A GENERALIZATION OF THE DANDELIN THEOREM

FEDOR K. NILOV

Abstract. We prove three apparently new theorems related to the doubly tangent circles of conics including a generalization of the Dandelin theorem on spheres inscribed in a cone. Also we discuss the focal properties of doubly tangent circles of conics.

1. Introduction

The main result of this paper is the following generalization of the celebrated Dandelin theorem on spheres inscribed in a cone. We say that a sphere is *inscribed* in a quadric surface of revolution if the sphere touches the quadric along a circle. Evidently, the center of such sphere lies on the axis of revolution.

Theorem 1.1. *Let an inclined plane intersect a quadric surface of revolution in a conic. Let a sphere be inscribed in the quadric and tangent to the plane. Then the tangency point of the sphere and the plane is a focus of the conic. The plane containing the contact points of the sphere and the quadric intersects the inclined plane in a directrix of the conic; see Figure 1.*

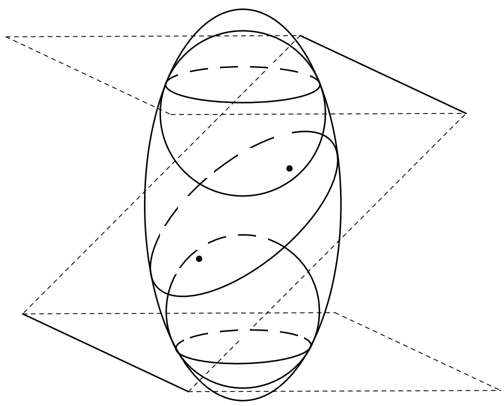

Fig. 1. A generalization of the Dandelin theorem.

Note that the particular case in which the quadric is a one-sheeted hyperboloid of revolution was already considered by Dandelin; see [3, §11]. We use the generalized "focus-directrix" property of conics stated in Section 2 to prove Theorem 1.1.

Using similar approach we prove the following theorem on ellipses doubly tangent to two given nested circles.

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Theorem 1.2. *Let a family of ellipses be such that each ellipse in the family is doubly tangent to two given nested circles. Then*
(a) Similarity property. *All the ellipses in the family are similar;*
(b) Tangency property. *For each given circle the tangency points of the circle and each ellipse in the family are collinear with a limiting point of the pencil of circles generated by the two given circles; see Figure 2.*

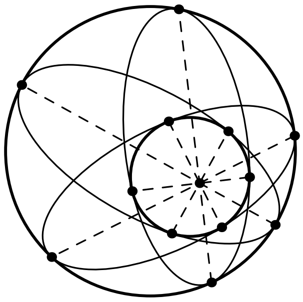

Fig. 2. Tangency property.

(c) Foci property. *The foci of all the ellipses in the family lie on a fixed circle concentric with the larger given circle; see Figure 3.*

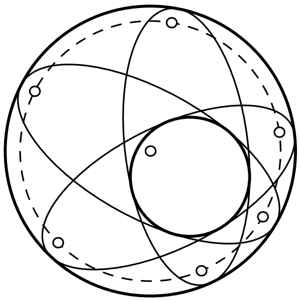

Fig. 3. Foci property.

(d) Orthogonality property. *The four common points of any two ellipses in the family lie on two perpendicular lines intersecting in a limiting point of the pencil of circles generated by the two given circles; see Figure 4.*

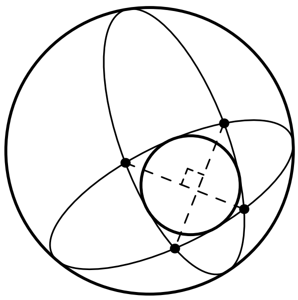

Fig. 4. Orthogonality property.

The following theorem was firstly proved by E.H. Neville in 1936; see [5] and [4].

