ON CIRCLES TOUCHING THE INCIRCLE

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Abstract. For a given triangle, we deal with the circles tangent to the incircle and passing through two its vertices. We present some known and recent properties of the points of tangency and some related objects. Further we outline some generalizations for polygons and polytopes.

1. Case of triangle

Throughout this section, we use the following notation[1].

Let ABC be a triangle. The circles γ and Γ are its incircle and circumcircle with centers I and O and radii r and R, respectively. The sides BC, CA, and AB touch γ at points A_1, B_1, and C_1, respectively.

Now construct the circle ω_A passing through B and C and tangent to γ at some point X_A. Define the circles ω_B, ω_C and the points of tangency X_B, X_C in a similar way. This paper is devoted to the investigation of the objects related to the constructed circles.

Let M_A be the second meeting point of ω_A and X_AA_1; define the points M_B and M_C analogously.

We start with the following description of the points M_A, M_B, and M_C.

Theorem 1. *Line OM_A is the perpendicular bisector of BC. Further, point M_A is the radical center of γ, B, and C (here we regard B and C as degenerate circles).*

Proof. Consider the homothety with center X_A taking γ to ω_A. This homothety takes BC to the line t_A touching ω_A at M_A, hence $t_A \parallel BC$. Thus M_A is the midpoint of arc BC, and OM_A is the perpendicular bisector of BC.

Consequently, there exists an inversion ι with center M_A which takes BC to ω_A. We have $\iota(A_1) = X_A$, $\iota(B) = B$, $\iota(C) = C$, hence $M_AA_1 \cdot M_AX_A = M_AB^2 = M_AC^2$. This means that M_A has equal powers with respect to the circles γ, B, and C. □

[1]All the results from this section allow various generalizations which are shown in the next section. So, throughout this section we put into footnotes the statements and approached that work only in this particular case, and fail up to further generalizations.

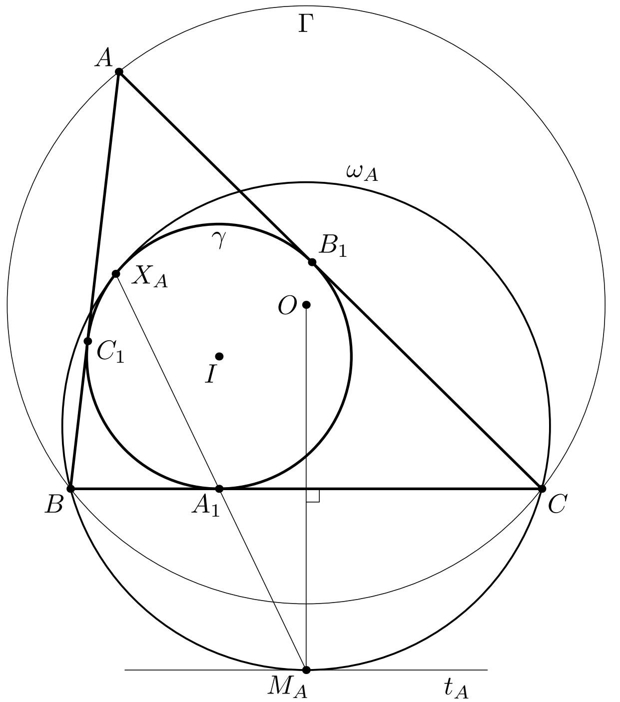

Fig. 1.

Remark 1. One could finish the proof without using inversion in the following way. In triangles M_ABA_1 and M_AX_AB we have $\angle A_1BM_A = \frac{1}{2}\overset{\frown}{M_AC} = \frac{1}{2}\overset{\frown}{M_AB} = \angle BX_AM_A$. Hence $\triangle M_ABA_1 \sim \triangle M_AX_AB$, and $M_AA_1 \cdot M_AX_A = M_AB^2$.

