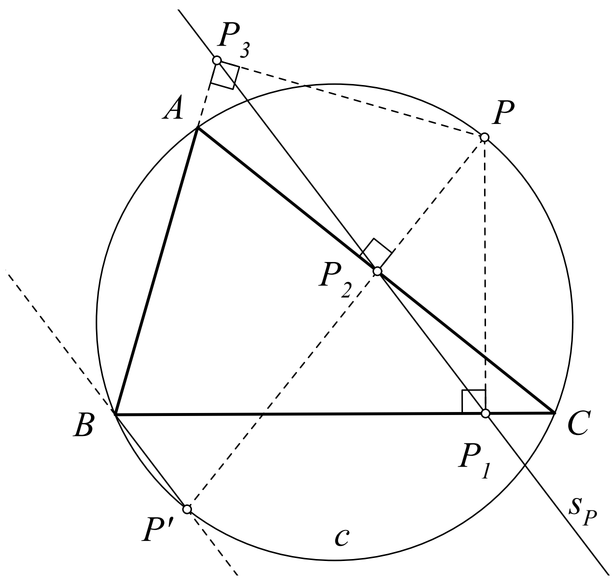

Fig. 22. Simson lines

Theorem 15. *If for a point P on the circumcircle c of triangle ABC and its projection on a side, say P_2 on CA, line PP_2 is extended to cut c at a second point P', then line $P'B$ is parallel to the Simson line s_P of P.*

Another fact used in the discussion is also the following ([1, p.240], [28, p.231], [14, p.101]),([15, p.563],[8, p.224]) (See Figure 23).

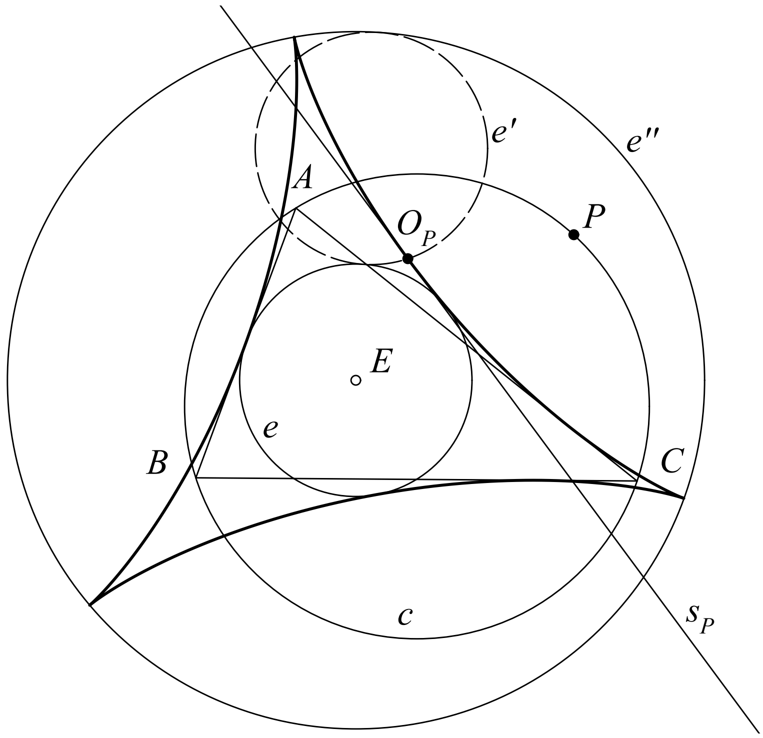

Fig. 23. Deltoid: the envelope of Simson lines

Theorem 16. *The evelope of all Simson lines s_p of triangle ABC is an algebraic curve of degree four, called deltoid. This curve is generated by a point O_p of a circle e' equal to the Euler circle e of the triangle, which roles inside a circle e'' concentric to e and of triple radius.*

8.4. **Orthopoles.** The orthopole of a line ε w.r. to a triangle ABC results by a process of double projection of each vertex of the triangle. Vertex A is projected on line ε to the point A' and point A', in turn, is projected on the opposite side BC, to point A''. Analogously

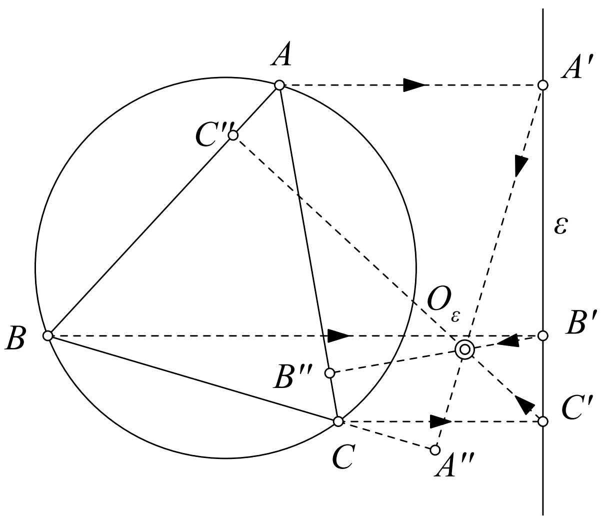

Fig. 24. The orthopole of a line w.r. to a triangle

are defined the points B', B'' and C', C'' (See Figure 24). It is proved that the lines $A'A''$, $B'B''$ and $C'C''$ intersect at a point O_ε which is the orthopole of the line ε w.r. to ABC ([21, p.17], [16, p.49], [19, p.106], [18]). The main properties used in this article are expressed by the following two theorems. The first of them relating the orthopole to Simson lines and the deltoid they envelope (See Figure 25).

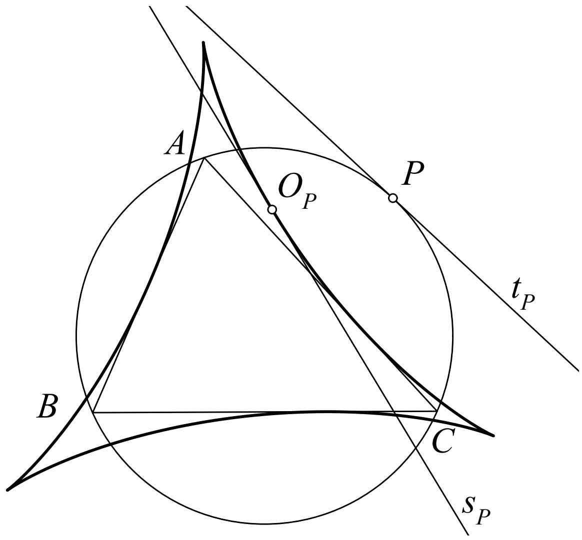

Fig. 25. The deltoid contact point O_P of the Simson line s_P

Theorem 17. *The orthopole O_P of the tangnet t_P of a point P on the circumcircle of triangle ABC is on the Simson line s_P of that point and coincides with its contact point with the deltoid-envelope of all Simson lines of the triangle.*

The second property is concerned with the orthopoles w.r. to a fixed triangle and all the lines through a given point D. Their orthopoles define a conic called the *orthopolar* conic of point D w.r. to the triangle (See Figure 26).

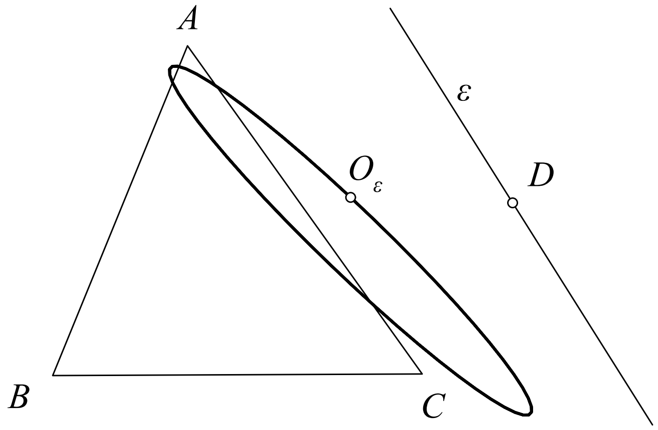

Fig. 26. The orthopolar conic of point D w.r. to the triangle ABC

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Theorem 18. *The orthopoles w.r. to the triangle ABC of all lines through the point D generate a conic.*

Given the triangle ABC, there is a particular orthopolar conic inscribed in the triangle. Next theorem describes how this is done ([4], [13], [19, p.125], [16, p.46], [18]) (See Figure 27).

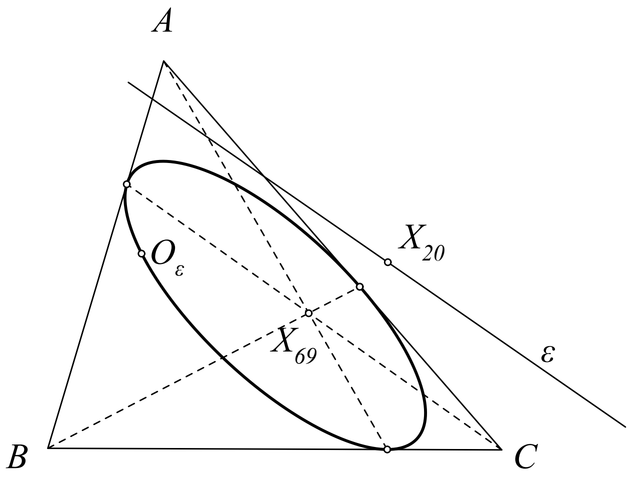

Fig. 27. The orthopolar conic of point X_{20}

Theorem 19. *Given the triangle ABC, the orthopolar conic of its triangle center X_{20} (called the Delongchamps point of the triangle), is a conic inscribed in the triangle, with perspector the triangle center X_{69}. This is an ellipse tangent to the sides of the triangle whose center coincides with the circumcenter of the triangle.*

The standard reference for general triangle centers is Kimberling's encyclopedia of triangle centers [20]. For the particular centers X_{20} and X_{69}, the previously cited facts and many other interesting properties can be found in the articles [17], [25], [29].

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References

[1] J. Aarts. *Plane and solid geometry.* Springer Verlag, Berlin, 2008.

[2] A. V. Akopyan and A. A. Zaslavsky. *Geometry of conics.* American Mathematical Society, New York, 2007.

[3] A. Akopyan. Geometry of the Cardioid. *American mathematical monthly*:1–6, 2015.

[4] N. Altshiller-Court. On the De Longchamps Circle of the Triangle. *American math. monthly*, 33:368–375, 1926.

[5] R. Archibald. The Cardioid and Tricuspid: Quartics with Three Cusps. *The annals of mathematics*, 4:95–104, 1903.

[6] H. Baker. *Principles of geometry, vols i-vi.* Cambridge university press, Cambridge, 1930.

[7] M. Berger. *Geometry vols i, ii.* Springer Verlag, Heidelberg, 1987.

[8] H. Brocard. Demonstration de la proposition de Steiner relative a l' enveloppe de la droite de Simson. *Bulletin de la s.m.f.*, 1:224–226, 1872.

[9] J. Butchart. Rotation of the Tangent to a Hypocycloid. *American mathematical monthly*, 55:152–153, 1948.

[10] J. Butchart. Some Properties of the Limacon and Cardioid. *American mathematical monthly*, 52:384–387, 1945.

[11] M. Chasles. *Traite de sections coniques.* Gauthier-Villars, Paris, 1865.

[12] N. A. Court. *College geometry.* Dover Publications Inc., New York, 1980.

[13] H. Coxeter. Some Applications of Trilinear Coordinates. *Linear algebra and its applications*, 226-228:375–378, 1995.

[14] M. Cremona. Sur l' hypocycloide a trois rebroussements. *Crelle's journal.*, 64:98–105, 1865.

[15] Z. Fueredi and I. Palasti. Arrangements of Lines with a Large Number of Triangles. *Proceedings of the american mathematical society*, 92:561–566, 1984.

[16] W. Gallatly. *The modern geometry of the triangle.* Francis Hodgsonn, London, 1913.

[17] R. Goormaghtigh. On Some Loci Connected with the Orthopole-Geometry. *American mathematical monthly*, 37:370–371, 1930.

[18] R. Goormaghtigh. The Orthopole. *Tohoku mathematical journal, first series*, 27:77–125, 1926.

[19] R. Honsberger. *Episodes in nineteenth and twentieth century euclidean geometry.* The Mathematical Association of America, New Library, Washington, 1995.

[20] C. Kimberling. Encyclopedia of triangle centers. URL: `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.

[21] T. Lalesco. *La geometrie du triangle.* Librairie Vuibert, Paris, 1952.

[22] E. Lockwood. *A book of curves.* Cambridge university press, Cambridge, 1961.

[23] G. Loria. *Ebene kurven.* B.G. Teubner, Leipzig, 1902.

