NEUBERG LOCUS AND ITS PROPERTIES

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Abstract. In this article we discuss the famous Neuberg Locus. We also explore some special properties of the cubic, and provide purely synthetic proofs to them.

1. Introduction

In this article we will discuss the Neuberg Locus Problem. It is related with concurrency of some specific lines in the triangles PBC, PCA, PAB given any triangle ABC and a point P on its plane. The Neuberg cubic can be defined as the locus of all such points in the plane of ABC which satisfy the concurrency of the Euler lines, or the Brocard Axes.

We try to convince the reader that the problem, and all the properties are interrelated. So let us come to know about the actual problem:

The Neuberg Problem. *Given a triangle ABC and a point P, suppose, A', B', C' are the reflections of P on BC, CA, AB. Prove that AA', BB', CC' are concurrent iff PP^* is parallel to the Euler line of ABC where P^* is the isogonal conjugate of P wrt ABC. The locus of such points P is the Neuberg Cubic of ABC. But we will not use any property of cubics in our proof. So let us call the locus of P which satisfies the above concurrency fact as Neuberg locus (we are avoiding the term cubic).*

The proof that we will be discussing is synthetic. We also prove a lot of properties of the Neuberg locus which did not have well-known synthetic proofs.

2. Some Useful Lemmas

Before going into the proof of the main problem, we will mention some lemmas that will prove to be handy. These are very useful lemmas and properties which can turn out to be helpful in a lot of concurrency situations.

Lemma 1 (Definition of $\Gamma(ABC)$)**.** *Given two triangles ABC and $A'B'C'$(not homothetic), consider the set of all triangles(no two of them homothetic), so that both ABC and $A'B'C'$ are orthologic to them. Then the locus of the center of orthology of ABC and this set of triangles lie on a conic passing through A, B, C or the line at infinity.*

Proof. On the sides of BC, CA, AB take points C_1, A_1, B_1 so that $\triangle A_1B_1C_1$ is homothetic to $\triangle A'B'C'$.

Through C_1 draw parallel to AC to intersect AB at A_2. Cyclically define B_2, C_2 on BC, AC. Applying the converse of Pascal's theorem on the hexagon $A_1C_2B_1A_2C_1B_2$ we get that A_1, B_1, C_1, A_2, B_2, C_2 lie on a conic(call this conic $\Gamma(A_1B_1C_1)$). $A_3 = A_2C_1 \cap B_2A_1$, $B_3 = B_2A_1 \cap B_1C_2$, $C_3 = C_2B_1 \cap A_2C_1$. Clearly, ABC and A_3, B_3, C_3 are homothetic.

So AA_3, BB_3, CC_3 concur. Since A_2A_1 is the harmonic conjugate of AA_3 wrt AB, AC, so if we draw parallels to A_1A_2, B_1B_2, C_1C_2 through A, B, C, then they intersect BC, CA, AB at three collinear points. So there exists a conic passing through A, B, C so that the tangent at A is parallel to A_1A_2 and similar for B, C. Call this conic $\Gamma(ABC)$. Take any point X on $\Gamma(ABC)$.

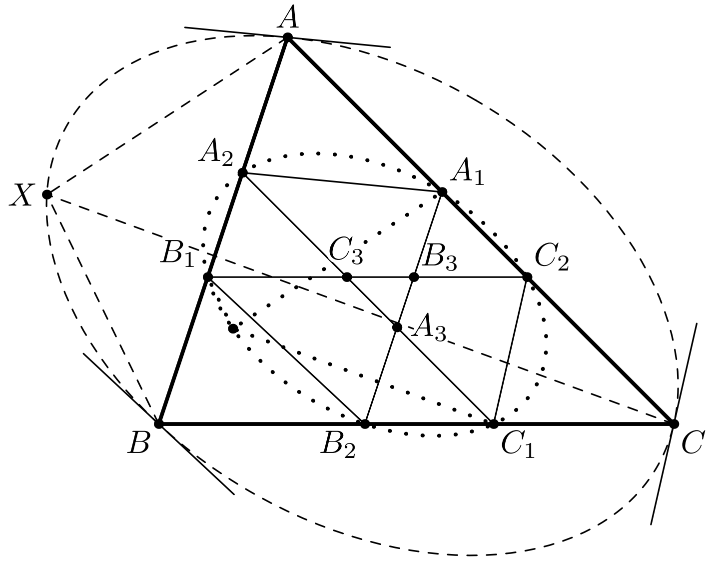

Fig. 1.

