IX GEOMETRICAL OLYMPIAD IN HONOUR OF I. F. SHARYGIN

The Correspondence Round

Below is the list of problems for the first (correspondence) round of the IX Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of 8–11 grades (these are four elder grades in Russian school). In the list below, each problem is indicated by the numbers of school grades, for which it is intended. However, the participants may solve problems for elder grades as well (solutions of problems for younger grades will not be considered).

Your work containing solutions for the problems, written in Russian or in English, should be sent not later than by April 1, 2013 to *geomolymp@mccme.ru* in `pdf`, `doc` or `jpg` files.

Winners of the correspondence round will be invited to take part in the final round to be held in Dubna town (near Moscow, Russia) in Summer 2013 . More about Sharygin Olympiad see on `www.geometry.ru/olimpgeom.htm`.

(1) (8) Let ABC be an isosceles triangle with $AB = BC$. Point E lies on side AB, and ED is the perpendicular from E to BC. It is known that $AE = DE$. Find $\angle DAC$.

(2) (8) Let ABC be an isosceles triangle ($AC = BC$) with $\angle C = 20°$. The bisectors of angles A and B meet the opposite sides in points A_1 and B_1 respectively. Prove that triangle A_1OB_1 (where O is the circumcenter of ABC) is regular.

(3) (8) Let ABC be a right-angled triangle ($\angle B = 90°$). The excircle inscribed into angle A touches the extensions of sides AB, AC in points A_1, A_2 respectively; points C_1, C_2 are defined similarly. Prove that the perpendiculars from A, B, C to C_1C_2, A_1C_1, A_1A_2 respectively concur.

(4) (8) Let ABC be a nonisosceles triangle. Point O is its circumcenter, and point K is the center of the circumcircle w of triangle BCO. The altitude of ABC from A meets w in point P. Line PK intersects the circumcircle of ABC in points E and F. Prove that one of segments EP and FP is equal to segment PA.

(5) (8) Four segments join some point inside a convex quadrilateral with its vertices. Four obtained triangles are equal. Can we assert that this quadrilateral is a rhombus?

(6) (8–9) Diagonals AC, BD of trapezoid $ABCD$ meet in point P. The circumcircles of triangles ABP, CDP intersect line AD for the second time in points X, Y. Point M is the midpoint of segment XY. Prove that $BM = CM$.

(7) (8–9) Let BD be a bisector of triangle ABC. Points I_a, I_c are the incenters of triangles ABD, CBD. Line I_aI_c meets AC in point Q. Prove that $\angle DBQ = 90°$.

(8) (8–9) Let X be an arbitrary point inside the circumcircle of triangle ABC. Lines BX and CX meet the circumcircle in points K and L respectively. Line LK intersects BA and AC in points E and F respectively. Find the locus of points X such that the circumcircles of triangles AFK and AEL touch.

(9) (8–9) Let T_1 and T_2 be the touching points of the excircles of triangle ABC with sides BC and AC respectively. It is known that the reflection of the incenter of ABC in the midpoint of AB lies on the circumcircle of triangle CT_1T_2. Find $\angle BCA$.

(10) (8–9) The incircle of triangle ABC touches the side AB in point C'; the incircle of triangle ACC' touches sides AB and AC in points C_1, B_1; the incircle of triangle BCC' touches the sides AB and BC in points C_2, A_2. Prove that lines B_1C_1, A_2C_2 and CC' concur.

(11) (8–9) a) Let $ABCD$ be a convex quadrilateral. Let $r_1 \le r_2 \le r_3 \le r_4$ be the radii of the incircles of triagles ABC, BCD, CDA, DAB. Can the inequality $r_4 > 2r_3$ hold?

b)The diagonals of a convex quadrilateral $ABCD$ meet in point E. Let $r_1 \le r_2 \le r_3 \le r_4$ be the radii of the incircles of triangles ABE, BCE, CDE, DAE. Can the inequality $r_2 > 2r_1$ be correct?

(12) (8–11) On each side of triangle ABC, two distinct points are marked. It is known that these points are the feet of the altitudes and the bisectors.

a) Using only a ruler determine which points are the feet of the altitudes and which points are the feet of the bisectors.

b) Solve p.a) drawing only three lines.