[24] R. Proctor. *The geometry of cycloids.* Longmans, Green and Co., London, 1878.

[25] O. Ramler. The Orthopole Loci of some One-Parameter Systems of Lines. *American mathematical monthly*, 37:130–136, 1930.

[26] J. Runkle. *Elements of plane analytic geometry.* Ginn and Company, Boston, 1890.

[27] G. Salmon. *A treatise on conic sections.* Longmans, Green and Co., London, 1917.

[28] J. Steiner. Ueber eine besondere Curve dritter Klasse (und vierten Grades). *Crelle's journal*, 53:231–237, 1857.

[29] A. Vandeghen. Soddy's Circles and the De Longchamps Point of a Triangle. *The american mathematical monthly*, 71:176–179, 1964.

[30] N. B. Victor Gutenmacher. *Lines and curves.* Birkhaeuser, Boston, 2004.

[31] P. Yiu. *Introduction to the geometry of the triangle*, 2013. URL: `http://math.fau.edu/Yiu/Geometry.html`.

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SOME PROPERTIES OF INTERSECTION POINTS OF EULER LINE AND ORTHOTRIANGLE

DANYLO KHILKO

Abstract. We consider the points where the Euler line of a given triangle ABC meets the sides of its orthotriangle, i.e. the triangle whose vertices are feet of the altitudes of ABC. In this note we study properties of these points and how they relate to the known objects.

A notable construction occured in [2, Problem 3] and [1, Problem G6]. A problem inspired by this construction initiated the research which we present in this paper. While solving this problem, we discovered several facts about intersection points of the Euler line and the sides of the orthotriangle. Further investigation of these points resulted in facts which we find intersting on their own and decided to share them.

The following notation will be used.

Let ABC be an acute triangle. Its altitudes AH_A, BH_B, CH_C intersect at the orthocenter H. Denote the midpoints of the sides AB, BC, CA by M_C, M_A, M_B, respectively, and the circumcenter of ABC by O. Let X_A be the foot of the perpendicular from A to H_BH_C. Define the points X_B, X_C analogously.

Let us remind reader some classical facts first. The points H_A, H_B, H_C, M_A, M_B, M_C lie on a circle (*the nine-point circle*) centered at O_9 which is the midpoint of OH. The lines AX_A, BX_B, CX_C meet at the point O. The next lemma can be used to prove various facts including IMO2013 3 and IMOSL2012 G6.

Lemma 1. *The circumcircles of the triangles $M_AX_BX_C$, $M_BX_AX_C$, $M_CX_AX_B$ intersect at the point O.*

Proof. Consider the circle ω with diameter OH_A. Since AX_A, BX_B, CX_C meet at O, we have $\angle H_AX_BO = \angle H_AX_CO = \angle H_AM_AO = 90°$. Then the points H_A, M_A, X_B, X_C lie on ω. Similarly we have that the circumcircles of the triangles $M_AX_BX_C$, $M_BX_AX_C$, $M_CX_AX_B$ intersect at the point O. □

One might wonder, whether a similar statement holds for the triangles $M_AM_CX_B$, $M_AM_BX_C$, $M_BM_CX_A$.

The following theorem provides the answer.

Theorem 1. *The circumcircles of the triangles $M_AM_BX_C$, $M_AM_CX_B$, $M_BM_CX_A$ have a common point which belongs to the Euler line.*

Before proving Theorem 1, we establish an auxiliary result. It introduces the key object of the proof, which is the main object of this exposition.

Proposition 1. *Let OH intersects the lines H_BH_C, H_AH_C, H_AH_B at the points K_A, K_B, K_C. Then the points M_B, M_C, X_A, K_A are cyclic. The same holds for the fours of points M_A, M_C, X_B, K_B; M_B, M_A, X_C, K_C.*

We need the following lemma which was proposed on the All-Russian Mathematical Olympiad [3, 2004–2005, District round, Grade 11, Problem 4]

Lemma 2. *Let the lines H_BH_C and M_BM_C meet at the point T_A. Then $AT_A \perp OH$.*

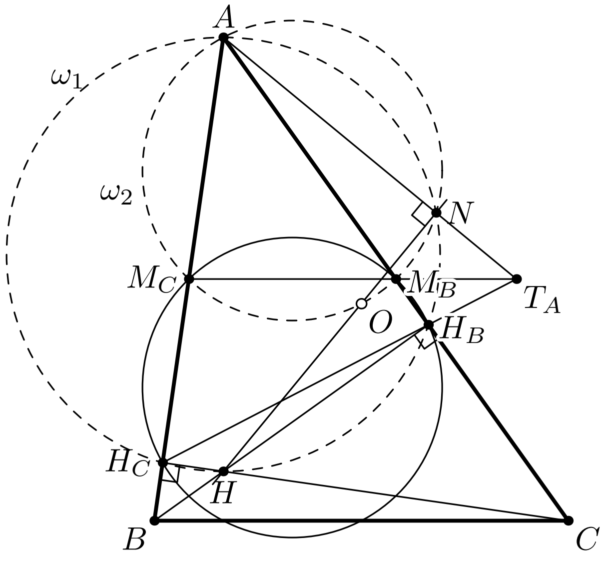

Fig. 1.

Proof. Denote by ω_1 the circumcircle of AH_BH_C (see Fig. 1)). Let OH intersect ω_1 again at the point N. Then $\angle AH_CH = \angle AH_BH = \angle ANH = 90°$. We have that $90° = \angle ANO = \angle AM_BO = \angle AM_CO$. Hence N lies on the circumcircle of AM_BM_C, denoted by ω_2. Consider the circles ω_1, ω_2 and the nine-point circle of ABC. The line AN is the radical axis of ω_1 and ω_2. The line H_BH_C is the radical axis of ω_1 and the nine-point circle. Finally, the line M_BM_C is the radical axis of ω_2 and the nine-point circle. This implies that the line AN passes through T_A. Then $AT_A \perp OH$. □

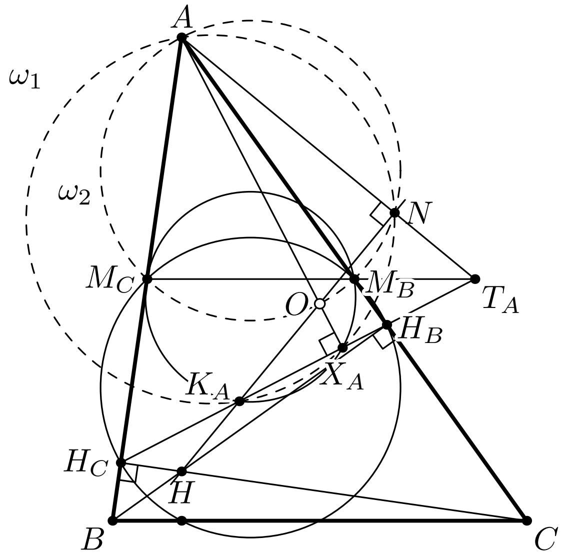

Fig. 2.

Proof of Proposition 1. We will show that $T_AM_B \cdot T_AM_C = T_AX_A \cdot T_AK_A$ (see Fig. 2). By Lemma 2, $\angle ANH = \angle ANK_A = \angle AX_AK_A = 90°$. Then A, N, X_A, K_A are concyclic. We obtain the following equation

$$T_AN \cdot T_AA = T_AX_A \cdot T_AK_A.$$

Also we have $T_AN \cdot T_AA = T_AM_B \cdot T_AM_C$ Hence

$$T_AM_B \cdot T_AM_C = T_AX_A \cdot T_AK_A.$$

Thus the points M_B, M_C, X_A, K_A are concyclic. □

Now we are ready to prove Theorem 1.

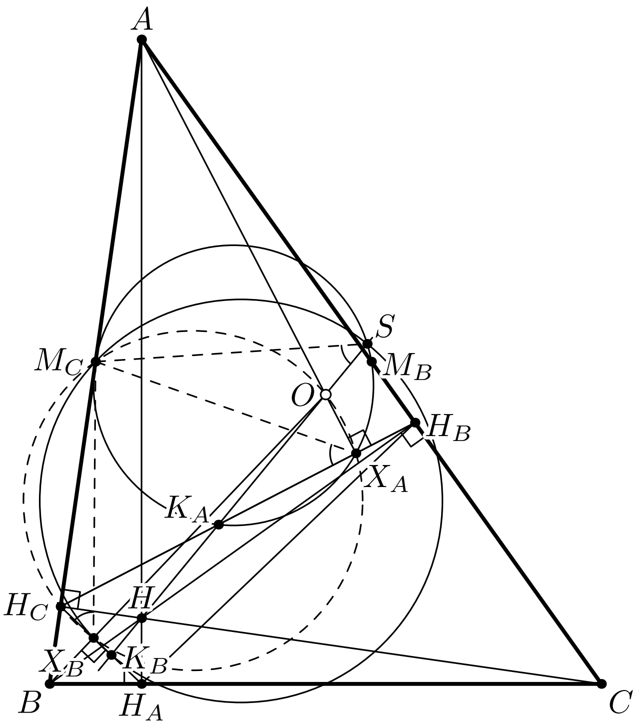

Fig. 3.

Proof of Theorem 1. Let the circumcircle of the quadrilateral $M_CX_BK_BM_A$ meets OH again at S (see Fig. 3). It suffices to show that the point S belongs to the circumcircle of $M_CX_AK_AM_B$, a similar statement for $M_AX_CK_CM_B$ will follow. We will work with oriented angles between lines. Denote by $\angle(l, m)$ the angle of the counterclockwise rotation which maps a line l to one parallel to a line m. See more in [4].

We have

$$\angle(M_CX_B, X_BK_B) = \angle(M_CS, SK_B) = \angle(M_CS, SK_A).$$

From Lemma 2 we obtain that

$$\angle(M_CX_B, X_BK_B) = \angle(M_CX_B, X_BH_C) = \angle(M_CX_A, X_AH_C) = \angle(M_CX_A, X_AK_A).$$

Hence we conclude that

$$\angle(M_CS, SK_A) = \angle(M_CX_A, X_AK_A),$$

and the proof is completed. □

Remark 1. *It is possible to prove the first part of Theorem 1 about three circles by angle chasing using Lemma1, however, this way does not imply that the intersection point of the circles lies on the line OH.*

Having proved Theorem 1, we establish further properties of the points K_A, K_B, K_C.

Theorem 2. *The circumcircles of the triangles $K_AH_AO_9$, $K_BH_BO_9$ and $K_CH_CO_9$ have another common point different from O_9.*

Firstly, we remind that in Lemma 2 we have defined the point T_A as the common point of H_BH_C and M_BM_C. Define the points T_B and T_C analogously.

Proposition 2. *The points K_A, H_A, T_A and O_9 are cyclic.*

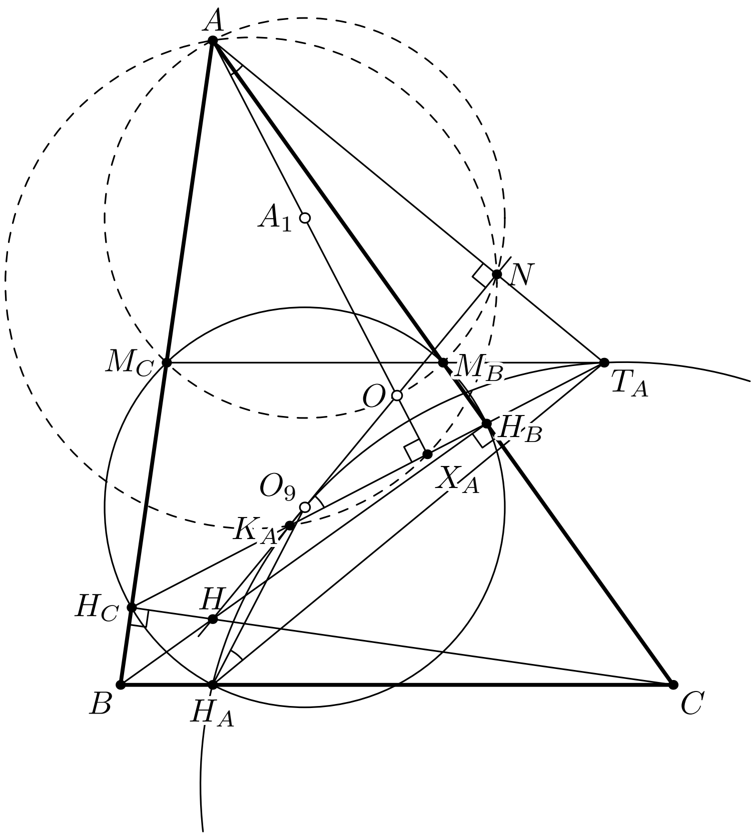

Fig. 4.