The Neville Theorem. *If three ellipses are such that each pair selected from them has one common focus and two intersection points, then their three common chords are concurent; see Figure 5.*

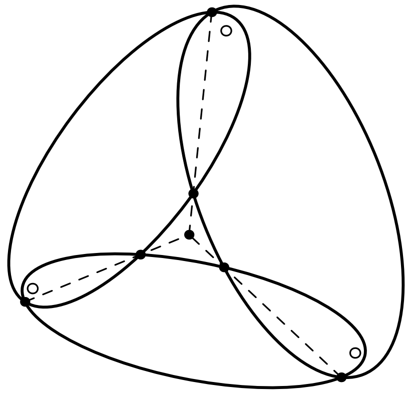

Fig. 5. The Neville theorem.

We prove the following generalization of the Neville theorem.

Theorem 1.3. *Let three ellipses be such that each pair selected from them has a common doubly tangent circle with the center lying on the major axes of the ellipses. Suppose that each pair of the ellipses has four common points. Then the intersection points of the three ellipses lie on* 6 *lines forming a complete quadrangle; see Figure 6.*

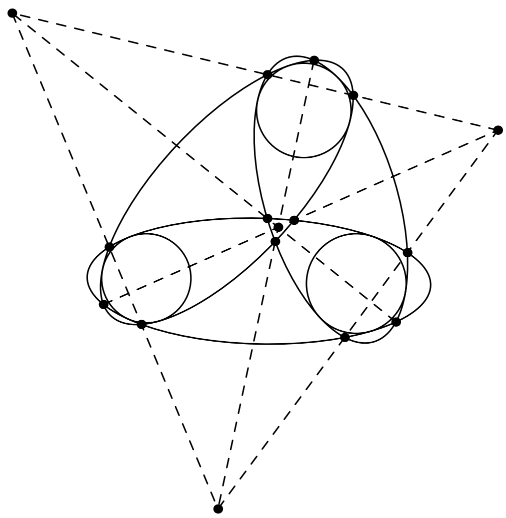

Fig. 6. A generalization of the Neville theorem.

The paper is organized as follows. In Section 2 we discuss generalized focal properties of conics. In Section 3 we prove Theorem 1.1, Theorem 1.2, and Theorem 1.3.

2. Generalized focal properties of conics

In this section we discuss generalized focal properties involving *doubly tangent circles* of conics. We start with the following definition.

Definition. Consider a noncircular conic. A circle having two tangency points with the conic is called a *doubly tangent circle* of this conic.

The following proposition is obvious.

Proposition. *See Figure 7. Each conic distinct from a parabola or a circle has two families of doubly tangent circles. The centers of the circles in the first family lie on the major axis. The centers of the circles in the second family lie on the minor axis. Any parabola has exactly one family of doubly tangent circles.* □

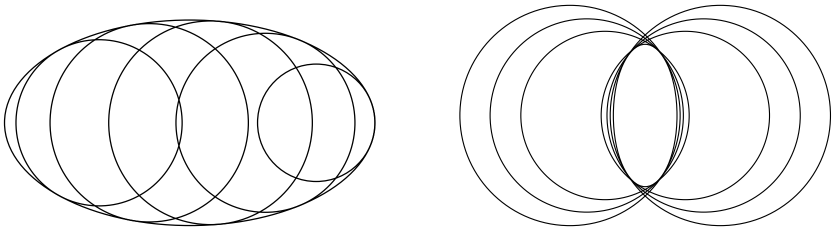

Fig. 7. Two families of doubly tangent circles of an ellipse.

Let P be a point and ω be a circle. Denote by r the radius of ω. Denote by d the distance between P and the center of ω. In what follows we use the following notations. We define the *tangent distance* $t(P, \omega)$ from P to ω to be $\sqrt{|d^2 - r^2|}$. If P lies in the exterior of ω, then $t(P, \omega)$ is the length of a tangent segment from P to ω. Let λ be a line. Denote by $d(P, \lambda)$ the distance from P to λ.