Through A_1 draw a line ℓ_a parallel to AX and similarly define ℓ_b, ℓ_c. Note that

$$(l_a, A_1A_2; A_1C_2, A_1B_2) = (AX, A_1A_2; AC, AB) =$$
$$= (BX, BA; BC, B_1B_2) = (l_b, B_1A_2; B_1C_2, B_1B_2);$$

So $\ell_a \cap \ell_b$ lies on $\Gamma(A_1B_1C_1)$ and similar for others. So ℓ_a, ℓ_b, ℓ_c concur on $\Gamma(A_1B_1C_1)$. Its easy to prove that its not true if X doesn't lie on $\Gamma(ABC)$.

In this article we will use the notation $\Gamma(ABC)$ wrt $A'B'C'$ for any two triangles ABC and $A'B'C'$ in this sense. □

Lemma 2 (Sondat's Theorem). *Given two triangles ABC and $A'B'C'$, such that they are orthologic and perspective, prove that the two centers of orthology are collinear with the perspector. This is known as the Sondat's Theorem.*

Proof. Suppose, $AP \perp B'C'$, $BP \perp C'A'$, $CP \perp A'B'$. Similarly define P' such that $A'P' \perp BC$, $B'P' \perp CA$, $C'P' \perp AB$. Suppose that, AA', BB', CC' concur at some point Q.

Note that, if we define $\Gamma(ABC)$ wrt $\triangle A'B'C'$ same as Lemma 1, then we get that, its the rectangular hyperbola passing through A, B, C, P and the orthocenter of ABC. Now note that, BPC and $B'P'C'$ are orthologic and A, A' are the two centers of orthology.

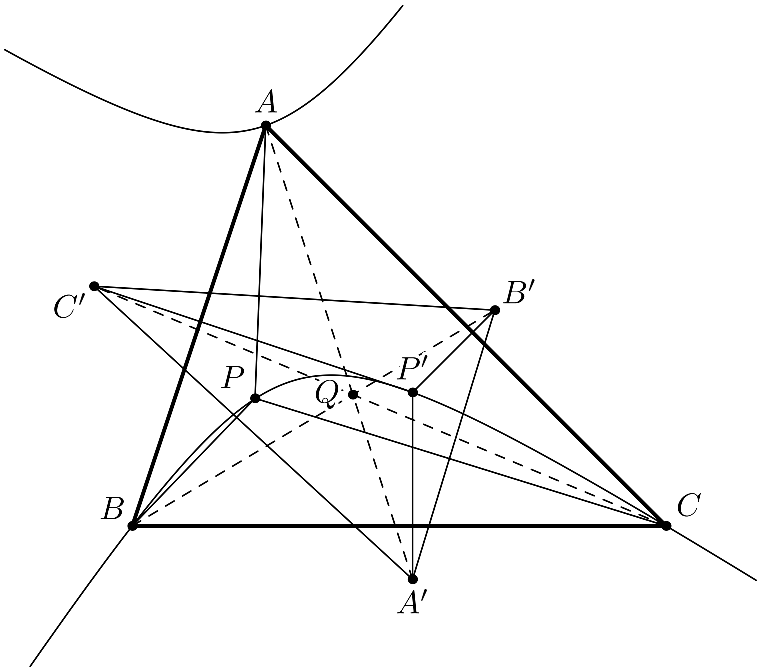

Fig. 2.

Note that, $\Gamma(ABC)$ passes through the orthocenter of $\triangle BPC$, but $\Gamma(BPC)$ wrt $\triangle B'P'C'$ is the rectangular hyperbola passing through B, C, P, A and the orthocenter of $\triangle BPC$.