Let ℓ_A be the radical axis of circles γ and A; hence ℓ_A passes through the midpoints of AB_1 and AC_1. Define the lines ℓ_B and ℓ_C in a similar way. From Theorem 1 and analogous statements for M_B and M_C, we obtain the following

Corollary 2. $\ell_A = M_BM_C$, $\ell_B = M_CM_A$, $\ell_C = M_AM_B$.

Theorem 3. *The circumcenter of triangle $M_AM_BM_C$ is O. Next, triangles $M_AM_BM_C$ and $A_1B_1C_1$ are homothetical with some center K.*

Proof. By Corollary 2, we have $M_BM_C \perp AI$. Next, by Theorem 1 we get $M_BO \perp AC$ and $M_CO \perp AB$. Consequently, $\angle OM_BM_C = \angle CAI = \angle IAB = \angle M_BM_CO$; therefore, $M_BO = M_CO$. Analogously, we get $M_AO = M_BO$, and the points M_A, M_B, and M_C lie on some circle Ω with center O.

Consider now the homothety h with positive coefficient which takes γ to Ω; thus $h(I) = O$. Notice that the vectors $\overrightarrow{IA_1}$ and $\overrightarrow{OM_A}$ are codirectional, so $h(A_1) = M_A$; analogously, $h(B_1) = M_B$ and $h(C_1) = M_C$, as desired[2]. □

[2]Let us present an alternative proof. For the second claim of Theorem, it suffices to show that the respective sides of triangles $M_AM_BM_C$ and $A_1B_1C_1$ are parallel. From Corollary 2 we get $\ell_B = M_AM_C$. Since $\ell_B \perp BI$, we have $\ell_B \parallel A_1C_1$. Similarly, $\ell_A \parallel B_1C_1$ and $\ell_C \parallel A_1B_1$.