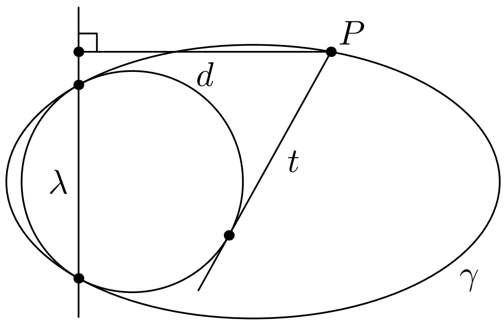

Fig. 8. *Generalized "focus-directrix" property of conics.* For an arbitrary point $P \in \gamma$ we have $t/d = \text{const}$.

Theorem 2.1. Generalized "focus-directrix" property. (Cf. [3, Theorem 1]) See Figure 8. *Let γ be a noncircular conic. Let ω be an arbitrary doubly tangent circle of γ. Let λ be the line passing through the tangency points of ω and γ. If the center of ω lies on the major axis of γ, then for an arbitrary point $P \in \gamma$ we have $\frac{t(P,\omega)}{d(P,\lambda)} = \varepsilon$, where ε is the eccentricity of γ. If the center of ω lies on the minor axis of γ, then for an arbitrary point $P \in \gamma$ we have $\frac{t(P,\omega)}{d(P,\lambda)} = \varepsilon'$, where $\varepsilon' = \frac{\varepsilon}{\sqrt{|1-\varepsilon^2|}}$.*

Proof. It suffices to consider the following 2 cases.

Case 1. *The center of ω lies on the major axis of the conic γ* or *γ is a hyperbola and the center of ω lies on the minor axis.* The proof is found in [3, Theorem 1].

Case 2. *The conic γ is an ellipse and the center of ω lies on the minor axis.* Denote by Π the plane containing the circle ω. Consider the sphere Σ such that ω is a great circle of Σ; see Figure 9. Consider a circle $\theta \subset \Sigma$ such that γ is the orthogonal projection of θ onto the plane Π. It is easy to see that the eccentricity ε of γ is equal to $\sin \varphi$, where φ is the angle between Π and the plane containing θ. Denote by O the center of ω. Consider the point N on θ such that $PN \perp \Pi$. Note that $|PN| = \sqrt{|ON|^2 - |OP|^2} = t(P,\omega)$. Therefore,

$$\frac{t(P,\omega)}{d(P,\lambda)} = \frac{|PN|}{d(P,\lambda)} = \tan\varphi = \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} = \varepsilon'.$$

Generalized "focus-directrix" property is proved. □

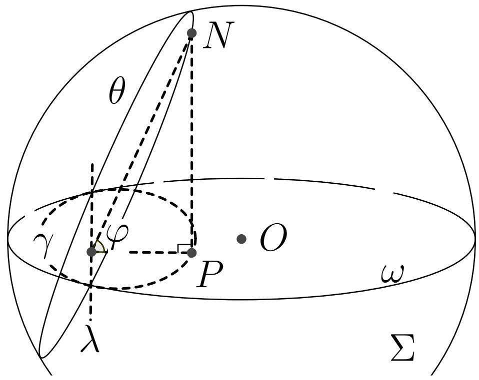

Fig. 9.

Theorem 2.2. Generalized "bifocal" property. (Cf. [3, Theorem 3]) See Figure 10. *Let γ be a noncircular conic. Let ω_1 and ω_2 be two arbitrary doubly tangent circles of γ such that the centers of the circles lie on the same axis of symmetry of γ. Let λ_i be the line passing through the tangency points of γ and ω_i. If a point $P \in \gamma$ moves between the lines λ_1 and λ_2, then $t(P,\omega_1)+t(P,\omega_2) = \text{const}$. Otherwise $|t(P,\omega_1) - t(P,\omega_2)| = \text{const}$.*

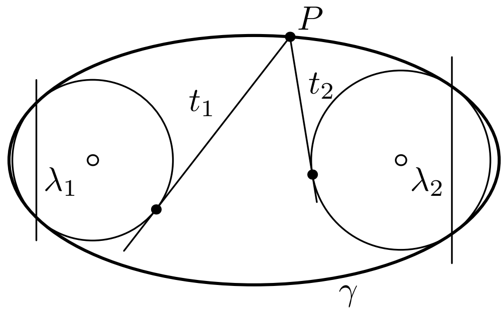

Fig. 10. *Generalized "bifocal" property of conics.* If P moves between the lines λ_1 and λ_2, then $t_1 + t_2 = \text{const}$. Otherwise $|t_1 - t_2| = \text{const}$.