Proof. In the proof of Proposition 1 we have obtained that the points A, N, X_A and K_A are cyclic. Therefore,

$$\angle(X_AA, AT_A) = \angle(X_AA, AN) = \angle(X_AK_A, K_AN) = \angle(T_AK_A, K_AO_9).$$

We claim that $\angle(X_AA, AT_A) = \angle(T_AH_A, H_AO_9)$ (see Fig. 4). Indeed, the points A and H_A are symmetric with respect to M_AM_B. Then the nine-point circle and the circumcircle of AM_AM_B are symmetric with respect to M_AM_B. Hence O_9 is symmetric to the point A_1 which is the center of the circumcircle AM_AM_B. As O belongs to this circle and $\angle ANO = 90^\circ$ we have that A_1 lies on AO i. e. AX_A. So we have $\angle(X_AA, AT_A) = \angle(T_AH_A, H_AO_9)$. Then

$$\angle(T_AH_A, H_AO_9) = \angle(X_AA, AN) = \angle(T_AK_A, K_AO_9),$$

and we are done. $\square$

Consider the points T_A, T_B, T_C.

Lemma 3. *The points T_A, T_B, C are collinear.*

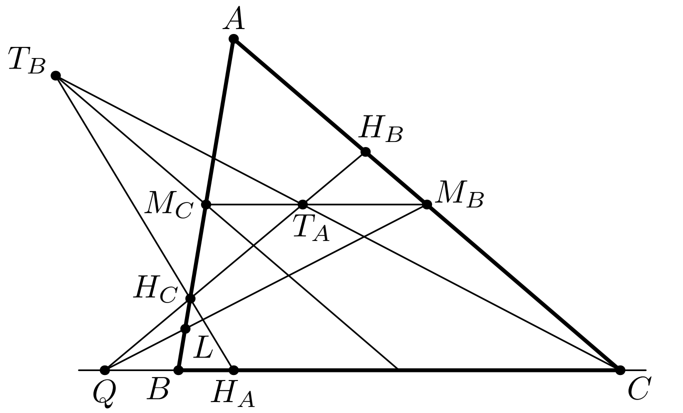

Fig. 5.

Proof. We need some additional notation, which will be used only in this proof. Denote by Q the intersection point of BC and H_BH_C and by L the intersection point of QM_B and AB (see Fig. 5).

Let us apply Desargues' theorem for the triangles H_AH_CQ and the one formed by the lines M_CM_A, M_CM_B, M_BC (this triangle has one vertex at infinity). Then the following statements are equivalent: M_CH_C, M_BQ and the line parallel to CM_B passing through H_A are concurrent and the intersection points of QH_C and M_BM_C, H_AH_C and M_AM_C, QH_A and M_BC are collinear. Notice that H_AH_C meets M_AM_C at T_B, QH_C meets M_CM_B at T_A and QH_A meets M_BC at C. So in order to prove that T_B, T_A, C are collinear we will prove the first statement obtained by Desargues' theorem. It is sufficient to prove that $LH_A \parallel AC$. By Menelaus' theorem

$$\frac{BQ}{QC} \cdot \frac{CM_B}{M_BA} \cdot \frac{AL}{LB} = 1.$$

Then

$$\frac{BL}{AL} = \frac{BQ}{QC}.$$

It is a well-known fact that

$$\frac{BQ}{QC} = \frac{BH_A}{H_AC}.$$

Hence

$$\frac{BL}{LA} = \frac{BH_A}{H_AC},$$

and $LH_A \parallel AC$.

□

Remark 2. *A similar fact to Lemma 3 will hold if one replace the points H_A, H_B, H_C by some points A_1, B_1, C_1 which lie on the respective sides of ABC and AA_1, BB_1, CC_1 are concurrent.*

The next fact describes other properties of T_A, T_B, T_C.

Lemma 4. *The circumcircles of the triangles $T_BT_AH_C$, $T_BT_CH_A$ and $T_CT_AH_B$ have a common point P.*

Proof of Lemma 4. The statement follows from Miquel's theorem applied to the triangle $H_AH_BH_C$ and the points T_C, T_A, T_B (see Fig. 6). □

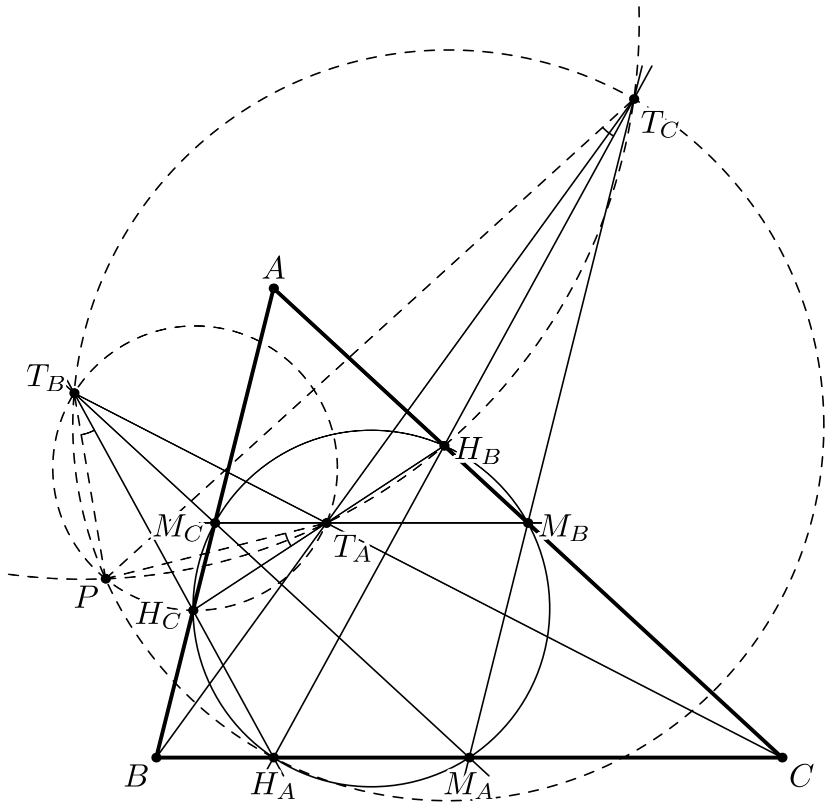

Fig. 6.

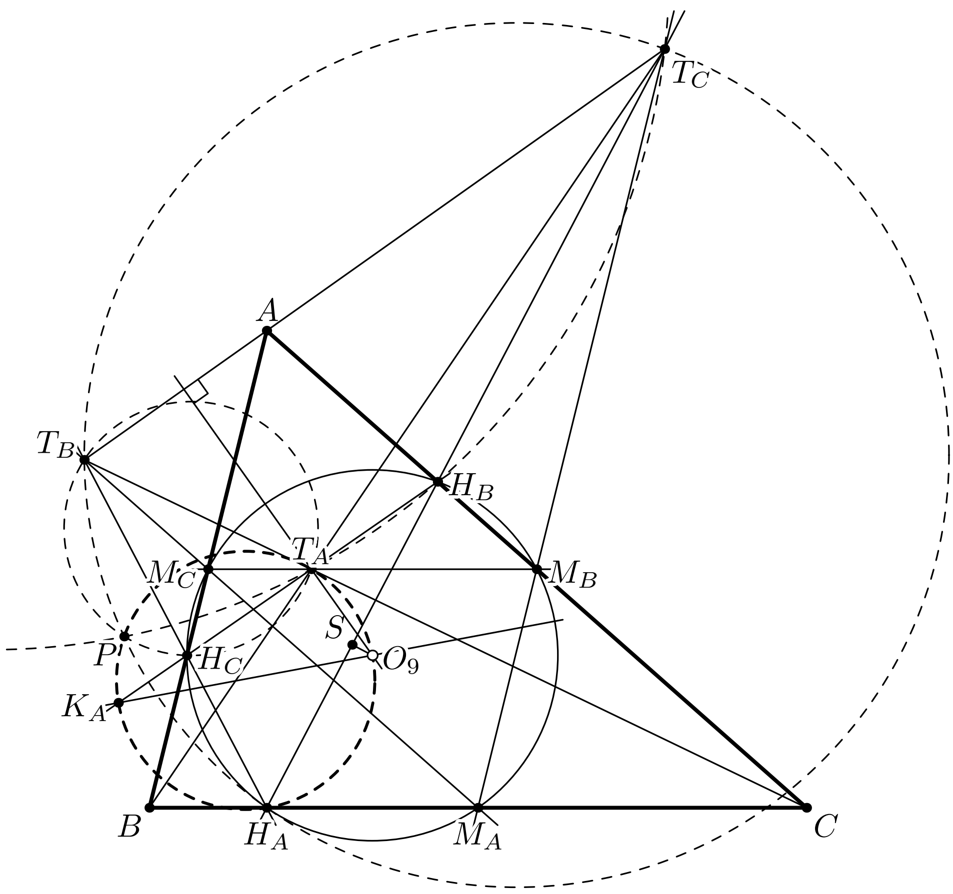

Fig. 7.

Now we claim that the point P lies on the circumcircle of the triangle $K_AH_AO_9$. This fact combined with that for $K_BH_BO_9$ and $K_CH_CO_9$ is equivalent to Theorem 2.

Proof of Theorem 2. In order to prove that the circumcircle of $O_9K_AH_A$ passes through P, we will show that $\angle(T_AP, PH_A) = \angle(T_AO_9, O_9H_A)$ (See Fig. 7).

Firstly, we will prove that $O_9T_A \perp T_BT_C$. Let H_BM_C and H_CM_B intersect at U. It is a well-known fact that AU is the polar line of T_A with respect to the nine-point circle. Applying Pascal's theorem on the hexagon $M_BM_AM_CH_BH_AH_C$ we obtain that T_B, T_C, U are collinear. Hence T_BT_C is the polar line of T_A, consequently, $O_9T_A \perp T_BT_C$. Let S be the foot of the perpendicular from O_9 to H_AH_B. Then $\angle(T_AO_9, O_9S) = \angle(T_BT_C, T_CS)$. Also we have $\angle(SO_9, O_9H_A) = \angle(H_BH_C, H_CH_A)$. So

$$\begin{aligned}
&\angle(T_AO_9, O_9H_A) = \angle(T_AO_9, O_9S) + \angle(SO_9, O_9H_A) = \\
&= \angle(T_BT_C, T_CS) + \angle(H_BH_C, H_CH_A) = \\
&= \angle(T_BT_C, T_CH_A) + \angle(T_AH_C, H_CT_B) = \\
&= \angle(T_BP, PH_A) + \angle(T_AP, PT_B) = \angle(T_AP, PH_A)
\end{aligned}$$

and we are done. □

Theorem 3. *T_BT_C, M_BM_C and K_AH_A are concurrent.*

Proposition 3. *K_AH_A is tangent to the circumcircle of $T_AT_BH_A$ at H_A.*

Proof. We have

$$\begin{aligned}
&\angle(K_AH_A, H_AP) = \angle(K_AT_A, T_AP) = \\
&= \angle(H_CT_A, T_AP) = \angle(H_CT_B, T_BH_C) = \angle(H_AT_B, T_BP).
\end{aligned}$$

□

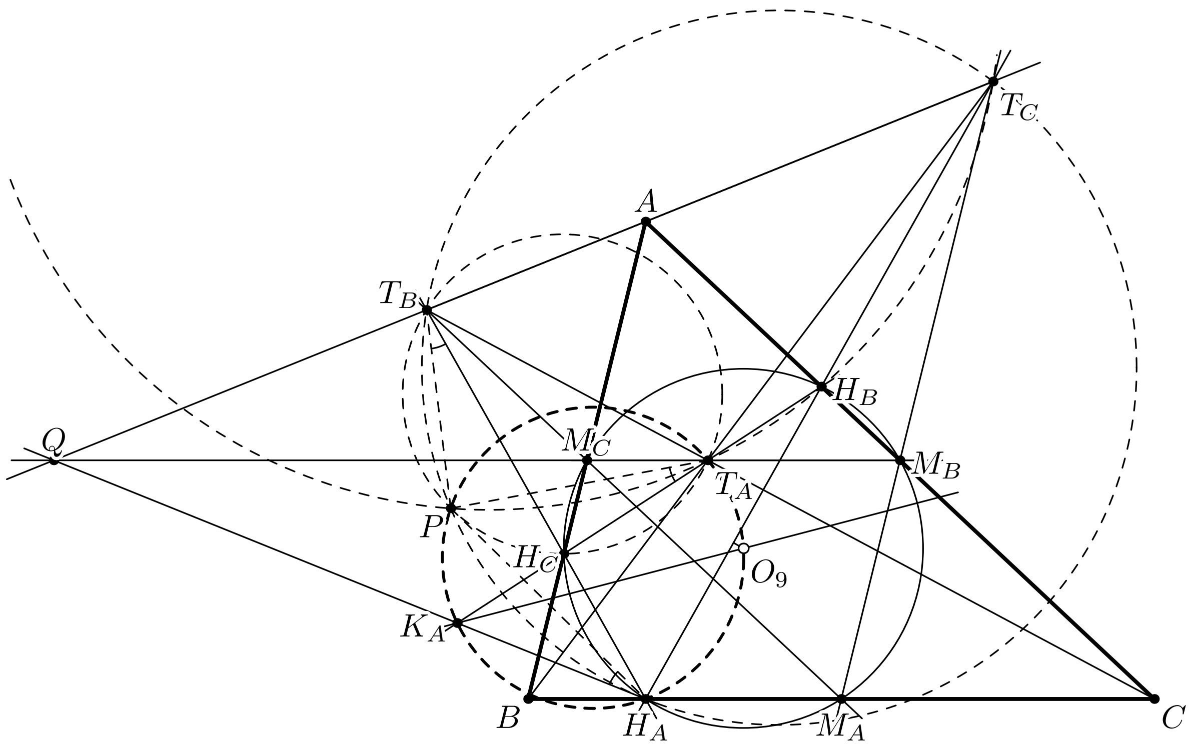

Fig. 8.

