IX GEOMETRICAL OLYMPIAD IN HONOUR OF I. F. SHARYGIN

Final round. Ratmino, 2013, August 1-2

8 grade. First day

8.1. Let $ABCDE$ be a pentagon with right angles at vertices B and E and such that $AB = AE$ and $BC = CD = DE$. The diagonals BD and CE meet at point F. Prove that $FA = AB$.

8.2. Two circles with centers O_1 and O_2 meet at points A and B. The bisector of angle O_1AO_2 meets the circles for the second time at points C and D. Prove that the distances from the circumcenter of triangle CBD to O_1 and to O_2 are equal.

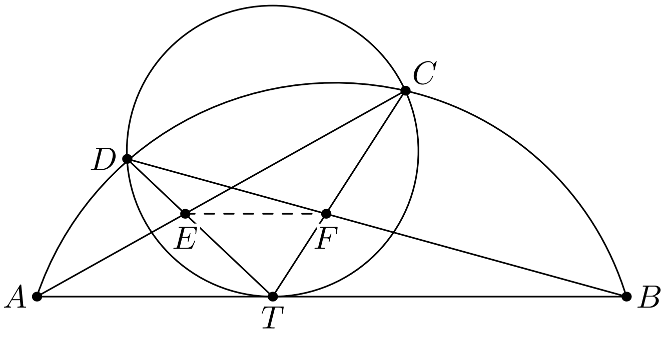
8.3. Each vertex of a convex polygon is projected to all nonadjacent sidelines. Can it happen that each of these projections lies outside the corresponding side?

8.4. The diagonals of a convex quadrilateral $ABCD$ meet at point L. The orthocenter H of the triangle LAB and the circumcenters O_1, O_2, and O_3 of the triangles LBC, LCD, and LDA were marked. Then the whole configuration except for points H, O_1, O_2, and O_3 was erased. Restore it using a compass and a ruler.

8 grade. Second day

8.5. The altitude AA', the median BB', and the angle bisector CC' of a triangle ABC are concurrent at point K. Given that $A'K = B'K$, prove that $C'K = A'K$.

8.6. Let α be an arc with endpoints A and B (see fig.). A circle ω is tangent to segment AB at point T and meets α at points C and D. The rays AC and TD meet at point E, while the rays BD and TC meet at point F. Prove that EF and AB are parallel.



8.7. In the plane, four points are marked. It is known that these points are the centers of four circles, three of which are pairwise externally tangent, and all these three are internally tangent to the fourth one. It turns out, however, that it is impossible to determine which of the marked points is the center of the fourth (the largest) circle. Prove that these four points are the vertices of a rectangle.

8.8. Let P be an arbitrary point on the arc AC of the circumcircle of a fixed triangle ABC, not containing B. The bisector of angle APB meets the bisector of angle BAC at point P_a; the bisector of angle CPB meets the bisector of angle BCA at point P_c. Prove that for all points P, the circumcenters of triangles PP_aP_c are collinear.

9 grade. First day

9.1. All angles of a cyclic pentagon $ABCDE$ are obtuse. The sidelines AB and CD meet at point E_1; the sidelines BC and DE meet at point A_1. The tangent at B to the circumcircle of the triangle BE_1C meets the circumcircle ω of the pentagon for the second time at point B_1. The tangent at D to the circumcircle of the triangle DA_1C meets ω for the second time at point D_1. Prove that $B_1D_1 \parallel AE$.

9.2. Two circles ω_1 and ω_2 with centers O_1 and O_2 meet at points A and B. Points C and D on ω_1 and ω_2, respectively, lie on the opposite sides of the line AB and are equidistant from this line. Prove that C and D are equidistant from the midpoint of O_1O_2.

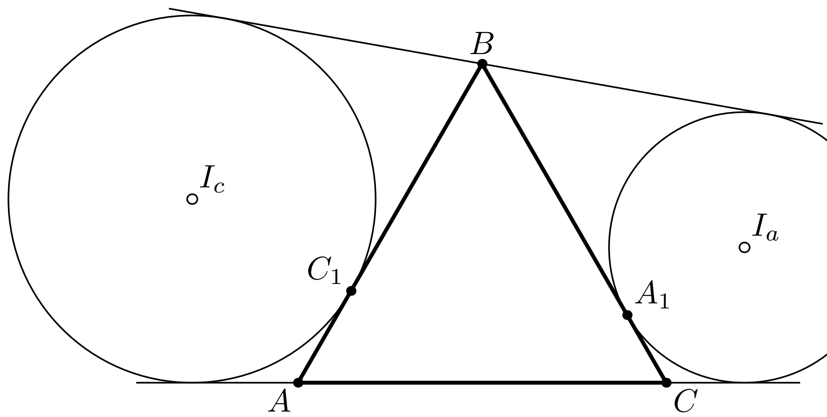
9.3. Each sidelength of a convex quadrilateral $ABCD$ is not less than 1 and not greater than 2. The diagonals of this quadrilateral meet at point O. Prove that $S_{AOB} + S_{COD} \leqslant 2(S_{AOD} + S_{BOC})$.