So $\Gamma(BPC)$ wrt $\triangle B'P'C'$ is same as $\Gamma(ABC)$ wrt $\triangle A'B'C'$. But note that, Q lies on $\Gamma(ABC)$. So Q lies on $\Gamma(BPC)$ wrt $B'P'C'$. So $PQ \parallel P'Q$. So P, P', Q are collinear. □

Lemma 3 (Complete quadrilateral isogonality)**.** *Given two points P, Q and their isogonal conjugates wrt ABC be P', Q', then suppose, $X = PQ' \cap QP'$ and $Y = PQ \cap P'Q'$, then Y is the isogonal conjugate of X wrt ABC.*

Proof. The projective transformation that takes a point P to Q keeping the triangle ABC fixed, sends Q' to some point K. So

$$(AB, AC; AP, AQ') = (AB, AC; AQ, AK),$$

Since the transformation preserves cross-ratio.

Also $(AB, AC; AP, AQ') = (AC, AB; AP', AQ)$ since reflecting on the angle-bisector of $\angle BAC$ doesn't change the cross-ratio of the lines. And

$$(AC, AB; AP', AQ) = (AB, AC; AQ, AP').$$

So $AP' \equiv AK$. Similarly $BP' \equiv BK$. So $K \equiv P'$. So the transformation maps Q' to P'. Thence the projective transformation mapping P to Q' maps Q to P'.

Therefore $(P'A, P'B; P'C, P'Q') = (QA, QB; QC, QP)$, which leads to

$$(P'A, P'B; P'C, P'Y) = (QA, QB; QC, QY).$$

So A, B, C, Q, Y, P' lie on the same conic, which is the isogonal conjugate of the line PQ' wrt ABC.

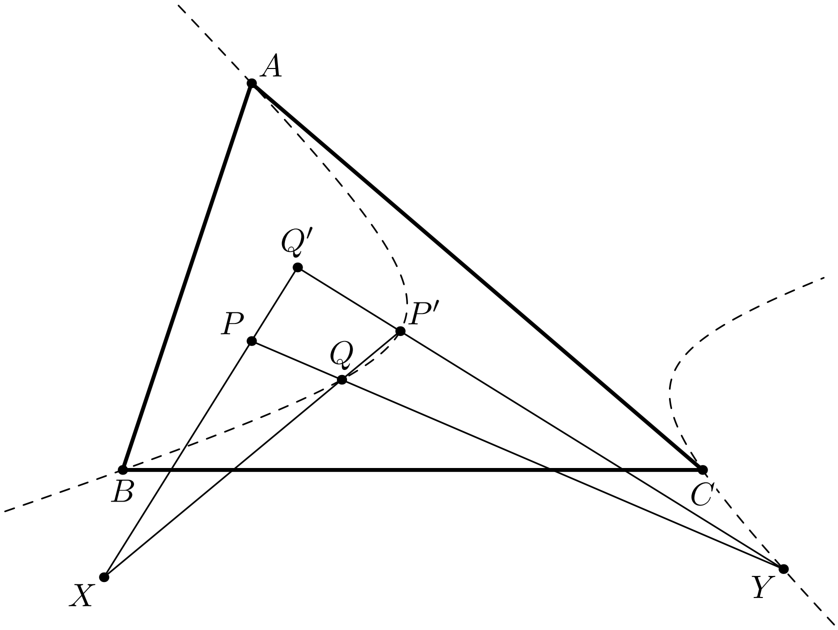

Fig. 3.

So isogonal conjugate of Y lies on PQ' and similarly it lies on QP'. So X is the isogonal conjugate of Y.

Also note that, if two points X, Y are on PQ' and PQ, such that X, Y are isogonal conjugates, then $X = PQ' \cap QP'$ and $Y = PQ \cap P'Q'$.

□

Lemma 4 (Isotomic of Lemoine Axis)**.** *Isotomic line of the Lemoine axis of a triangle ABC is perpendicular to the Euler line of ABC.*

Proof. Suppose, $A'B'C'$ is the Lemoine axis of ABC. Consider the circles with diameter AA', BB', CC'. Clearly, the orthocenter of ABC, say H, lies on their radical axis. Also the circumcenter of ABC (call it O) lies on their radical axis. So they are co-axial with the line OH as their radical axis.