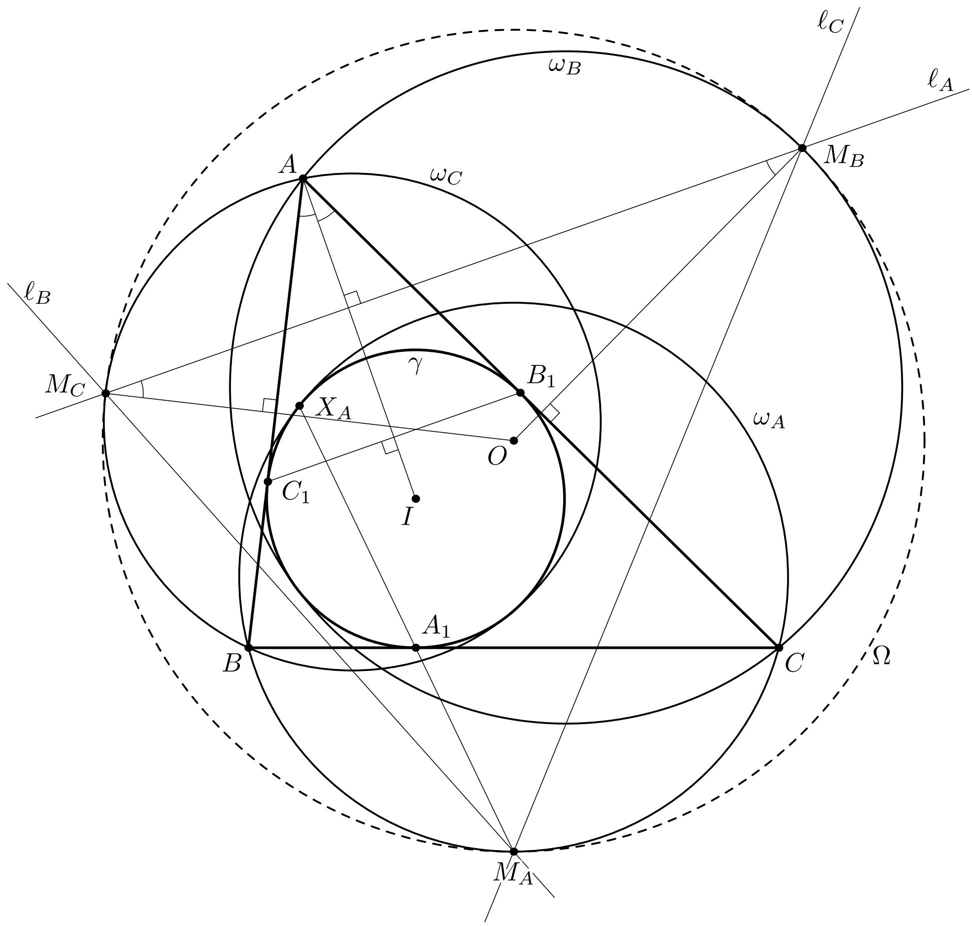

Fig. 2.

Till the end of this section, we fix the notation of Ω as the circumcircle of $M_AM_BM_C$, h as the homothety taking $A_1B_1C_1$ to $M_AM_BM_C$, and K as its center. Denote by $R(\Omega)$ the radius of Ω.

From the homothety h, we immediately obtain the following.

Corollary 4. *Lines $A_1X_A = X_AM_A$, $B_1X_B = X_BM_B$, and $C_1X_C = X_CM_C$ are concurrent at K.*

Corollary 5. *Points K, I, and O are collinear, and $\frac{KO}{KI} = \frac{R(\Omega)}{r}$.*

Now we describe point K in some other terms.

Theorem 6. *K is the radical center of ω_A, ω_B, and ω_C.*

For the first part, consider the homothety h taking $\triangle A_1B_1C_1$ to $\triangle M_AM_BM_C$. Let Ω be the circumcircle of $M_AM_BM_C$; then $h(\gamma) = \Omega$. Hence $h(I)$ is the center of Ω. Next, h takes IA_1 to the line passing through M_A and perpendicular to BC, which is the perpendicular bisector of BC. Similarly, $h(IB_1)$ is the perpendicular bisector of AC. Since the perpendicular bisectors of BC and AC pass through O, we get $h(I) = O$.

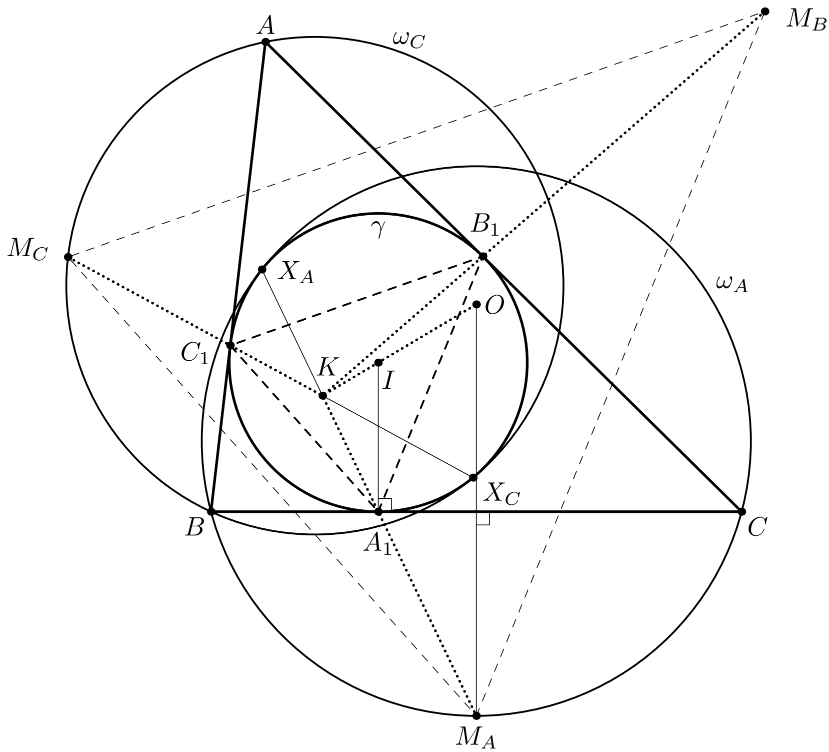

Fig. 3.