Proof of Theorem 3. Let T_BT_C intersect K_AH_A at Q (see Fig. 8). By Proposition 3 QH_A is tangent to the circumcircle of the triangle $T_BH_AT_C$. Also H_AA bisects $\angle T_BH_AT_C$. Hence $AQ = QH_A$ since

$$\angle QAH_A = \angle T_BT_CH_A + \angle AH_AT_C = \angle T_BH_AQ + \angle T_BH_AA = \angle QH_AA.$$

So Q belongs to the perpendicular bisector of AH_A, which is obviously M_AM_B. □

The author is grateful to Vladyslav Vovchenko for a short proof of the Lemma 3 and to Georgiy Shevchenko for careful reading of the manuscript.

References

[1] 53-rd International Mathematical Olympiad, Shortlisted Problems with Solutions. 2012. URL: `https://www.imo-official.org/problems.aspx`.

[2] 54-th International Mathematical Olympiad. 2013. URL: `https://www.imo-official.org/problems.aspx`.

[3] N. K. Agakhanov, I. I. Bogdanov, P. A. Kozhevnikov, O. K. Podlipsky, and D. A. Tereshin. *All-russian mathematical olympiad 1993–2006*. M.:MCCME, 2007.

[4] V. V. Prasolov. *Problems in plane geometry*. Russian. M.:MCCME, 2006.

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ON GENERALIZED BROCARD ELLIPSE

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Abstract. For a fixed point P and a fixed circle Ω consider a conic (the generalized Brocard ellipse) that touches lines XY inclined at a fixed angle to PX, where $X \in \Omega$. For this construction, we prove some facts that allow to obtain more properties of harmonic quadrilaterals.

1. Introduction

Consider next construction (*the Brocard construction*).

Let Ω be a circle with center O and radius R, let P be a fixed point, $P \neq O$, $P \notin \Omega$; and let φ be a fixed (oriented) angle, $0 < \varphi < \frac{\pi}{2}$. For an arbitrary point $X \in \Omega$ take $Y \in \Omega$ such that $\angle(YX, XP) = \varphi$.

For this construction, we recall the following facts from [1, section 4.6] or [3] (see also some proofs in Appendix).

(1) Lines XY are tangent to a conic $\varepsilon = \varepsilon(\Omega, P, \varphi)$. If P lies inside Ω, ε could be considered as *the generalized Brocard ellipse*.

(2) Point P is one of two foci of ε, and the second one is a point Q such that $OP = OQ$ and $\angle(OQ, OP) = 2\varphi$ (see Fig. 1).

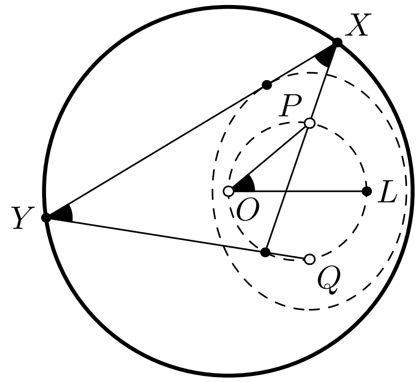

Fig. 1.

(3) The length of the great axis of generalized Brocard ellipse $\varepsilon(\Omega, P, \varphi)$ is equal to $2R \sin \varphi$.

We can reformulate this fact in the following from. Let us fix Ω and φ, and take points P_1, $L_1, Q_1 \in \Omega$ such that $\angle(OQ_1, OL_1) = \angle(OL_1, OP_1) = \varphi$. Note that $P_1Q_1 = 2R \sin \varphi$. While P and Q move along lines OP_1 and OQ_1 (so that $OP = OQ$) conics $\varepsilon(\Omega, P, \varphi)$ touch fixed lines passing through P_1 and Q_1 parallel to OL_1.

(4) Let K be a limit point of the pencil formed by Ω and *the Brocard circle* (POQ). Let lines XK and YK meet Ω for the second time at X' and Y'. Then all chords $X'Y'$ have equal lengths (Fig. 2).

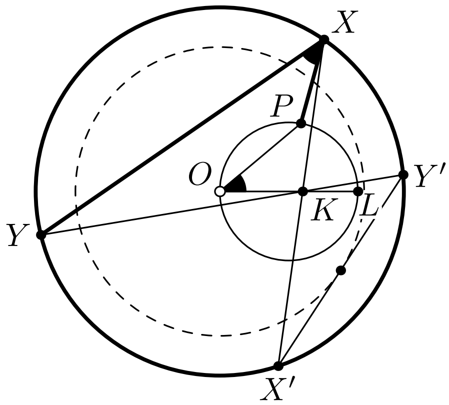

Fig. 2.

From this it follows that there exists a projective transformation π that preserves Ω and takes ε to a circle centered at O. Therefore, ε twice touches Ω. The touching points are points S_1 and S_2 that are common points of circles Ω and (POQ) (the touching points may be complex). Indeed, S_1O is the bisector of $\angle PS_1Q$, and $PS_1 + QS_1 = 2R\sin\varphi$ (one can perform a rotation taking triangle OQS_1 to OPS_1' so that S_1, P, S_1' are collinear; thus $PS_1 + QS_1 = PS_1 + PS_1' = S_1S_1' = 2OS_1\cos\angle OS_1P = 2R\sin\varphi$).

Note that π restricted to Ω is the central projection with center K, thus $\pi(K) = K$. Let L_1L_2 be the diameter of Ω containing K and L. It is easy to show equality of double ratio $(L_1, K, L, L_2) = (L_2, K, O, L_1)$ that gives $\pi(L) = O$.

(5) Let $A_1 \dots A_n$ be a polygon inscribed into Ω and such that $\angle(A_{i+1}A_i, A_iP) = \varphi$ for $i = 1, \dots, n$ (here $A_{n+1} = A_1$). Then $\angle(A_{i-1}A_i, A_iQ) = -\varphi$, and there exist an infinite set of polygons having these properties. Such polygons are called *Brocard polygons*, and the points P, Q are its *Brocard points*. All sides of the Brocard Polygon touch the same ellipse with foci P, Q. We will call this ellipse *the Brocard ellipse*. The Brocard polygon can be transformed to a regular polygon by an inversion (with center K) or by a projective transformation. In particular, a quadrilateral is a Brocard quadrilateral iff it is harmonic.

2. On points of tangency

In this section we continue using the notation from the Introduction. Projective arguments could be used to obtain the following description of points where ε touches XY.

Proposition 1. *Suppose $X, Y \in \Omega$ are points such that XY touches conic $\varepsilon(\Omega, P, \varphi)$ at point T, and let R be the pole of XY with respect to Ω; then L, T, and R are collinear.*[1]

Proof. Consider a projective transformation π from the fact 4 of the Introduction. We know that $\pi(\Omega) = \Omega$, $\pi(L) = O$, $\pi(\varepsilon) = \omega$, where ω is a circle centered at O. Let $\pi(X) = X' \in \Omega$, $\pi(Y) = Y' \in \Omega$, $\pi(T) = T'$, $\pi(R) = R'$. Note that $X'Y'$ touches ω at T', and R' is the pole of $X'Y'$ with respect to Ω. Obviously O, T', R' are collinear (belong to the perpendicular bisector of $X'Y'$), hence L, T, R are also collinear. $\square$

Below we present two more proofs of the previous Proposition using elementary geometry arguments only. The first of these two proofs was found by Nikita Nesterov.

[1]This fact reformulated in elementary geometry terms was proposed as a problem at Mathematical competition XVII Kolmogorov Cup [2, Third round, Senior level, Problem 5].

Alternative proof 1. Let $T' = RL \cap XY$. It is sufficient to establish equality $\angle(QT', T'Y) = \angle(XT', T'P)$ (see Fig. 3).

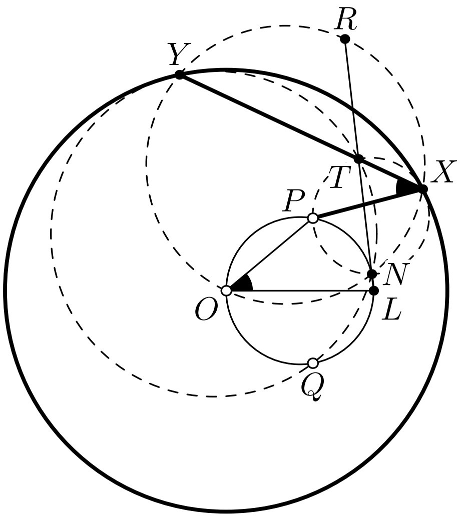

Fig. 3.

Let RL intersect (POQ) for the second time at N. We have $RN = LN \perp ON$, hence N belongs to circle $(OXRY)$ with diameter OR. From this circle $\angle(YN, NR) = \angle(RN, NX)$. Further, $\angle(T'N, NP) = \angle(LN, NP) = \angle(LO, OP) = \varphi = \angle(T'X, XP)$. We obtain that P, X, T', N are concyclic. Similarly, Q, Y, T', N are concyclic. From circles $(PXT'N)$ and $(YQT'N)$ we have $\angle(YQ, QT') = \angle(YN, NT') = \angle(YN, NR) = \angle(RN, NX) = \angle(T'N, NX) = \angle(T'P, PX)$. Note that triangles $QT'Y$ and $PT'X$ have two pairs of equal angles ($\angle(YQ, QT') = \angle(T'P, PX)$ and $\angle(QY, YT') = \angle(T'X, XP) = \varphi$), hence the remaining angles $\angle(QT', T'Y)$ and $\angle(XT', T'P)$ are equal. $\square$

Remark 1. *One could easily check that circles $(PXT'N)$ and $(YQT'N)$ (from the proof above) are tangent to Ω, and R is the radical center of circles $(PXT'N)$, $(YQT'N)$, Ω.*

Alternative proof 2. Let Q' be the reflection of Q in XY, and $T' = PQ' \cap XY$. It is sufficient to prove that R, T', L are collinear (see Fig. 4).

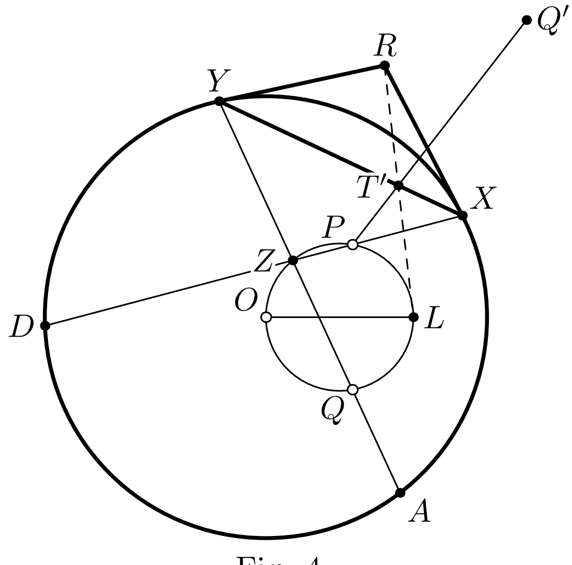

Fig. 4.

Let YZ and XZ intersect Ω for the second time at A and D, respectively. Then $AXYD$ is an inscribed trapezoid having axis of symmetry ZO. Note that O is the midpoint of the arc AZX in the circle (AZX) (since $\angle AZX = \angle YZD = \overset{\frown}{AX} = \angle AOX$).