However, using Gaussian line theorem we get that the line joining the midpoints of AA', BB', CC' is parallel to the isotomic line of $A'B'C'$ wrt ABC. So the isotomic line of $A'B'C'$ wrt ABC is perpendicular to OH. □

Lemma 5 (An isotomic property)**.** *Let ABC be a triangle, and let $\overleftrightarrow{A_1B_1C_1}$ and $\overleftrightarrow{A_2B_2C_2}$ be two isotomic lines such that $A_1, A_2 \in BC$ and similar for others. If L is the point where the line through A parallel to BC meets $\overleftrightarrow{A_1B_1C_1}$ then we have $\dfrac{B_1L}{C_1L} = \dfrac{A_2B_2}{A_2C_2}$.*

Proof. Note that, $\dfrac{B_1L}{C_1L} = \dfrac{AB_1}{AC_1}.\dfrac{AB}{AC}$; and also $\dfrac{B_2A_2}{A_2C_2} = \dfrac{B_2C}{C_2B}.\dfrac{AB}{AC}$. But since B_1, B_2 are isotomic points wrt A, so $AB_1 = CB_2$ and $AC_1 = BC_2$. So,

$$\frac{B_1L}{LC_1} = \frac{B_2A_2}{A_2C_2}.$$

□

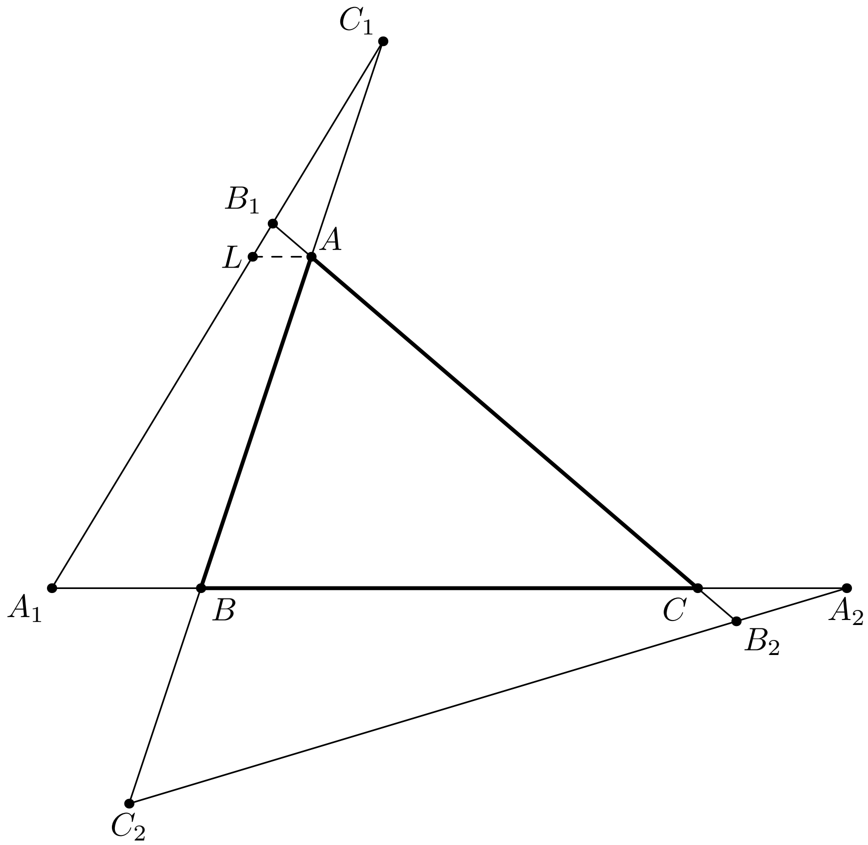

Fig. 4.

3. The Neuberg Problem

3.1. **Properties of the Neuberg locus.** At first we will prove some properties of the Neuberg locus. The properties are as follows.

Property 1 (P, P^* lie on Neuberg locus)**.** *Given a triangle ABC, if a point P lies on the Neuberg locus of ABC, then isogonal conjugate of P wrt ABC also lies on the Neuberg locus of ABC.*

Proof. Suppose, A', B', C' are the reflections of P on BC, CA, AB. Suppose, $A_1 = PA' \cap \odot A'B'C'$. Similarly define B_1, C_1. Under inversion wrt P with power $PA' \cdot PA_1$, A goes to the reflection of P on B_1C_1 and similar for others. Now note that, if A_2, B_2, C_2 are the reflections of P' on BC, CA, AB, then $\triangle A_2B_2C_2$ is homothetic to $\triangle A_1B_1C_1$. So if A_3, B_3, C_3 are the reflections of P' on B_2C_2, C_2A_2, A_2B_2, then $\odot P'A_2A_3$, $\odot P'B_2B_3$, $\odot P'C_2C_3$ are co-axial.