(13) (9–10) Let A_1 and C_1 be the touching points of the incircle of triangle ABC with BC and AB respectively, A' and C' be the touching points of the excircle inscribed into angle B with the extensions of BC and AB respectively. Prove that the orthocenter H of triangle ABC lies on A_1C_1 iff lines $A'C_1$ and BA are perpendicular.

(14) (9–11) Let M, N be the midpoints of diagonals AC, BD of right-angled trapezoid $ABCD$ ($\angle A = \angle D = 90°$). The circumcircles of triangles ABN, CDM meet line BC in points Q, R. Prove that the distances from Q, R to the midpoint of MN are equal.

(15) (9–11) a) Triangles $A_1B_1C_1$ and $A_2B_2C_2$ are inscribed into triangle ABC so that $C_1A_1 \perp BC$, $A_1B_1 \perp CA$, $B_1C_1 \perp AB$, $B_2A_2 \perp BC$, $C_2B_2 \perp CA$, $A_2C_2 \perp AB$. Prove that these triangles are equal.

b) Points $A_1, B_1, C_1, A_2, B_2, C_2$ lie inside triangle ABC so that A_1 is on segment AB_1, B_1 is on segment BC_1, C_1 is on segment CA_1, A_2 is on segment AC_2, B_2 is on segment BA_2, C_2 is on segment CB_2 and angles $BAA_1, CBB_1, ACC_1, CAA_2, ABB_2, BCC_2$ are equal. Prove that triangles $A_1B_1C_1$ and $A_2B_2C_2$ are equal.

(16) (9–11) The incircle of triangle ABC touches BC, CA, AB in points A', B', C' respectively. The perpendicular from incenter I to the median from vertex C meets line $A'B'$ in point K. Prove that $CK \parallel AB$.

(17) (9–11) An acute angle between the diagonals of a cyclic quadrilateral is equal to ϕ. Prove that an acute angle between the diagonals of any another quadrilateral having the same sidelengths is less than ϕ.

(18) (9–11) Let AD be a bisector of triangle ABC. Points M and N are the projections of B and C to AD. The circle with diameter MN intersects BC in points X and Y. Prove that $\angle BAX = \angle CAY$.

(19) (10–11) a) The incircle of triangle ABC touches AC and AB in points B_0 and C_0 respectively. The bisectors of angles B and C meet the medial perpendicular to the bisector AL in points Q and P respectively. Prove that lines PC_0, QB_0 and BC concur.

b) Let AL be the bisector of triangle ABC. Points O_1 and O_2 are the circumcenters of triangles ABL and ACL respectively. Points B_1 and C_1 are the projections of C and B to the bisectors of angles B and C respectively. Prove that lines O_1C_1, O_1B_1 and BC concur.

c) Prove that two points obtained in pp. a) and b) coincide.

(20) (10–11) Let C_1 be an arbitrary point on side AB of triangle ABC. Points A_1 and B_1 of rays BC and AC are such that $\angle AC_1B_1 = \angle BC_1A_1 = \angle ACB$. Lines AA_1 and BB_1 meet in point C_2. Prove that all lines C_1C_2 have a common point.

(21) (10–11) Given are a circle ω and a point A outside it. One of two lines drawn through A intersects ω in points B and C, the second one intersect it in points D and E (D lies between A and E). The line passing through D and parallel to BC, meets ω for the second time in point F, and line AF meets ω in point T. Let M be the common point of lines ET and BC, and N be the reflection of A in M. Prove that the circumcircle of triangle DEN passes through the midpoint of segment BC.

(22) (10–11) The common perpendiculars to the opposite sidelines of a non-planar quadrilateral are mutually perpendicular. Prove that they are complanar.

(23) (10–11) Two convex polygons A and B don't intersect. Polygon A have exactly 2012 planes of symmetry. What is the maximal number of symmetry planes of the union of A and B when B has a) 2012, b) 2013 symmetry planes?

c) What is the answer to the question of p.b) when the symmetry planes are replaced by the symmetry axes?