Now let us fix $AXYD$ (thus points O, Z, R are fixed), and let P and Q move at a constant speed along XD and AY, respectively (with equal values of velocity; thus vectors of velocity of P and Q are symmetric with respect to direction XY). At any fixed moment we have $\angle(OP, PZ) = \angle(OQ, QZ)$, thus P, O, Q, Z are concyclic. Vectors of velocity of P and Q' are equal, hence T' is moving linearly. The center of the circle (POQ) is moving linearly along the perpendicular bisector of OZ, therefore L (that is the intersection point of (POQ) and the bisector ℓ of angle AZX) is moving linearly along ℓ.

Now it is sufficient to specify two cases of location of P, Q, and check that R, T', L are collinear in these cases. One of such cases is $P = X$, $Q = A$. In this cases $T' = X$, and it is easy to see that $L = RX \cap \ell$. The second case is analogous: $P = D$, $Q = Y$. $\square$

Remark 2. *In fact the arguments the last proof of Proposition 1 could be used to prove the existence of the generalized Brocard ellipse. Note that* $PQ' = XA = 2R\sin\varphi$ *in accordance with fact 3 from the Introduction.*

3. Case of a quadrilateral

3.1. **Inscribed quadrilateral.** We start with the following constructions for an inscribed quadrilateral (further in the case of harmonic quadrilateral we show the connection of this construction and the generalized Brocard ellipse).

From here to the end of this section we use the following notation. Let $ABCD$ be a quadrilateral inscribed to a circle Ω centered at O (see Fig. 5)

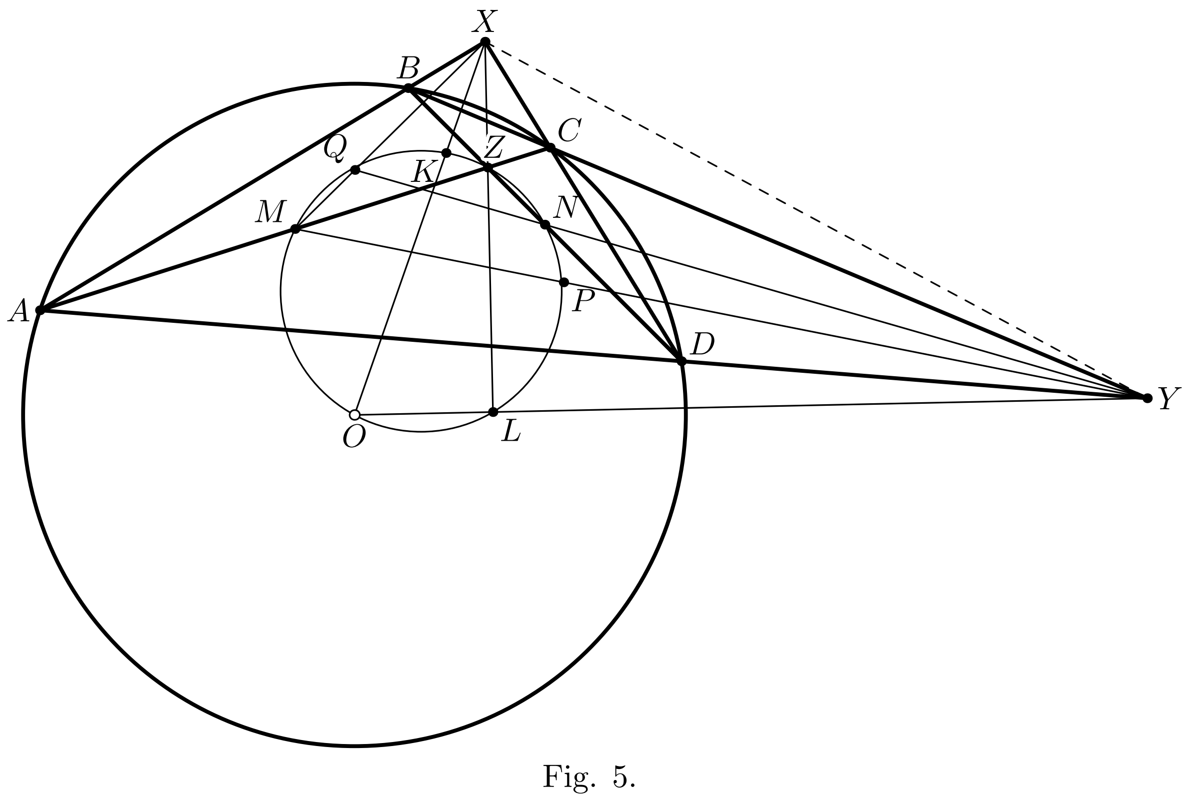

Fig. 5.

Let $X = AB \cap CD$, $Y = BC \cap DA$, $Z = AC \cap BD$, let $K = OX \cap YZ$, $L = OY \cap XZ$. Note that X and Y belong to the polar line of Z, hence $OZ \perp XY$. Similarly $OX \perp YZ$,

$OY \perp ZX$, thus O, X, Y, Z is an orthocentric quadruple. From that it follows that $OX \perp YK$, $OY \perp XL$.

Let M, N, T be midpoints of AC, BD, XY, respectively. Note that M, N, T are collinear (Gauss-Newton line for lines AB, BC, CD, DA), and points O, Z, K, L, M, N belong to the circle ω with diameter OZ.

Let $P = XN \cap YM$, $Q = XM \cap YN$.

Proposition 2. *Line XY is the radical axis of circles Ω and ω.*

Proof. Let U and V be the intersection points of Ω and XZ. Since XZ is the polar line of Y with respect to Ω, we have $OU \perp YU$, $OV \perp YV$, hence U and V belong to circle (OKY) with diameter OY. From that it follows that $XK \cdot XO = XU \cdot XV$, thus X has equal powers with respect to circles Ω and ω. Similarly, Y has equal powers with respect to Ω and ω. □

Proposition 3. *Quadruples $(ABOK)$, $(ABZL)$, $(BCOL)$, $(BCZK)$, $(CDOK)$, $(CDZL)$, $(DAOL)$, $(DAZK)$ are cyclic.*

Proof. From Proposition 2 it follows that $XK \cdot XO = XA \cdot XB = XZ \cdot XL$. This means that quadruples $(ABOK)$, $(ABZL)$ are cyclic. Similarly for the other quadruples. □

Further we need the following

Lemma 1. *Let OXY be a triangle. Suppose that XL, YK are its altitudes, Z is the orthocenter, T is the midpoint of XY (see Fig. 6).*

Let a line through T intersect circle $\omega = (OZKL)$ at M, N. Let $P = XN \cap YM$, $Q = XM \cap YN$. Then lines ON, ZM, KQ, LP are concurrent (or parallel) at some point W, and lines OM, ZN, KP, LQ are concurrent (or parallel) at some point W'. Moreover, $PQ \parallel XY \parallel WW'$; and WW' coincides to XY iff P and Q belong to $(OZKL)$.

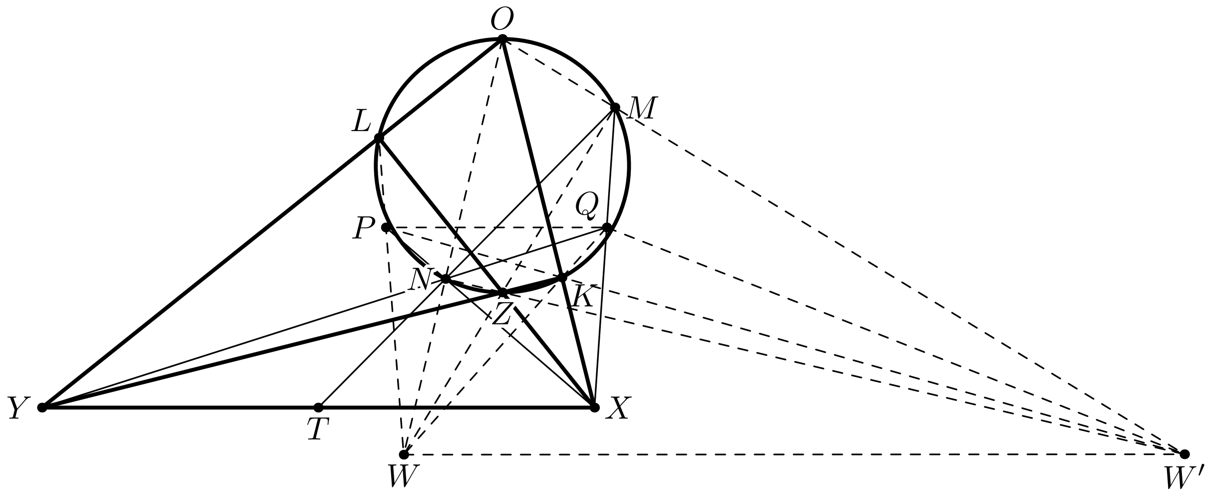

Fig. 6.

Proof. Let $\Pi(O)$, $\Pi(Z)$ be the pencils of lines passing through O and Z respectively. Then the correspondence $F : \Pi(O) \to \Pi(Z)$ transforming ON to MZ save the cross-ratios. Hence by Sollertinsky lemma (see lemma 4.4 in [1] or the Appendix) $W = ON \cap MZ$ lies in a conic Γ passing through O and Z. Point $W' = OM \cap NZ$ also belongs to Γ. It is easy to show that T is the pole of KL (with respect to ω), hence K, $L \in \Gamma$. Applying Pascal theorem to six points O, Z, W, W', K, L obtain that $P = WL \cap W'K$ belongs to XN and to YM, while $Q = WK \cap W'L$ belongs to XM and to YN.

In triangle XYM cevians XP and YQ intersect at N that belongs to median MT, hence $PQ \parallel XY$, and MN bisects the segment PQ. Since T is the pole of KL with respect to ω, PQ, KL, and MN are concurrent at some point U (both lines PQ, KL intersect line MNT at a point S such that quadruple M, S, N, T is harmonic).

Note that WW' is the polar line of $OZ \cap MN$, hence $XY \parallel WW'$. Moreover, $XY = WW'$ iff MN passes through $U = OZ \cap KL$. In this case P and Q are symmetric in OZ, and quadruple $LP \cap XY$, $LQ \cap XY$, X, Y is harmonic. By these conditions the pair P, Q is defined uniquely. But points of intersection of PQ and ω also satisfy these conditions, this means that $P, Q \in \omega$. □

Remark 3. *From the proof we see that P, Q belong to a conic E passing through K, L.*

Let us sketch one another proof of Lemma that works if circle $(OZKL)$ does not intersect line XY.

Proof. Consider a projective transformation s preserving circle ω and taking XY to infinity. Then $O^*K^*Z^*L^*$ and $M^*P^*N^*Q^*$ are rectangles with parallel sides (here for images of points we use the same letters provided with a star). Further, O^*, Z^*, K^*, L^*, M^*, $N^* \in \omega$, and $M^*N^* \perp K^*L^*$ ($MKNL$ is harmonic, and the same is true for $M^*K^*N^*L^*$). Points $O^*L^* \cap M^*Q^*$, $O^*K^* \cap M^*P^*$, $K^*L^* \cap M^*N^*$ are collinear since they are projections of M^* to lines O^*L^*, O^*K^*, K^*L^* (Simson line). By Desargue's Theorem triangles $O^*K^*L^*$ and $M^*P^*Q^*$ are perspective. Similarly, triangles $Z^*K^*L^*$ and $N^*P^*Q^*$ are perspective, thus O^*M^*, Z^*N^*, K^*P^*, L^*Q^* are concurrent at a point W'^*. Analogously O^*N^*, Z^*M^*, K^*Q^*, L^*P^* are concurrent at a point W^*.

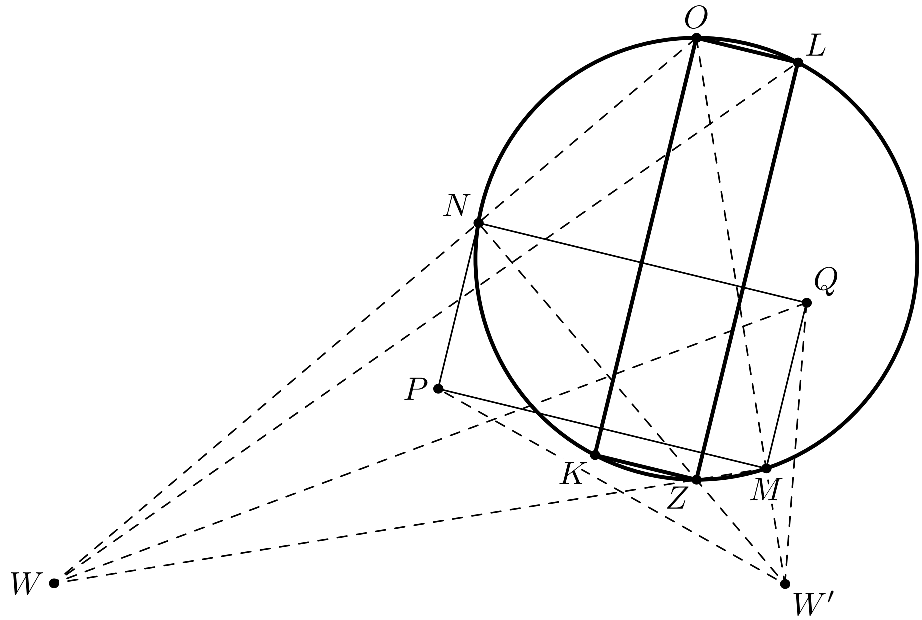

Fig. 7.