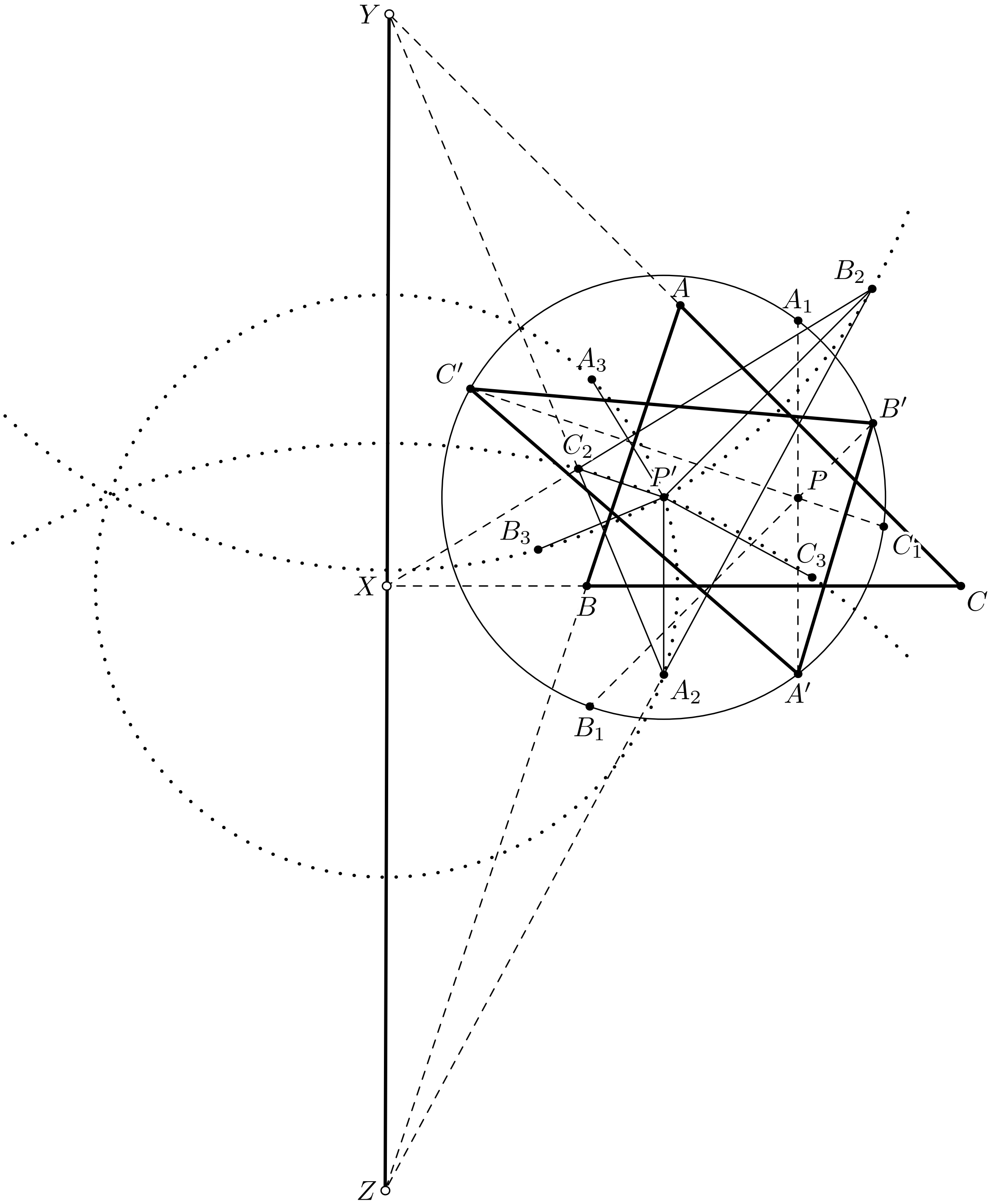

Fig. 5.