Proof. Assume that the centers of the circles ω_1 and ω_2 lie on the major axis of the conic γ. Let ε be the eccentricity of γ. Denote by d the distance between the parallel lines λ_1 and λ_2. If the point P lies between the lines λ_1 and λ_2, then $t(P,\omega_1)+t(P,\omega_2) = \varepsilon\cdot(d(P,\lambda_1)+d(P,\lambda_2)) = \varepsilon\cdot d = \text{const}$, where the first equality follows from the generalized "focus-directrix" property. If the point P does not lie between the lines λ_1 and λ_2, we have $|t(P,\omega_1)-t(P,\omega_2)| = \varepsilon\cdot|d(P,\lambda_1)-d(P,\lambda_2)| = \varepsilon\cdot d = \text{const}$. If the centers of the circles ω_1 and ω_2 lie on the minor axis of the conic γ, the proof is analogous. □

Analogous focal properties of conics on the sphere and in the hyperbolic plane are discussed in [6].

3. Proofs

Proof of Theorem 1.1. Assume that the center of the given sphere lies on the major axis of axial sections of the given quadric. Denote by F the tangency point of the given sphere and the inclined plane; see Figure 11. Denote by λ the intersection line of the inclined plane and the plane containing the contact points of the sphere and the quadric. Denote by φ the angle between these planes. Let P be an arbitrary point on the given conic. Consider the plane containing the axis of the quadric and passing through P. Let this plane intersect the quadric, the given sphere, and the plane containing the contact points in the conic γ_P, in the circle ω_P, and in the line λ_P, respectively. Evidently, the circle ω_P is doubly tangent to γ_P, and λ_P is the line passing through the tangency points. We have $\frac{d(P,\lambda_P)}{d(P,\lambda)} = \sin\varphi$. Note that $|PF| = t(P,\omega_P)$, because $|PF|$ and $t(P,\omega_P)$ are the lengths of tangent segments from the point P to the given sphere. By the generalized "focus-directrix" property, the ratio $\frac{t(P,\omega_P)}{d(P,\lambda_P)}$ equals the eccentricity ε of γ_P and does not depend on the point P lying on the given conic. Therefore, we have

$$\frac{|PF|}{d(P,\lambda)} = \frac{t(P,\omega_P)}{d(P,\lambda_P)} \cdot \frac{d(P,\lambda_P)}{d(P,\lambda)} = \varepsilon \cdot \sin\varphi = \text{const}.$$

Thus the point F and the line λ are a focus and a directrix of the given conic. If the center of the given sphere lies on the minor axis of axial sections of the given quadric, then the proof is analogous. □

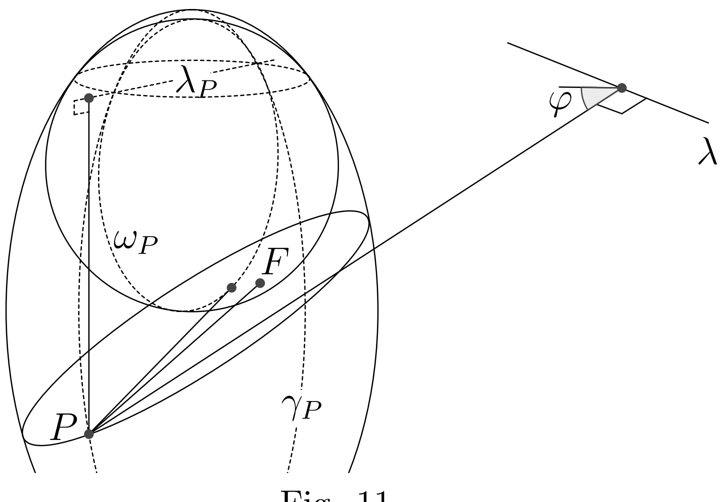

Fig. 11.