Proof. From the homothety h we get $\frac{KA_1}{KC_1} = \frac{KM_A}{KM_C}$. Hence the relation $KX_A \cdot KA_1 = KX_C \cdot KC_1$ implies $KX_A \cdot KM_A = KX_C \cdot KM_C$. Thus we obtain the equality of powers of K with respect to ω_A and ω_C. The same is true for ω_A and ω_B. □

Remark 2. In fact, this theorem is an instance of a more general fact. Consider two fixed circles γ and Ω, and let ω be a variable circle tangent to γ at X and to Ω at M (with a fixed type of tangencies — internal or external). Then, by Monge's theorem, all the lines KM pass through a fixed point K which is a center of homothety taking γ to Ω. Moreover, K is the radical center of all such circles ω: if Y is the second common point of XM and γ, then $KX \cdot KM$ is proportional to $KX \cdot KY$ which is fixed.

Remark 3. Theorem 6 combined with the first statement of Corollary 5 forms the statement of a problem on the Romanian Masters of Mathematics 2012 olympiad proposed by F. Ivlev [6, Problem 3].

Next, we introduce some more objects related to our construction. Let m_A be the line passing through A and perpendicular to IA. Define lines m_B and m_C

similarly. Finally, let us define the points[3] $I_A = m_B \cap m_C$, $I_B = m_C \cap m_A$, and $I_C = m_A \cap m_B$.

Theorem 7. *M_A is the midpoint of the segment A_1I_A.*

Proof. By its definition, line ℓ_B is the midline between parallel lines A_1C_1 and m_B; hence m_B intersects A_1M_A at point I'_A such that $M_AI'_A = M_AA_1$. Similarly, m_C also intersects A_1M_A at the same point, hence $I'_A = I_A$, and M_A is the midpoint of A_1I_A. □

Let S_A be the circumcenter of triangle BIC. Define S_B and S_C similarly[4]. Denote by Γ_S the circumcircle of triangle $S_AS_BS_C$; let R_S be the radius of this circle[5].

Theorem 8. *The circle Γ_S has O as its center, and the equality $R_S = R(\Omega) - r/2$ holds[6]. Moreover, triangles $S_AS_BS_C$ and $M_AM_BM_C$ are homothetical with center O.*

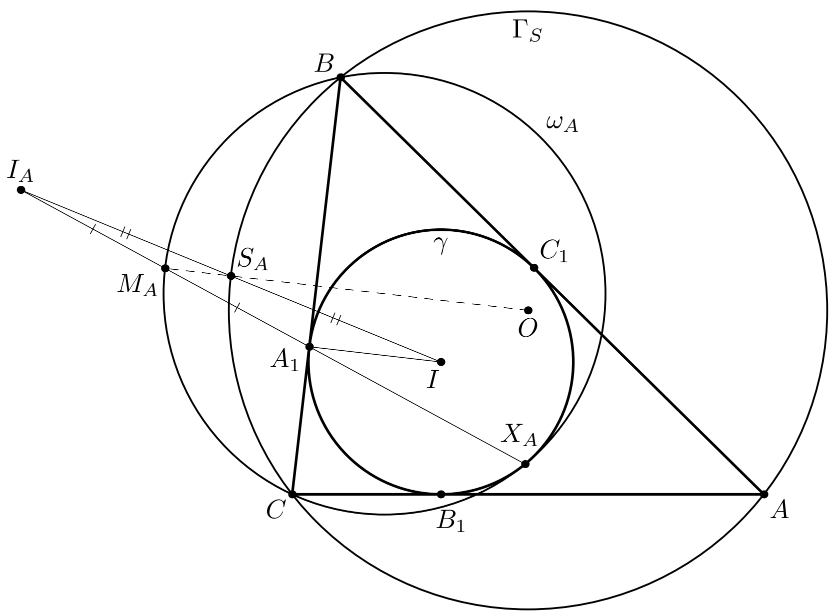

Fig. 4.

