GENERALIZATION OF A PROBLEM WITH ISOGONAL CONJUGATE POINTS

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ABSTRACT. In this note we give a generalization of the problem that was used in the All-Russian Mathematical Olympiad and a purely sythetic proofs.

The following problem was proposed by Andrey Badzyan on All-Russian Mathematical Olympiad (2004–2005, District round, Grade 9, Problem 4).

Problem 1. *Let ABC be a triangle with circumcircle (O) and incircle (I). M is the midpoint of AC, N is the midpoint of the arc AC which contains B. Prove that $\angle IMA = \angle INB$.*

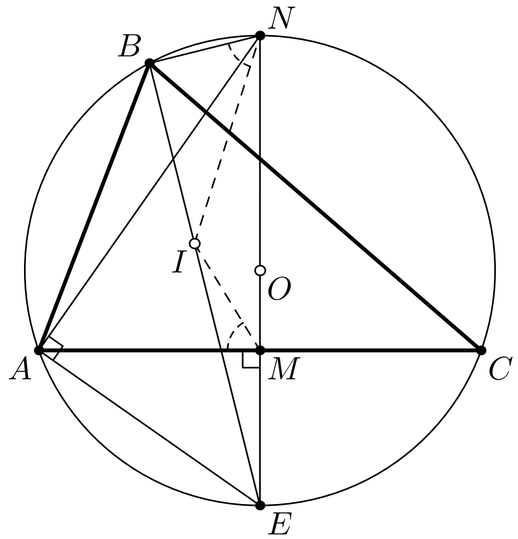

Fig. 1.

Official solution by the Committee. Denote by E be the midpoint of the arc AC which does not contain B. It is clear that B, I, E are collinear, since the line formed by these points is the angle bisector of $\angle ABC$.

Additionally, N, O, M, E are also collinear, since these points all belong to the perpendicular bisector of AC and it is well-known that $AE = EC = IE$.

Since $\angle NAE = \angle AME = 90^\circ$ it is easy to see that $\triangle AME \sim \triangle NAE$ which implies that $|ME| \cdot |NE| = |AE|^2 = |EI|^2$. Hence, we have $\triangle EIM \sim \triangle ENI$ from which we get $\angle IME = \angle EIN$.

Note the following

$$90^\circ + \angle IMA = \angle AME + \angle IMA = \angle IME = \angle EIN = \\ = \angle INB + \angle IBN = \angle INB + 90^\circ.$$

We get the required equality

$$\angle IMA = \angle INB.$$

□

Darij Grinberg in [1] gave a solution using the idea of excircle construction while another member named *mecrazywong* on the same forum suggested a different solution by making use of similarity and angle chasing. Now we give a generalized problem.

Problem 2. *Let ABC be a triangle with circumcircle (O). Suppose P, Q are two points lying in the triangle such that P is the isogonal conjugate of Q with respect to $\triangle ABC$. Denote by D the point of intersection of AP and (O) in which $D \neq A$. OD consecutively cuts BC at M and again cuts (O) at N. Prove that $\angle PMB = \angle QNA$.*

If points P and Q are coincide with the incenter I, Problem 2 is coincide with problem 1.

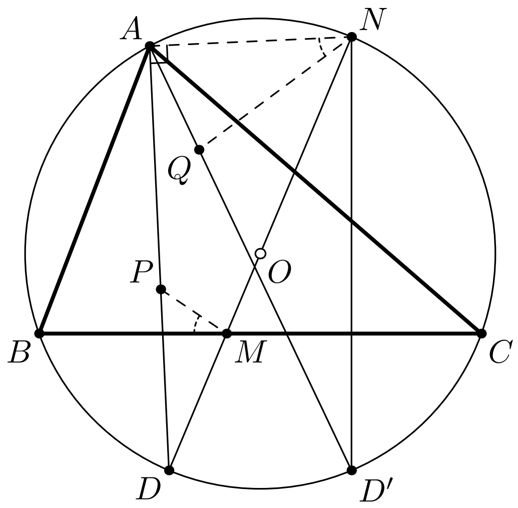

Fig. 2.

Proof. Denote the intersection of AQ and (O) by D'. Since $\angle DAB = \angle D'AC$ we have that $BCD'D$ is an isosceles trapezoid.

We have,

- $\angle PDB = \angle BD'Q$
- $\angle BPD = \angle BAP + \angle PBA = \angle CBD' + \angle QBC = \angle QBD'$.