Note that O, Z, W, W' is an orthocentric quadruple, hence $WW' \perp OZ$, $OZ \perp PQ$, and s preserves this orthogonality.

Finally, s takes XY to the line at infinity, thus WW' coincides to XY iff $O^*M^* \parallel N^*Z^*$ and $O^*N^* \parallel M^*Z^*$ iff M^*N^* is a diameter of ω iff PQ is a diameter of ω. □

Remark 4. *One can find more properties of the construction considered in Lemma, for example, the following. Let us rotate MN around T. It occurs that the locus of points P*

(or Q) is a conic Δ passing through K, L centered at the midpoint of KL (again it could be proved by Sollertinsky lemma). Note that Γ and Δ touch each other at K and L, and thier tangents at K and L are parallel to X and Y (one can show this considering the case when M and N tends to K or L). The asymptotes of Γ coincide to the axes of Δ and are parallel to the bisectors of angles between OZ and TF, where F is the midpoint of OZ (one can show this considering the case when MN tends to a diameter of ω, in this case W tends to infinity, and $KPLQ$ tends to a rectangle whose sides are parallel to asymptotes of Γ).

Now continue working with quadrilateral $ABCD$. By Lemma, lines ON, $ZM = AC$, KQ, LP are concurrent (or parallel) at some point W, and lines OM, $ZN = BD$, KP, LQ are concurrent (or parallel) at some point W'.

Proposition 4. *$W \in XY$ iff $P, Q \in \omega$ iff $ABCD$ is a harmonic quadrilateral.*

Proof. The first equivalence follows directly from the previous Lemma.

The pole of BD with respect to Ω lies in XY (since XY is the polar line of $Z \in BD$). We have $ON \perp BD$, hence $ON \cap XY$ is the pole of BD. Thus $W \in XY$ iff W is the pole of BD iff the pole of BD belongs to AC iff $ABCD$ is harmonic. $\square$

3.2. **The Brocard points of a harmonic quadrilateral.** Further assume that $ABCD$ is harmonic. By Proposition 4, in this case P, $Q \in \omega$, P and Q are symmetric in OZ, $W \in XY$. Let us mention that equality of arcs PZ and QZ means that $\angle(MX, AC) = \angle(AC, MY)$.[2]

Proposition 5. *Quadruples $(ABQM)$, $(ABPN)$, $(BCQN)$, $(BCPM)$, $(CDQM)$, $(CDPN)$, $(DAQN)$, $(DAPM)$ are cyclic.*

Proof. Now $P, Q \in \omega$, thus from Proposition 2 we have $XQ \cdot XM = XA \cdot XB = XP \cdot XN$. This means that quadruples $(ABQM)$, $(ABPN)$ are cyclic. Similarly for other quadruples. $\square$

Proposition 6. *Points P and Q are the Brocard points for $ABCD$, and each of angles $\angle(XM, AC)$, $\angle(AC, YM)$, $\angle(BD, XN)$, $\angle(YN, BD)$ equals to the Brocard angle.*

Proof. Denote $\varphi = \angle(QO, OZ) = \angle(OZ, OP)$. From $(ABQM)$ we have $\angle(QB, BA) = \angle(QM, MZ) = \varphi$. Similarly we get $\varphi = \angle(QC, CB) = \angle(QD, DC) = \angle(QA, AD) = \angle(PA, AB) = \angle(PB, BC) = \angle(PC, CD) = \angle(PD, DA)$. $\square$

Now the generalized Brocard ellipse $\varepsilon = \varepsilon(\Omega, P, \varphi)$ with foci P, Q is inscribed into quadrilateral $ABCD$. Let us mention that MN passes through the midpoint R of PQ that is the center of $\varepsilon(\Omega, P, \varphi)$ (OZ, PQ, KL, MN are concurrent at R). [3]

Proposition 7. *The ellipse ε of $ABCD$ touches BC at point $BC \cap XZ$.*

Proof. Line BC passes through Y, hence the pole of BC with respect to Ω lies in XZ. From Proposition 1 it follows that the tangency point lies in XZ. $\square$

Remark 5. *From 7 we see that Y is the pole of XZ with respect to ε. Similar statemets are true for X and Z. Thus X, Y, Z is an autopolar triple with respect to ε.*

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[2]This is in accordance with a known property of a harmonic quadrilateral: MX and MY are equally inclined to AC.

[3]This is in accordance with Newton's theorem: Gauss-Newton's line MN passes through centers of all conics inscribed to $ABCD$.

Proposition 8. *Quadruples* $(ACON)$, $(ACKQ)$, $(ACLP)$, $(BDOM)$, $(BDKP)$, $(BDLQ)$ *are cyclic (Fig. 8).*

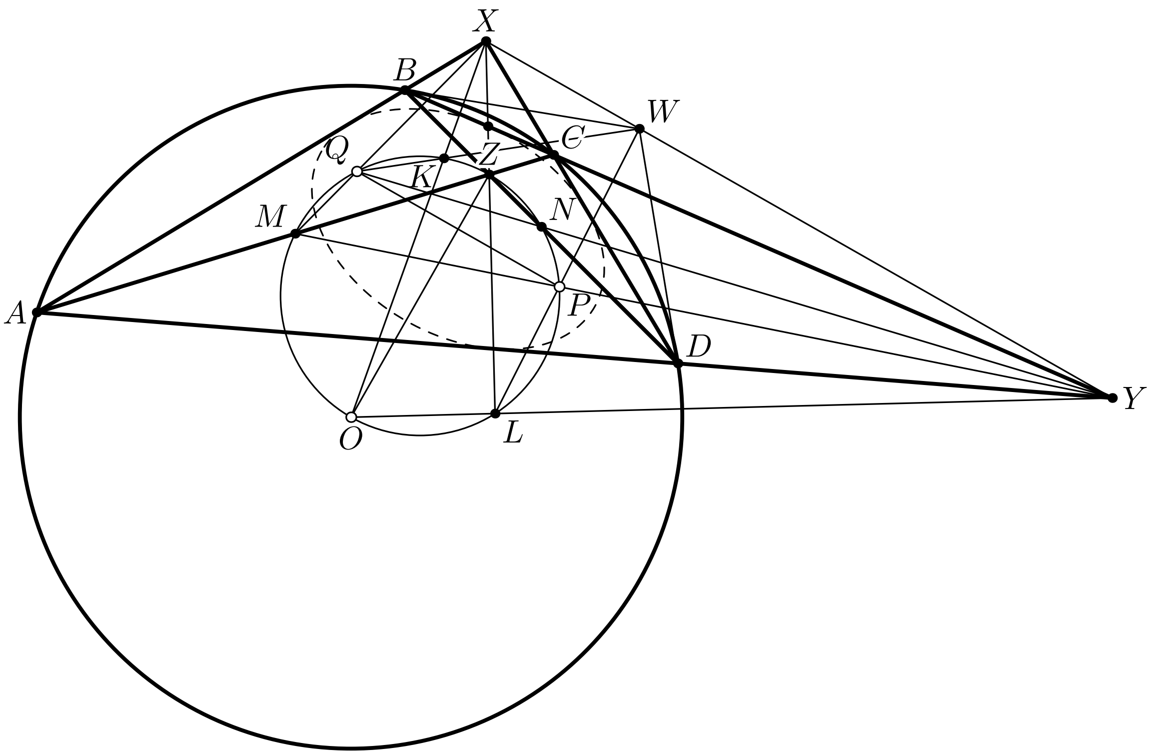

Fig. 8.

Proof. By Propositions 2 and 4, W has equal powers with respect to ω and Ω. Therefore, $WA \cdot WC = WO \cdot WN = WK \cdot WQ = WL \cdot WP$, hence quadruples $(ACON)$, $(ACKQ)$, $(ACLP)$ are cyclic. Similarly for quadruples $(BDOM)$, $(BDKP)$, $(BDLQ)$. □

It is known that in a harmonic quadrilateral there exists an inscribed ellipse $\varepsilon(M, N)$ with foci M, N (it could be shown from angle equalities $\angle(BM, BA) = \angle(BC, BN)$, $\dots$).

Proposition 9. *Ellipses* $\varepsilon(M, N)$ *and* ε *are similar.*

Proof. Consider a composition of symmetry in the bisector of angle AYB and homothety with center Y taking Q, P to M, N. (Such a transformaton exists since $\angle(BY, YN) = \angle(MY, YA)$ and $\angle(YM, MN) = \angle(PQ, QY)$.) This transformation takes ε with foci P, Q to the ellipse ε' with foci M and N toching YA and YB. From the uniqueness of such an ellipse it follows that $\varepsilon' = \varepsilon(M, N)$. □

4. Appendix

4.1. **Proofs or the properties of the Brocard construction. Assertion 1.** Let XY be the base of an isosceles triangle XYZ with sideline XZ passing through P. Then Z lies on circle POQ and sideline YZ passes through Q (Fig. 9).

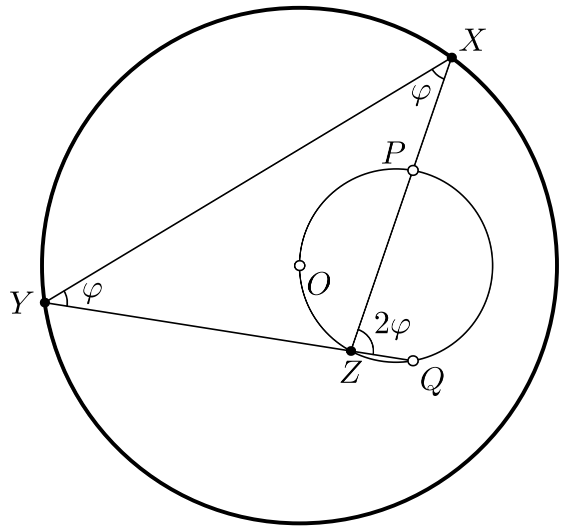

Fig. 9.

Proof. Let L be a point opposite to O in the *Brocard circle* (POQ) (L is the *generalized Lemoine point*). Suppose that PX intersects circle (POQ) for the second time at Z. Note that $\angle(QZ, ZL) = \angle(LZ, ZP) = \varphi$. We have $\angle(QZ, ZL) = \angle(YX, XP) = \varphi$, hence $ZL \parallel XY$. Therefore, $ZO \perp XY$ (since $ZO \perp ZL$). This means that X and Y are symmetric in ZO, hence $\angle(ZY, YX) = \angle(YX, XZ) = \varphi$. Further, $\angle(YZ, ZL) = \angle(ZY, YX) = \varphi = \angle(QZ, ZL)$, thus Y, Z, Q are collinear. □

Assertion 2. All lines XY touch the same ellipse with foci P and Q whose major axis equals $2R \sin \varphi$

Proof. Let P' be the reflection of P in XY (Fig. 10). Then triangle PXP' are POQ similar, thus triangles OPX and QPP' are also similar, i.e.

$$QP' = OX \frac{PQ}{OP} = 2R \sin \varphi$$

do not depend on X. Therefore the common point of lines XY and QP' lies on the ellipse with foci P, Q whose major axis equals $2R \sin \varphi$, and XY touches this ellipse. □

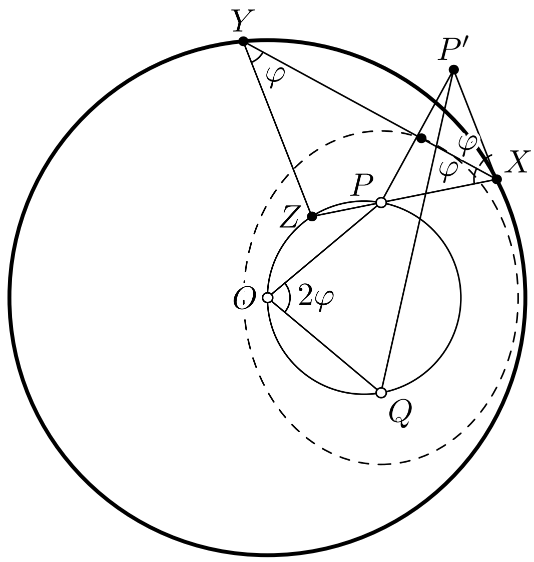

Fig. 10.