In the proof of Theorem 1.2 we use the following notations. Denote by α and β the given circles, with β lies inside α. Denote by L the limiting point of the pencil of circles generated by α and β.

Proof of Theorem 1.2(a). Consider an arbitrary ellipse from the given family; see Figure 12. Denote by P a tangency point of the ellipse and α. Denote by λ the line passing through the tangency points of the ellipse and β.

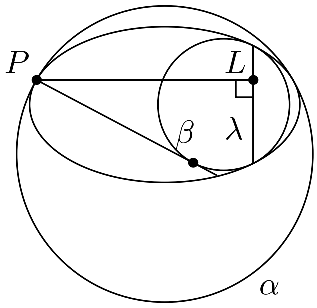

Fig. 12.

From Theorem 1.2(b) it follows that the lines PL and λ are perpendicular. By the generalized "focus-directrix" property, the ratio $\frac{t(P,\beta)}{|PL|}$ is equal to the eccentricity of the ellipse. By the well-known *geometric characterization of a hyperbolic pencil of circles* (see [2, Theorem 2.12]), this ratio does not depend on the point P lying on the circle α. Thus all the ellipses doubly tangent to α and β are similar. □

Proof of Theorem 1.2(b). Apply a projective transformation that preserves the circle α and takes L to the center O of α. Let this transformation take the circle β to an ellipse β'. Then O is a limiting point of the pencil of conics generated by α and β'. Evidently, O is the center of β'. So this transformation takes each ellipse γ from the given family to an ellipse γ' with the center O. The line passing through the tangency points of γ' and β' (or α) is passing through O. Thus the line passing through the tangency points of γ and β (or α) is passing through L. □

Proof of Theorem 1.2(c). Consider an arbitrary ellipse γ doubly tangent to α and β. Let F_1 and F_2 be the foci of γ; see Figure 13. Denote by ε the eccentricity of γ. Denote by I the center of β. Denote by r the radius of β. Let T be a tangency point of γ and β.

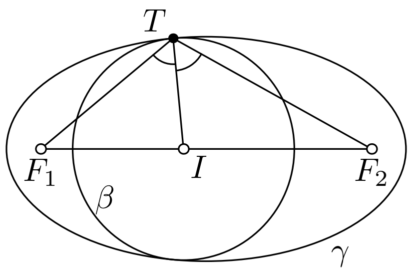

Fig. 13.

From the *optical property of an ellipse* (see [2, Theorem 1.1]) it follows that TI is the bisector of the angle F_1TF_2. Therefore,

$$\frac{|F_1I|}{|F_1T|} = \frac{|F_2I|}{|F_2T|} = \frac{|F_1I| + |F_2I|}{|F_1T| + |F_2T|} = \varepsilon.$$

Therefore, we have

$$|F_1I|\cdot|F_2I| = \frac{\varepsilon^2}{1-\varepsilon^2}\cdot(|F_1T|\cdot|F_2T| - |F_1I|\cdot|F_2I|) =$$
$$= \frac{\varepsilon^2}{1-\varepsilon^2}\cdot|TI|^2 = \frac{\varepsilon^2}{1-\varepsilon^2}\cdot r^2 = \text{const}.$$

Here the first equality follows from the previous equation; the second equality follows from the known formula for the length of the angle bisector; the fourth equality follows from Theorem 1.2(a). Since the perpendicular bisector of the segment F_1F_2 passes through the center of the circle α and the product $|F_1I|\cdot|F_2I|$ does not depend on an ellipse γ in the given family, it follows that the foci of all the ellipses in the family lie on a fixed circle concentric with α. □

Proof of Theorem 1.2(d). The assertion is a corollary of Theorem 1.2(a) and the following lemma.