9.4. A point F inside a triangle ABC is chosen so that $\angle AFB = \angle BFC = \angle CFA$. The line passing through F and perpendicular to BC meets the median from A at point A_1. Points B_1 and C_1 are defined similarly. Prove that the points A_1, B_1, and C_1 are three vertices of some regular hexagon, and that the three remaining vertices of that hexagon lie on the sidelines of ABC.

9 grade. Second day

9.5. Points E and F lie on the sides AB and AC of a triangle ABC. Lines EF and BC meet at point S. Let M and N be the midpoints of BC and EF, respectively. The line passing through A and parallel to MN meets BC at point K. Prove that $\frac{BK}{CK} = \frac{FS}{ES}$.

9.6. A line ℓ passes through the vertex B of a regular triangle ABC. A circle ω_a centered at I_a is tangent to BC at point A_1, and is also tangent to the lines ℓ and AC. A circle ω_c centered at I_c is tangent to BA at point C_1, and is also tangent to the lines ℓ and AC.



9.7. Two fixed circles ω_1 and ω_2 pass through point O. A circle of an arbitrary radius R centered at O meets ω_1 at points A and B, and meets ω_2 at points C and D. Let X be the common point of lines AC and BD. Prove that all the points X are collinear as R changes.

9.8. Three cyclists ride along a circular road with radius 1 km counterclockwise. Their velocities are constant and different. Does there necessarily exist (in a sufficiently long time) a moment when all the three distances between cyclists are greater than 1 km?

10 grade. First day

10.1. A circle k passes through the vertices B and C of a triangle ABC with $AB > AC$. This circle meets the extensions of sides AB and AC beyond B and C at points P and Q, respectively. Let AA_1 be the altitude of ABC. Given that $A_1P = A_1Q$, prove that $\angle PA_1Q = 2\angle BAC$.

10.2. Let $ABCD$ be a circumscribed quadrilateral with $AB = CD \neq BC$. The diagonals of the quadrilateral meet at point L. Prove that the angle ALB is acute.

10.3. Let X be a point inside a triangle ABC such that $XA \cdot BC = XB \cdot AC = XC \cdot AB$. Let I_1, I_2, and I_3 be the incenters of the triangles XBC, XCA, and XAB, respectively. Prove that the lines AI_1, BI_2, and CI_3 are concurrent.

10.4. We are given a cardboard square of area $1/4$ and a paper triangle of area $1/2$ such that all the squares of the side lengths of the triangle are integers. Prove that the square can be completely wrapped with the triangle. (In other words, prove that the triangle can be folded along several straight lines and the square can be placed inside the folded figure so that both faces of the square are completely covered with paper.)

10 grade. Second day

10.5. Let O be the circumcenter of a cyclic quadrilateral $ABCD$. Points E and F are the midpoints of arcs AB and CD not containing the other vertices of the quadrilateral. The lines passing through E and F and parallel to the diagonals of $ABCD$ meet at points E, F, K, and L. Prove that line KL passes through O.

10.6. The altitudes AA_1, BB_1, and CC_1 of an acute-angled triangle ABC meet at point H. The perpendiculars from H to B_1C_1 and A_1C_1 meet the rays CA and CB at points P and Q, respectively. Prove that the perpendicular from C to A_1B_1 passes through the midpoint of PQ.

10.7. In the space, five points are marked. It is known that these points are the centers of five spheres, four of which are pairwise externally tangent, and all these four are internally tangent to the fifth one. It turns out, however, that it is impossible to determine which of the marked points is the center of the fifth (the largest) sphere. Find the ratio of the greatest and the smallest radii of the spheres.

10.8. In the plane, two fixed circles are given, one of them lies inside the other one. For an arbitrary point C of the external circle, let CA and CB be two chords of this circle which are tangent to the internal one. Find the locus of the incenters of triangles ABC.