Proof. Notice that all the points M_A, S_A, and O lie on the perpendicular bisector to BC, hence they are collinear. Next, we have $\overrightarrow{OM_A} = \overrightarrow{OS_A} + \overrightarrow{S_AM_A}$. By the definition, S_A belongs to perpendicular bisectors of BI and CI. Hence S_A is the image of I_A under the homothety with center I and ratio $\frac{1}{2}$; thus S_A is the

[3]Notice that m_A, m_B, and m_C are external bisectors of $\angle A$, $\angle B$, and $\angle C$, respectively, while I_A, I_B, and I_C are the excenters of $\triangle ABC$.

[4]S_A, S_B, and S_C are the midpoints of arcs BC, CA, and AB of circle Γ, respectively.

[5]In our case of triangle, $\Gamma_S = \Gamma$.

[6]Thus $R(\Omega) = R + r/2$.

midpoint of II_A. Using Theorem 7, we conclude that S_AM_A is the midline in triangle I_AIA_1, hence $\overrightarrow{S_AM_A} = \overrightarrow{IA_1}/2$. Since the vectors $\overrightarrow{OS_A}$ and $\overrightarrow{IA_1}$ are codirectional, we get $OS_A = |\overrightarrow{OS_A}| = |\overrightarrow{OM_A}| - |\overrightarrow{IA_1}|/2 = R(\Omega) - r/2$. Similarly, we obtain that the segments OS_B and OS_C have the same length, so O is the center of Γ_S. Finally, the homothety with center O which takes Ω to Γ_S takes $M_AM_BM_C$ to $S_AS_BS_C$. □

Remark 4. Using the relation between ratios of similarity of triangles $A_1B_1C_1$, $I_AI_BI_C$, $M_AM_BM_C$, and $S_AS_BS_C$, one may obtain the relation between radii in alternative way: we have $R(\Omega) = R(M_AM_BM_C) = \big(R(I_AI_BI_C)+R(A_1B_1C_1)\big)/2$ and $R(I_AI_BI_C) = 2R(S_AS_BS_C) = 2R_S$; hence $R(\Omega) = R_S + r/2$.

Remark 5. The radius R_S could be calculated in terms of R, r, and OI: $R_S = \frac{R^2-OI^2}{2r}$ (see, for example, the Problem from Sharygin olympiad, 2005 [7, Problem 20]). Combining with Corollary 5 we get $\frac{KO}{KI} = \frac{R^2+r^2-OI^2}{2r^2}$ thus obtaining the ratio of our homothety[7].

At the end of this section, let us make some remarks on the results obtained above.