Lemma on two similar conics. *Let two similar conics have four common points. Suppose there exists a circle doubly tangent to the conics such that the center of this circle is the intersection point of either the major axes or the minor axes. Then the four common points of these conics lie on the bisectors of the angle between the lines passing through the tangency points of each conic and the circle; see Figure 14.*

Proof. Denote by γ_1 and γ_2 the given conics. Since γ_1 and γ_2 are similar, we see that the eccentricity of γ_1 is equal to the eccentricity of γ_2, i. e. $\varepsilon_1 = \varepsilon_2 = \varepsilon$. Denote by ω the given circle.

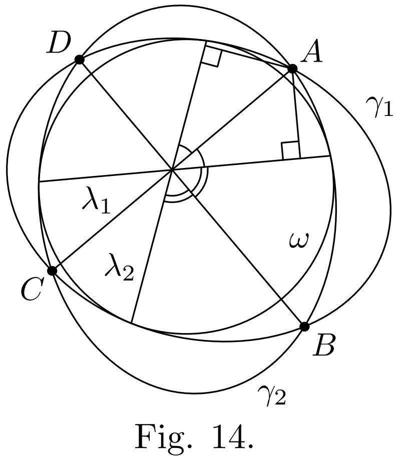

Fig. 14.

Assume that the center of ω lies on the major axes of γ_1 and γ_2. Denote by λ_i the line passing through the tangency points of γ_i and ω. Denote by A, B, C, and D the intersection points of γ_1 and γ_2. By the generalized "focus-directrix" property, we get

$$\frac{t(A,\omega)}{d(A,\lambda_1)} = \varepsilon = \frac{t(A,\omega)}{d(A,\lambda_2)}.$$

Therefore, $d(A,\lambda_1) = d(A,\lambda_2)$. Analogously, we have $d(B,\lambda_1) = d(B,\lambda_2)$, $d(C,\lambda_1) = d(C,\lambda_2)$, and $d(D,\lambda_1) = d(D,\lambda_2)$. Thus the points A, B, C, and D

lie on the bisectors of the angle between λ_1 and λ_2. If the center of ω lies on the minor axes of γ_1 and γ_2, the proof is analogous. □

Proof of Theorem 1.3. Denote by Π the plane containing the given ellipses γ_1, γ_2, and γ_3. Denote by ω_{ij} the common doubly tangent circle of the ellipses γ_i and γ_j. Consider the spheres Σ_{12}, Σ_{23}, and Σ_{31} such that ω_{12}, ω_{23}, and ω_{31} are great circles of these spheres, respectively. Consider three prolate spheroids Γ_1, Γ_2, and Γ_3 having γ_1, γ_2, and γ_3 as axial sections, respectively. The sphere Σ_{ij} is inscribed in Γ_i and Γ_j. By Monge's theorem, we have that the intersection $\Gamma_i \bigcap \Gamma_j$ consists of two ellipses $\tau_{ij}^p, p = 1, 2$. Obviously, the plane containing each ellipse τ_{ij}^p is perpendicular to Π. Thus the orthogonal projection of τ_{ij}^p onto Π is a segment of the straight line λ_{ij}^p passing through the intersection points of γ_i and γ_j. Since the ellipses τ_{ij}^p and τ_{jk}^q lie on Γ_j, it follows that they intersect in two real or imaginary points. Evidently, the real line passing through these points is perpendicular to Π and intersects Π in the common point of λ_{ij}^p and λ_{jk}^q. Note that the common points of τ_{ij}^p and τ_{jk}^q lie on τ_{ki}^r for some r. Thus the lines λ_{ij}^p, λ_{jk}^q, and λ_{ki}^r have the common point. By the same argument, the other corresponding lines are concurrent. □

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Moscow State University
E-mail address: nilovfk@gmail.com