So $\triangle PBD \sim \triangle BQD'$ and it is easy to conclude that

$$(1) \qquad \frac{|PD|}{|BD'|} = \frac{|BD|}{|QD'|} \Rightarrow |PD| \cdot |QD'| = |BD| \cdot |BD'|.$$

On the other hand,

- $\angle MBD = \angle BND'$ (since $\angle MBD = \frac{1}{2} m \overset{\frown}{CD} = \frac{1}{2} m \overset{\frown}{BD'} = \angle BND'$)
- $\angle BDM = \angle BD'N$

so $\triangle BMD \sim \triangle NBD'$. Hence

$$(2) \qquad |BD| \cdot |BD'| = |MD| \cdot |ND'|.$$

From (1) and (2) it is follows that $|PQ| \cdot |QD|' = |MD| \cdot |ND'|$, or $\dfrac{|PD|}{|MD|} = \dfrac{|QD'|}{|ND'|}$. Since $\angle PDM = \angle QD'N$ we get $\triangle PDM \sim \triangle ND'Q$, thus $\angle PMD = \angle NQD'$.

Also, from $\triangle BMD \sim \triangle NBD'$ we get $\angle BMD = \angle NBD' = \angle NAD'$.

Hence

$$\angle PMD - \angle BMD = \angle NQD' - \angle NAD' \Rightarrow \angle PMB = \angle QNA.$$

The proof is completed. □

From the above general problem, we get some corollaries

Corollary 1. *Let ABC be a triangle with bisector AD. Let M be the midpoint of BC. Suppose P and Q are two points on the segment AD such that $\angle ABP = \angle CBQ$, then the circumcenter of the triangle PQM lies on a fixed line when P, Q vary.*

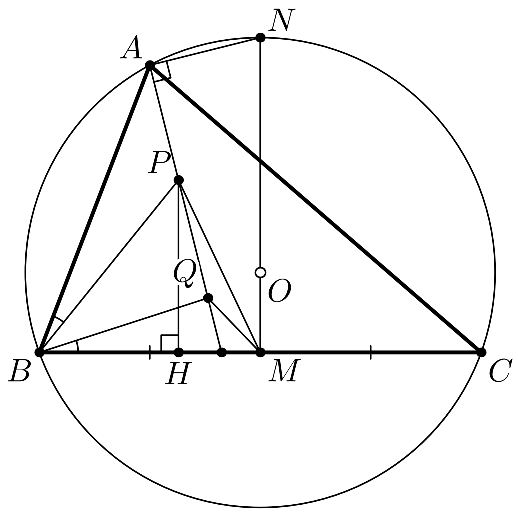

Fig. 3.

Proof. Let H the a point on BC such that $PH \perp BC$. Denote by N the midpoint of the arc BC which contains A. It is easy to see that P, Q are two isogonal conjugate points with repsect to triangle ABC. From our generalized problem, we have $\angle QNA = \angle PMB$ which yields $\angle AQN = \angle HPM = \angle PMN$ (note that $\angle NAD = 90°$), thus $QPMN$ is a concyclic quadrilateral. Therefore the circumcenter of triangle PQM lies on the perpendicular bisector of MN, which is a fixed line. We are done. □

Corollary 2. *From the generalized problem it is follows that $\angle PMN = \angle AQN$, thus if we denote the intersection of PM and AQ by T, then Q, M, N, T are concyclic. Moreover, $PM \parallel AQ$ if and only if $Q \in OM$.*

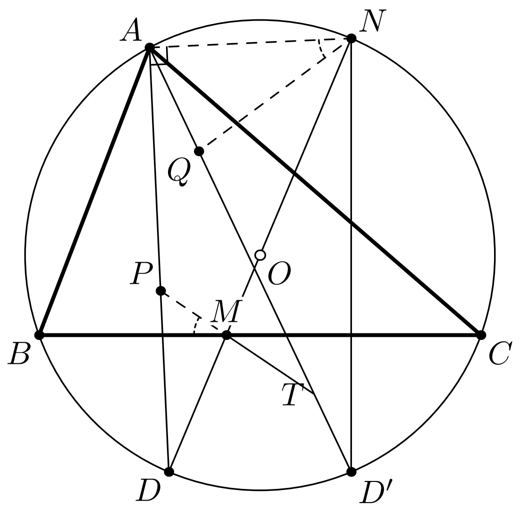

Fig. 4.

Proof. We have $\angle NMT = \angle NMC + \angle CMT = \angle MNB + \angle NBM + \angle PMB = \angle D'AC + \angle NAC + \angle QNA = \angle QAN + \angle QNA = \angle D'QN$. Hence Q, M, N, T are concyclic.

Therefore

$$PM \parallel AQ \iff (PM, AQ) = 0 \iff (NQ, ND) = 0 \iff NQ \equiv ND.$$

We are done.

□

Hence from the above corollary, we can make a new problem.

Problem 3. *Let ABC be a triangle with circumcircle (O). Let d be a line which passes through O and intersects BC at M. Suppose Q is a point on d and P is the isogonal conjugate of Q. Prove that AP and d intersect at a point lying on (O) if and only if $PM \parallel AQ$.*

The proof directly follows from Corollary 2.

References

[1] Incenter, circumcircle and equal angles, All-Russian MO Round 4, 2005. *http://www.artofproblemsolving.com/Forum/viewtopic.php?t=32163.*

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