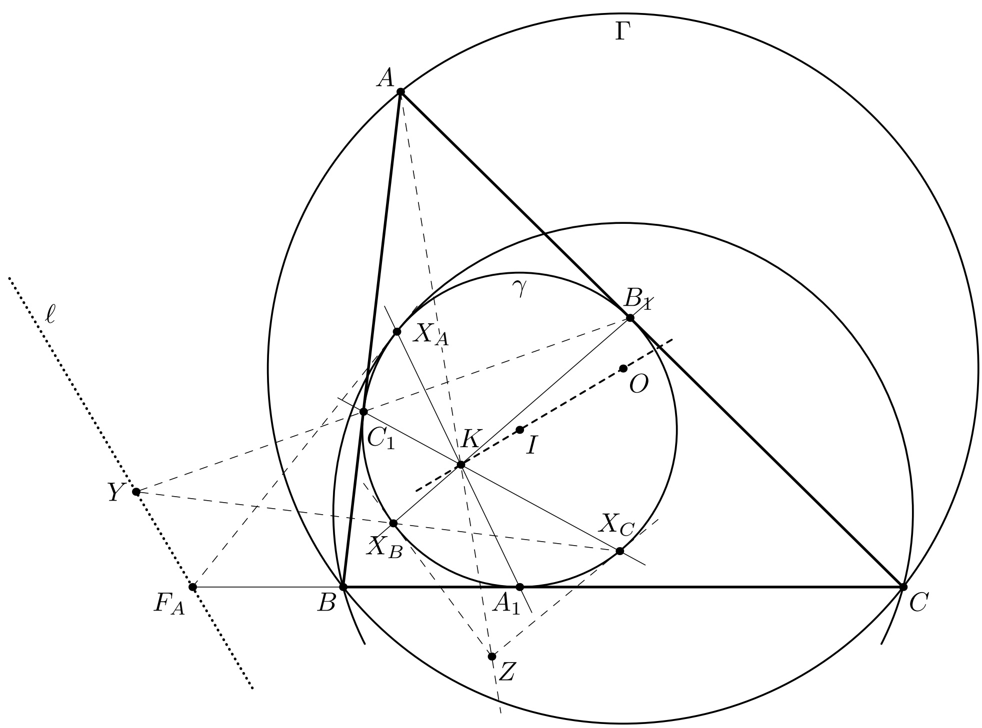

Fig. 5.

[7]By Euler formula, $OI^2 = R^2 - 2Rr$, hence $\frac{KO}{KI} = \frac{R}{r} + \frac{1}{2}$

Remark 6. Here, we present an alternative approach to considered construction (without using points M_A, M_B, and M_C) related to a result by S. Ilyasov and A. Akopyan [5, Problem M2244]. We present here an equivalent reformulation of this statement; for the completeness, we also provide its proof.

Proposition 1. *Let Γ and γ be two fixed circles, and let A, B be variable points on Γ. Suppose two circles ω_1 and ω_2 (each may degenerate to a line) passing through A and B are tangent to γ at X and Y, respectively. Then the line XY passes through a fixed point.*

Proof. Let ℓ be the radical axis of γ and Γ, and let K be its pole with respect to γ. We claim that XY passes through K.

Denote by F the point of intersection of AB with the common tangent to γ and ω_1 at X. Then F is the radical center of γ, Γ, and ω_1, hence it belongs to ℓ. Since the powers of F with respect to ω_1 and ω_2 are equal (in fact, they are equal to $FA \cdot FB$), this point lies also on the common tangent to γ and ω_2 at Y. Hence XY is the polar line of F with respect to γ, so it passes through K. □

In our case, one may apply this fact to the pairs of points (A, B), (B, C), and (C, A) obtaining that the lines $X_A A_1$, $X_B B_1$, and $X_C C_1$ are concurrent at K. This proves Corollary 4 with one more description of point K. Next, since $\ell \perp OI$ and $IK \perp \ell$, we have $K \in OI$, thus proving Corollary 5. Note that from this new description of K one could easily derive that $\frac{KO}{KI} = \frac{R^2+r^2-OI^2}{2r^2}$.

Further, let the tangents to γ at X_B and X_C intersect at point Z; this point is the radical center of ω_B, ω_C, and γ. Hence AZ is the radical axis of ω_B and ω_C. To finish an alternative proof of Theorem 6, it suffices to show that $K \in AZ$. For that, notice that the point Y of intersection of lines B_1C_1 and $X_B X_C$ lies on the polar line ℓ of K with respect to γ. Then the polar line of Y contains the poles of lines B_1C_1, ℓ, and $X_B X_C$, which are A, K, and Z, respectively.

One may also notice that this approach allows to generalize some of the facts to the case of curved triangle ABC (when its sides are the circular arcs). Namely, in this case the lines A_1X_A, B_1X_B, and C_1X_C are also concurrent at some point collinear with O and I.