Assertion 3. The length of chord $X'Y'$ do not depend on X.

Proof. We have

$$X'Y' = XY\frac{KY'}{KX} = \frac{XY \cdot (R^2 - OK^2)}{KX \cdot KY}.$$

Since K is the limit point of the pencil of circles the ratio of KX^2 and the power of X wrt POQ does not depend on X. Therefore the ratio $XY/(KX \cdot KY)$ is proportional to

$$\frac{XY}{\sqrt{XZ \cdot XP \cdot YZ \cdot YQ}} = \frac{XY}{XZ\sqrt{XP \cdot YQ}} = \frac{2\cos\varphi}{\sqrt{XP \cdot YQ}}.$$

Since XP and YQ are two lines passing through the foci of the ellipse and forming a fixed angle with a tangent to it their product is constant. Therefore the length of $X'Y'$ is also constant. □

4.2. **The Sollertinsky lemma.** Let A, B be two fixed point and let $f : \Pi(A) \to \Pi(B)$ be a transformation conserving the cross-ratios. Then the locus of points $\ell \cap f(\ell)$, $\ell \in \Pi(A)$ is a conic passing through A, B. When $f(AB) = AB$ this conic is degenerated to the union of AB and some other line.

Proof. Take three lines x, y, $z \in \Pi(A)$ and let $X = x \cap f(x)$, $Y = y \cap f(y)$, $Z = z \cap f(z)$. Consider a conic Γ passing through A, B, C, X, Y. For an arbitrary point W of Γ we have $(AX, AY, AZ, AW) = (BX, BY, BZ, BW)$. Therefore $f(AW) = BW$ and W lies on the desired locus. The converse statement (i.e. each point of the locus belongs to Γ) could be proved in the same manner. □

References

[1] A. Akopyan and A. Zaslavsky. *Geometricheskie svoystva krivikh vtorogo poryadka, second edition.* MCCME, 2011.

[2] XVII Kolmogorov Cup. URL: `http://cdoosh.ru/kolm/kolm.html`.

[3] A. Zaslavskiy, O. Zaslavskiy, P. Kozhevnikov, and A. Akopyan. Brocard points, 25 summer conference International mathematical Tournament of towns. 2013. URL: `http://www.turgor.ru/lktg/2013/3/index.htm`.

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PROBLEM SECTION

In this section we suggest to solve and discuss problems provided by readers of the journal. The authors of the problems do not have purely geometric proofs. We hope that interesting proofs will be found by readers and will be published. Please send us your solutions by email: `editor@jcgeometry.org`, as well as interesting "unsolved" problems for publishing in this Problems Section.

Y. Diomidov, V. Kalashnykov, **Trilinear polar and Poncelet's rotation**

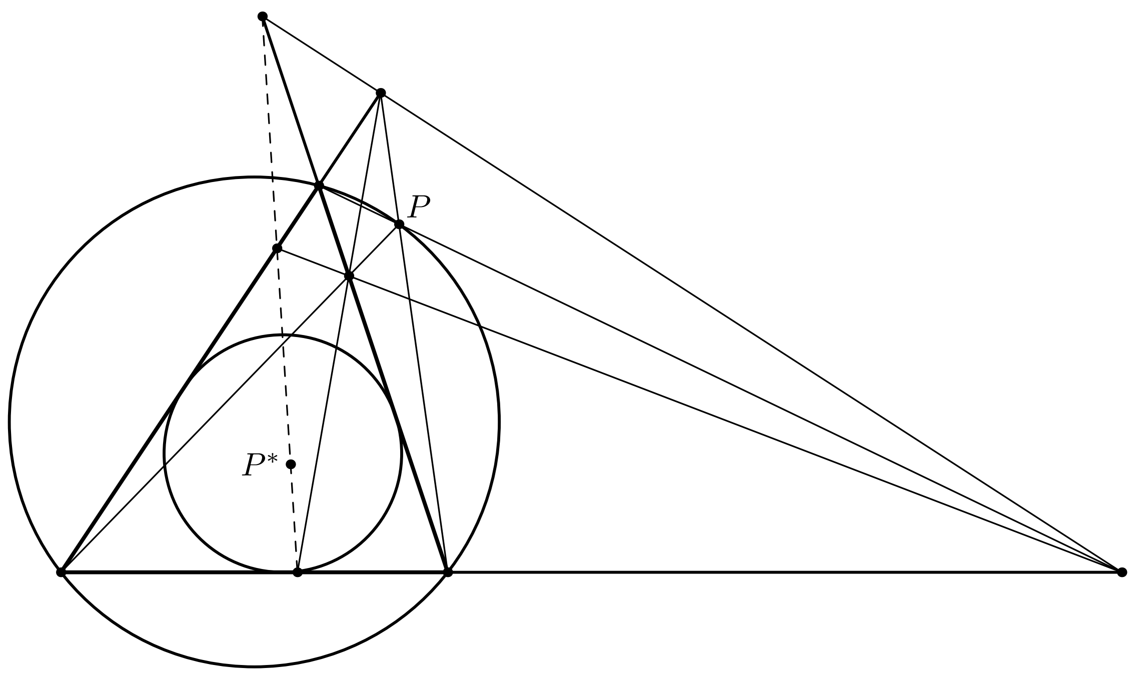
For formulation of the statement of this problem let us recall one corollary of Poncelet's theorem:

Let ω and Ω be inscribed and circumscribed circles of a triangle. Then for any point A on Ω there exists a triangle T with vertex at A inscribed in Ω and circumscribed around ω.

The rotation of the triangle T with the point A we call *Poncelet's rotation*.

For the following problem the authors do not have a geometrical proof.

Open Problem. *Let T be a Poncelet triangle rotated between two circles and P be an fixed point on its circumcircle. Then the trilinear polar of P with respect T passes through a fixed point.*

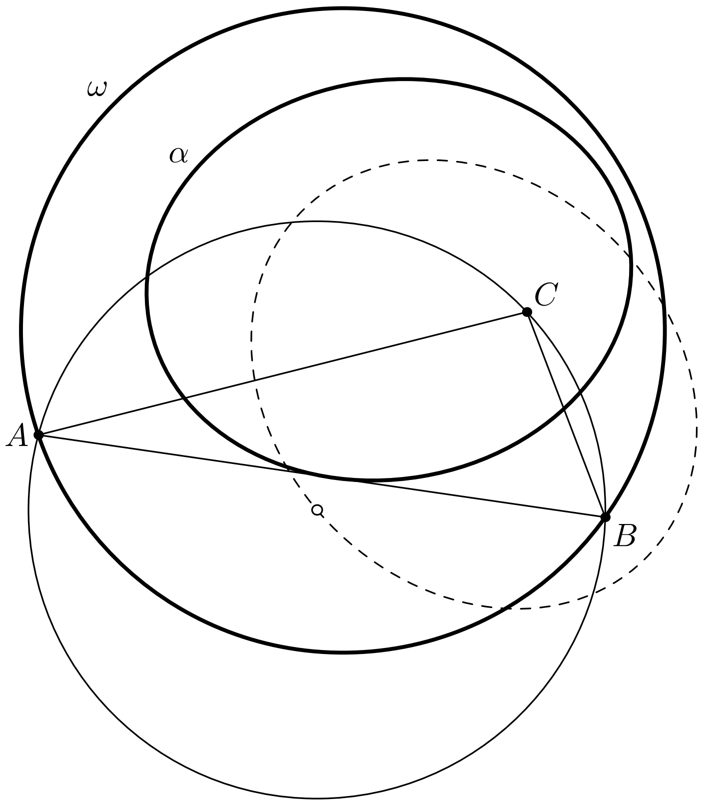


P. A. Kozhevnikov, A. A. Zaslavsky, **A conic of circumcenters**

Open Problem. *Let ω be a circle, and let α be an ellipse lying inside it. Let C be a fixed point in the plane. Describe the locus of circumcenters of triangles ABC, where AB is a chord of the circle touching the ellipse.*

Using elementary algebra approach it is not hard to show that this locus is a conic. Indeed, the locus is a polynomial curve and it is not hard to see that only two points of it lie on the infinite line.

However the authors can not describe the conic geometrically.

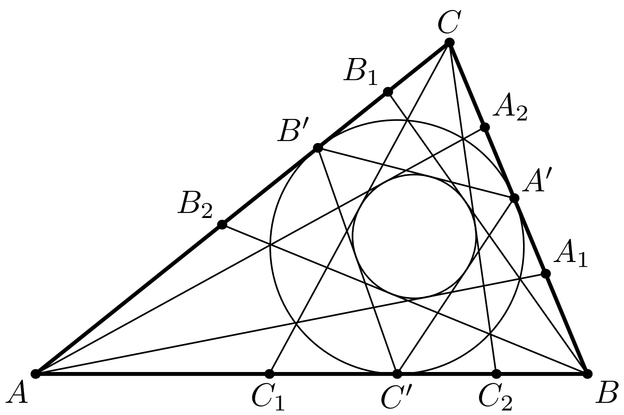


P. Dolgirev, **Interesting circle**

Open Problem. *Let $\triangle A'B'C'$ be the Gergonne triangle of a triangle ABC. Draw tangent lines from the vertices A, B, C to the incircle of the triangle $A'B'C'$ and denote by A_1, A_2, B_1, B_2, C_1, C_2 intersections of these tangent lines with sides of triangle ABC. Prove that the six points A_1, A_2, B_1, B_2, C_1, C_2 lie on a same circle.*

One of the ways to solve the problem is to show that this circle belongs to the pencil generated by incircles of triangles ABC and $A'B'C'$. For that it is enough to show that for each of these six points the ratio of lengths of tangents to these incircles is constant. But we do not know a synthetic arguments showing that.

Warning: the intersection of cevians AA_1 and BB_1 (and other similar intersection) does not lie on the incircle of $\triangle ABC$.



X GEOMETRICAL OLYMPIAD IN HONOUR OF I. F. SHARYGIN

The Correspondence Round

Below is the list of problems for the first (correspondence) round of the X Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of four elder grades. In Russian school, these are 8-11. In the list below, each problem is indicated by the numbers of Russian school grades, for which it is intended. Foreign students of the last grade have to solve the problems for 11th grade, students of the preceding grade solve the problems for 10th grade etc. However, the participants may solve problems for elder grades as well (solutions of problems for younger grades will not be considered).

A complete solution of each problem or of each its item if there are any, costs 7 points. An incomplete solution costs from 1 to 6 points according to the extent of advancement. If no significant advancement was achieved, the mark is 0. The result of a participant is the total sum of marks for all problems.

In your work, please start the solution for each problem in a new page. First write down the statement of the problem, and then the solution. Present your solutions in detail, including all significant arguments and calculations. Provide all necessary figures of sufficient size. If a problem has an explicit answer, this answer must be presented distinctly. Please, be accurate to provide good understanding and correct estimating of your work!

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

You may note the problems which you liked most (this is not obligatory). Your opinion is interesting for the Jury.

The solutions for the problems (in Russian or in English) must be delivered up to April 1, 2014. For this, please apply **since January 2, 2014** to **http://olimpsharygin.olimpiada.ru** and follow the instructions given there. **Attention:** the solution of each problem must be contained in a separate pdf, doc or jpg file. We recommend to prepare the paper using a computer or to scan it rather than to photograph it. In the last two cases, please check readability of the obtained file.

If you have any technical problem, please contact us by e-mail:
geomolymp@mccme.ru.

It is also possible to send the solutions by e-mail to **geompapers@yandex.ru**. In this case, please follow a few simple rules:

1. Each student sends his work in a separate message (with delivery notification). The size of the message must not exceed 10 Mb.

2. If your work consists of several files, send it as an archive.

3. If the size of your message exceeds 10 Mb, divide it into several messages.

4. In the subject of the message write "The work for Sharygin olympiad", and present the following personal data in the body of your message:

- last name;
- all other names;
- E-mail, phone number, post address;
- the current number of your grade at school;
- the last grade at your high school;
- the number of the last grade in your school system;
- the number and/or the name and the mail address of your school;
- full names of your teachers in mathematics at school and/or of instructors of your extra math classes (if you attend additional math classes after school).

If you have no possibility to deliver the work in electronic form, please apply to the Organizing Committee to find a specific solution for this case.

Winners of the correspondence round, the students of three grades before the last grade, will be invited to the final round in Summer 2014 in the city of Dubna, in Moscow region. (For instance, if the last grade is 12, then we invite winners from 9, 10, and 11 grade.) Winners of the correspondence round, the students of the last grade, will be awarded with diplomas of the Olympiad. The list of the winners will be published on **www.geometry.ru** at the end of May 2014. If you want to know your detailed results, please contact us by e-mail **geomolymp@mccme.ru**.