Remark 7. Let us mention two facts related to the considered construction.

a) The line containing points A_1, I_A, M_A, and X_A passes also through the midpoint of the altitude from A (see a problem on the Moscow mathematical olympiad 2001 [4, Problem 10.3], and also a problem proposed by Bulgaria for the IMO in 2002 [3, Problem 2002-G7, p. 319]). To prove this one can use the homothety with center A taking the incircle to the excircle.

b) Lines AX_A, BX_B, and CX_C are concurrent (see a problem by A. Badzyan in [1, Problem M2268]; this fact was independently noticed by D. Shvetsov).

Remark 8. All the results from this section could be reformulated if the incircle is replaced by one of the excircles.

2. A general case: polygons and polytopes

All Theorems and Corollaries as well as remarks from the previous section admit generalizations to the case when the base figure is a polygon which is simultaneously circumscribed and inscribed, and also to the case when the base figure is a polytope in space (or in n-dimensional space) which is simultaneously circumscribed and inscribed.

Let us start from the general set up. Further we use the terminology for the case of 3-dimensional space though everything is appropriate for n-dimensional case for all $n \geq 2$. For $n = 2$ one could replace *faces* by *sides*, *planes* by *lines*, and *spheres* by *circles*. For $n > 3$ one could replace *faces* by $(n-1)$-dimensional *hyperfaces*, *planes* by *hyperplanes*, *spheres* by $(n-1)$-*dimensional spheres*, and *circles* by $(n-2)$-*dimensional spheres*.

Let $\mathcal{P}$ be a convex polytope with vertices $P_1, \ldots, P_n$; let $F_1, \ldots, F_k$ be the planes determined by the faces of $\mathcal{P}$. Suppose that $\mathcal{P}$ has both an inscribed sphere γ and a circumscribed sphere Γ. Let I, O and r, R be the centers and the radii of these spheres, respectively.

For every $i \in \{1, 2, \ldots, k\}$, define $\Gamma_i = F_i \cap \Gamma$ (thus, Γ_i is the circumcircle of the face lying in F_i). Let γ touch F_i at point Q_i.

Now let us construct the sphere ω_i passing through Γ_i and tangent to γ at point X_i. Let M_i be the second meeting point of ω_i and X_iQ_i. Let $\mathcal{Q}$ and $\mathcal{M}$ be the convex polytopes with vertices $Q_1, \ldots, Q_k$ and $M_1, \ldots, M_k$, respectively. For every $j \in \{1, \ldots, n\}$, let ℓ_j be the radical plane of spheres γ and P_j (hence ℓ_j passes through the midpoints of P_jQ_i for all i such that $P_j \in F_i$).

Let m_j be the plane passing through P_j perpendicular to IP_j. Let S_i be the circumcenter of sphere passing through Γ_i and I.

Now we proceed with the analogues of results from the previous section.

In the proof of Theorem 11 below we clarify how to generalize the proof of Theorem 3 to the case of a polytope. All the other proofs of Theorems below are completely analogous to those in the case of triangle; thus we omit these proofs.

Theorem 9 (cf. Theorem 1)**.** *Line OM_i is the perpendicular to F_i. Further, point M_i has equal powers with respect to γ and any point $X \in \Gamma_i$ (here X is regarded as a degenerate sphere).*

Corollary 10 (cf. Corollary 2)**.** *If $P_j \in F_i$, then ℓ_j passes through M_i.*

Theorem 11 (cf. Theorem 3)**.** *Points $M_1, \ldots, M_k$ lie on some sphere Ω with center O. Next, polytopes $\mathcal{M}$ and $\mathcal{Q}$ are homothetical with some center K.*

Proof. To prove the first statement, it suffices to prove the equality $OM_i = OM_s$ for every pair of indices i, s such that F_i and F_s correspond to adjacent faces[8].

Let $F_{i,s}$ be the plane containing I and $F_i \cap F_s$; then $F_{i,s}$ is the bisector plane of F_i and F_s. By Theorem 1, each of M_i and M_s has equal powers with respect to γ and all the points in $\Gamma_i \cap \Gamma_s$, hence M_iM_s is perpendicular to the plane spanned

[8] In n-dimensional case, these two hyperfaces should have a common $(n-2)$-dimensional face.

by $\Gamma_i \cap \Gamma_s$ and I which is $F_{i,s}$. Finally, by the same Theorem we have $OM_i \perp F_i$ and $OM_s \perp F_s$, hence $\angle OM_iM_s = \angle OM_sM_i$, as required.

The proof of the second statement of Theorem is completely analogous to that in the case of triangle. □

Now, as in the case of triangle, we denote by h the obtained homothety taking $\mathcal{Q}$ to $\mathcal{M}$, by K its center, and by Ω the circumsphere of $\mathcal{M}$ (then $\Omega = h(\gamma)$). Denote also by $R(\Omega)$ the radius of Ω.

Remark 9. Notice that the first statement of Theorem 11 in the case of tetrahedron appeared as a problem proposed by F. Bakharev in All-Russian mathematical olympiad in 2003 [2, Problem 744].

Corollary 12 (cf. Corollary 4)**.** *Lines $Q_iX_i = X_iM_i$ ($i = 1, \dots, k$) are concurrent at point K.*

Corollary 13 (cf. Corollary 5)**.** *Points K, I, and O are collinear, and $\frac{KO}{KI} = \frac{R(\Omega)}{r}$.*

Theorem 14 (cf. Theorem 6)**.** *K has equal powers with respect to spheres ω_i ($i = 1, \dots, k$).*

Theorem 15 (cf. Theorem 7)**.** *Let $i \in \{1, \dots, k\}$. Consider a point I_i such that M_i is the midpoint of Q_iI_i. Then for all $j \in \{1, \dots, n\}$ with $P_j \in F_i$, plane m_j passes through I_i.*

Remark 10. From Theorem 15 we see that the convex polytope $\mathcal{I}$ with vertices $I_1, \dots, I_k$ is also homothetical to the polytopes $\mathcal{Q}$ and $\mathcal{M}$. A particular case of this fact (for an inscribed and circumscribed quadrilateral in the plane) appeared as a problem by S. Berlov, L. Emelyanov, and A. Smirnov in All-Russian mathematical olympiad in 2004 [2, Problem 755].

Theorem 16 (cf. Theorem 8)**.** *All the points S_i lie on the sphere Γ_S with center O and radius $R_S = R(\Omega) - r/2$.*

Remark 11. The alternative approach mentioned in the previous section also works in the general case. It uses the following generalized statement.

Proposition 2. *Let Γ and γ be two fixed spheres, and let $\Gamma' \subset \Gamma$ be a circle. Suppose two spheres ω_1 and ω_2 (each may degenerate to a plane) passing through Γ' are tangent to γ at X and Y, respectively. Then the line XY passes through a fixed point K that is the pole of radical plane of Γ and γ with respect to γ.*

References

[1] A. Badzyan. Problem M2268. *Kvant Magazine*, 3, 2012.

[2] N. K. Agakhanov, I. I. Bogdanov, P. A. Kozhevnikov, O. K. Podlipsky, and D. A. Tereshin. *All-russian mathematical olympiad 1993–2009.* M.:MCCME, 2007.

[3] D. Djukić, V. Janković, I. Matić, and N. Petrović. *The IMO compendium.* Springer, 2006.

[4] R. Fedorov. *Moscow mathematical olympiads, 2000-2005*. Vol. 7. American Mathematical Soc., 2011.

[5] S. Ilyasov and A. Akopyan. Problem M2244. *Kvant Magazine*, 5-6, 2011.

[6] F. Ivlev. Problem 3, Romanian Masters 2012. http://rmms.lbi.ro/rmm2012/.

[7] A. A. Zaslavsky and S. D. I. Protasov V. Yu. *Geometrical olympiad in honour of I. F. Sharygin*. M.:MCCME, 2007.

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