(1) (8) A right-angled triangle ABC is given. Its cathetus AB is the base of a regular triangle ADB lying in the exterior of ABC, and its hypothenuse AC is the base of a regular triangle AEC lying in the interior of ABC. Lines DE and AB meet at point M. The whole configuration except points A and B was erased. Restore the point M.

(2) (8) A paper square with sidelength 2 is given. From this square, can we cut out a 12-gon having all sidelengths equal to 1, and all angles divisible by $45°$?

(3) (8) Let ABC be an isosceles triangle with base AB. Line ℓ touches its circumcircle at point B. Let CD be a perpendicular from C to ℓ, and AE, BF be the altitudes of ABC. Prove that D, E, F are collinear.

(4) (8) A square is inscribed into a triangle (one side of the triangle contains two vertices and each of two remaining sides contains one vertex). Prove that the incenter of the triangle lies inside the square.

(5) (8) In an acute-angled triangle ABC, AM is a median, AL is a bisector and AH is an altitude (H lies between L and B). It is known that $ML = LH = HB$. Find the ratios of the sidelengths of ABC.

(6) (8–9) Given a circle with center O and a point P not lying on it. Let X be an arbitrary point of this circle, and Y be a common point of the bisector of angle POX and the perpendicular bisector to segment PX. Find the locus of points Y.

(7) (8–9) A parallelogram $ABCD$ is given. The perpendicular from C to CD meets the perpendicular from A to BD at point F, and the perpendicular from B to AB meets the perpendicular bisector to AC at point E. Find the ratio in which side BC divides segment EF.

(8) (8–9) Given a rectangle $ABCD$. Two perpendicular lines pass through point B. One of them meets segment AD at point K, and the second one meets the extension of side CD at point L. Let F be the common point of KL and AC. Prove that $BF \perp KL$.

(9) (8–9) Two circles ω_1 and ω_2 touching externally at point L are inscribed into angle BAC. Circle ω_1 touches ray AB at point E, and circle ω_2 touches ray AC at point M. Line EL meets ω_2 for the second time at point Q. Prove that $MQ \parallel AL$.

(10) (8–9) Two disjoint circles ω_1 and ω_2 are inscribed into an angle. Consider all pairs of parallel lines l_1 and l_2 such that l_1 touches ω_1, and l_2 touches ω_2 (ω_1, ω_2 lie between l_1 and l_2). Prove that the medial lines of all trapezoids formed by l_1, l_2 and the sides of the angle touch some fixed circle.

(11) (8–9) Points K, L, M and N lying on the sides AB, BC, CD and DA of a square $ABCD$ are vertices of another square. Lines DK and NM meet at point E, and lines KC and LM meet at point F. Prove that $EF \parallel AB$.

(12) (9–10) Circles ω_1 and ω_2 meet at points A and B. Let points K_1 and K_2 of ω_1 and ω_2 respectively be such that K_1A touches ω_2, and K_2A touches ω_1. The circumcircle of triangle K_1BK_2 meets lines AK_1 and AK_2 for the second time at points L_1 and L_2 respectively. Prove that L_1 and L_2 are equidistant from line AB.

(13) (9–10) Let AC be a fixed chord of a circle ω with center O. Point B moves along the arc AC. A fixed point P lies on AC. The line passing through P and parallel to AO meets BA at point A_1; the line passing through P and parallel to CO meets BC at point C_1. Prove that the circumcenter of triangle A_1BC_1 moves along a straight line.

(14) (9–11) In a given disc, construct a subset such that its area equals the half of the disc area and its intersection with its reflection over an arbitrary diameter has the area equal to the quarter of the disc area.

(15) (9–11) Let ABC be a non-isosceles triangle. The altitude from A, the bisector from B and the median from C concur at point K.

a) Which of the sidelengths of the triangle is medial?

b) Which of the lengths of segments AK, BK, CK is medial?

(16) (9–11)Given a triangle ABC and an arbitrary point D.The lines passing through D and perpendicular to segments DA, DB, DC meet lines BC,

AC, AB at points A_1, B_1, C_1 respectively. Prove that the midpoints of segments AA_1, BB_1, CC_1 are collinear.

(17) (10–11) Let AC be the hypothenuse of a right-angled triangle ABC. The bisector BD is given, and the midpoints E and F of the arcs BD of the circumcircles of triangles ADB and CDB respectively are marked (the circles are erased). Construct the centers of these circles using only a ruler.

(18) (10–11) Let I be the incenter of a circumscribed quadrilateral ABCD. The tangents to circle AIC at points A, C meet at point X. The tangents to circle BID at points B, D meet at point Y. Prove that X, I, Y are collinear.

(19) (10–11) Two circles ω_1 and ω_2 touch externally at point P. Let A be a point of ω_2 not lying on the line through the centers of the circles, and AB, AC be the tangents to ω_1. Lines BP, CP meet ω_2 for the second time at points E and F. Prove that line EF, the tangent to ω_2 at point A and the common tangent at P concur.

(20) (10–11) A quadrilateral $KLMN$ is given. A circle with center O meets its side KL at points A and A_1, side LM at points B and B_1, etc. Prove that if the circumcircles of triangles KDA, LAB, MBC and NCD concur at point P, then

a) the circumcircles of triangles KD_1A_1, LA_1B_1, MB_1C_1 and NC_1D_1 also concur at some point Q;

b) point O lies on the perpendicular bisector to PQ.

(21) (10–11) Let $ABCD$ be a circumscribed quadrilateral. Its incircle ω touches sides BC and DA at points E and F respectively. It is known that lines AB, FE and CD concur. The circumcircles of triangles AED and BFC meet ω for the second time at points E_1 and F_1. Prove that $EF \parallel E_1F_1$.

(22) (10–11) Does there exist a convex polyhedron such that it has diagonals and each of them is shorter than each of its edges?

(23) (11) Let A, B, C and D be a triharmonic quadruple of points, i.e

$$AB \cdot CD = AC \cdot BD = AD \cdot BC.$$

Let A_1 be a point distinct from A such that the quadruple A_1, B, C and D is triharmonic. Points B_1, C_1 and D_1 are defined similarly. Prove that

a) A, B, C_1, D_1 are concyclic;

b) the quadruple A_1, B_1, C_1, D_1 is triharmonic.

(24) (11) A circumscribed pyramid $ABCDS$ is given. The opposite sidelines of its base meet at points P and Q in such a way that A and B lie on segments PD and PC respectively. The inscribed sphere touches faces ABS and BCS at points K and L. Prove that if PK and QL are complanar then the touching point of the sphere with the base lies on BD.

X GEOMETRICAL OLYMPIAD IN HONOUR OF I. F. SHARYGIN

Final round. Ratmino, 2014, July 31 and August 1

8 grade. First day

8.1. (J. Zajtseva, D. Shvetsov) The incircle of a right-angled triangle ABC touches its catheti AC and BC at points B_1 and A_1, the hypotenuse touches the incircle at point C_1. Lines C_1A_1 and C_1B_1 meet CA and CB respectively at points B_0 and A_0. Prove that $AB_0 = BA_0$.

8.2. (B. Frenkin) Let AH_a and BH_b be the altitudes, AL_a and BL_b be the bisectors of a triangle ABC. It is known that $H_aH_b \parallel L_aL_b$. Is the equality $AC = BC$ correct?

8.3. (A. Blinkov) Points M and N are the midpoints of sides AC and BC of a triangle ABC. Angle MAN is equal to 15°, and angle BAN is equal to 45°. Find angle ABM.

8.4. (T. Kazitsyna) Tanya cut out a triangle from the checkered paper as shown in the picture. Later the lines of the grid faded. Can Tanya restore them without any instruments only folding the triangle (she remembered the triangle sidelengths)?

8 grade. Second day

8.5. (A. Shapovalov) A triangle with angles equal to 30, 70 and 80 degrees is given. Cut it into two triangles in such a way that the bisector of one of them and the median of the second one from the endpoints of the cutting segment are parallel (it is sufficient to find one solution).

8.6. (V. Yasinsky) Two circles k_1 and k_2 with centers O_1 and O_2 touche externally at point O. Points X and Y on k_1 and k_2 respectively are such that rays O_1X and O_2Y are codirectional. Prove that two tangents from X to k_2 and two tangents from Y to k_1 touche the same circle passing through O.

8.7. (Folklor) Two points on a circle are joined by a broken line shorter than the diameter of the circle. Prove that there exists a diameter which does not intersect this broken line.

8.8. (Tran Quang Hung) Let M be the midpoint of the chord AB of a circle (O). Suppose that K is the reflection of M about the center of the circle, and P is a variable point on the circumference of the circle. Let Q be the intersection of the perpendicular of AB through A and the perpendicular of PK through P. Given that H is the projection of P onto AB, prove that QB bisects PH

9 grade. First day

9.1. (V. Yasinsky) Let $ABCD$ be a cyclic quadrilateral. Prove that $AC > BD$ if and only if $(AD - BC)(AB - CD) > 0$.

9.2. (F. Nilov) In the quadrilateral $ABCD$ angles A and C are right. two circles with diameters AB and CD meet at points X and Y. Prove that line XY passes through the midpoint of AC.

9.3. (E. Diomidov) An acute angle A and a point E inside it are given. Construct such points B, C on the sides of the angle that E be the nine points center of triangle ABC.

9.4. (Mahdi Etesami Fard) Let H be the orthocenter of a triangle ABC. If H lies on incircle of ABC , prove that three circles with centers A, B, C and radii AH, BH, CH have a common tangent.

9 grade. Second day

9.5. (D. Shvetsov) In a triangle ABC $\angle B = 60°$, O is the circumcenter, BL is the bisector. The circumcirle of triangle BOL meets the circumcircle of ABC at point D. Prove that $BD \perp AC$.

9.6. (A. Polyansky) Let I be the incenter of a triangle ABC, M, N be the midpoints of arcs ABC and BAC of its circumcircle. Prove that points M, I, N are collinear if and only if $AC + BC = 3AB$.

9.7. (N. Beluhov) Nine circles are drawn around an arbitrary triangle as in the figure. All circles tangent to the same side of the triangle have equal radii. Three lines are drawn, each one connecting one of the triangle's vertices to the center of one of the circles touching the opposite side, as in the figure. Show that the three lines are concurrent.

9.8. (N. Beluhov, S. Gerdgikov) A convex polygon P lies on a flat wooden table. You are allowed to drive some nails into the table. The nails must not go through P, but they may touch its boundary. We say that a set of nails blocks P if the nails make it impossible to move P without lifting it off the table. What is the minimum number of nails that suffices to block any convex polygon P?

10 grade. First day

10.1. (I. Bogdanov, B. Frenkin) The vertices and the circumcenter of an isosceles triangle lie on four different sides of a square. Find the angles of this triangle.

10.2. (A. Zertsalov, D. Skrobot) A circle, its chord AB and the midpoint W of the minor arc AB are given. Take an arbitrary point C on the major arc AB. The tangent to the circle at C meets the tangents at A and B at points X and Y respectively. Lines WX and WY meet AB at points N and M. Prove that the length of segment NM doesn't depend on point C.

10.3. (A. Blinkov) Do there exist convex polyhedra with an arbitrary number of diagonals (a diagonal joins two vertices of a polyhedron and doesn't lie on its surface)?

10.4. (A. Garkavyj, A. Sokolov) A triangle ABC and a point D are given. The circle with center D, passing through A, meets AB and AC at points A_b and A_c respectively. Points B_a, B_c, C_a and C_b are defined similarly. How many does there exist such points D, that points A_b, A_c, B_a, B_c, C_a and C_b are concyclic?

10 grade. Second day

10.5. (A. Zaslavsky) An altitude from one vertex of a triangle, a bisector from the second one and a median from the remaining vertex were drawn, the common points of these three lines were marked, and after this all except for three marked points was erased. Restore the triangle

10.6. (E. H. Garsia) The incircle of a triangle ABC touches AB at point C'. The circle with diameter BC' meets the incircle and the bisector of angle B at points A_1 and A_2 respectively. The circle with diameter AC' meets the incircle and the bisector of angle A at points B_1 and B_2 respectively. Prove that lines AB, A_1B_1, A_2B_2 concur.

10.7. (S. Shosman, O. Ogievetsky) Prove that the smallest angle between the faces of an arbitrary tetrahedron is not greater than the angle between the faces of a regular tetrahedron.

10.8. (N. Beluhov) Given is a cyclic quadrilateral $ABCD$. The point L_a lies in the interior of $\triangle BCD$ and is such that its distances to the sides of this triangle are proportional to the corresponding sides. The points L_b, L_c, and L_d are defined analogously. Show that $L_aL_bL_cL_d$ is cyclic if and only if $ABCD$ is an isosceles trapezoid.