But, note that the center of $\odot P'A_2A_3$ is the intersection point of B_2C_2 and BC and similar for others. So $B_2C_2 \cap BC$, $C_2A_2 \cap CA$, $A_2B_2 \cap AB$ are collinear. So by Desargues' Theorem, $\triangle ABC$ and $\triangle A_2B_2C_2$ are perspective. So P' lies on the Neuberg locus of ABC. $\square$

Property 2 (Neuberg locus of pedal triangle)**.** *Given a triangle ABC and a point P, prove that P lies on the Neuberg locus of ABC iff P lies on the Neuberg locus of the pedal triangle of P wrt ABC.*

Proof. Suppose, P^* is the isogonal conjugate of P wrt ABC. A', B', C' be the reflections of P^* on BC, CA, AB. Note that, P is the circumcenter of $\triangle A'B'C'$.

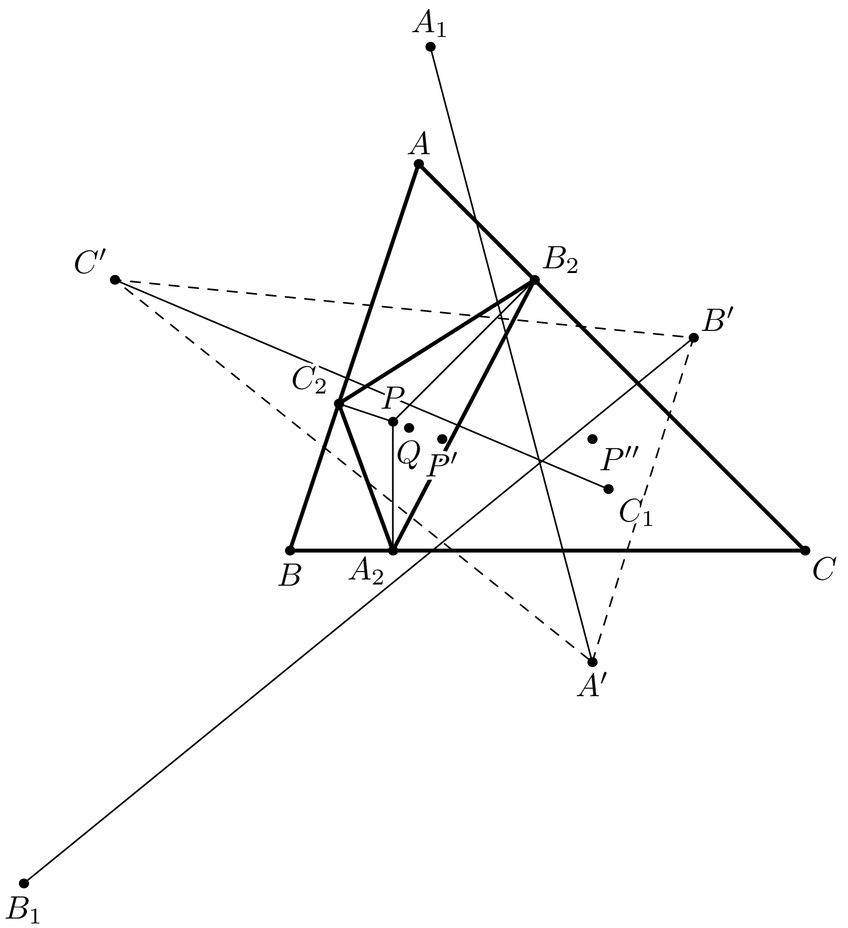

Fig. 6.

Let P' be the isogonal conjugate of P^* wrt $A'B'C'$. Let A_1, B_1, C_1 be the circumcenters of $\triangle B'P'C'$, $\triangle C'P'A'$, $\triangle A'P'B'$. Then,

$$\angle PAB' = 180^\circ - \angle B'P^*C' \text{and}$$

$$\angle PB'A_1 = \angle PB'C' + \angle C'B'A_1 = 90^\circ - \angle B'A'C' + \angle B'P'C' - 90^\circ.$$

So $\angle PB'A_1 = B'P'C' - B'A'C' = 180^\circ - \angle B'P^*C'$, leading to $PA \cdot PA_1 = PB'^2 = PA'^2$. Then, $A'A_1$ is anti-parallel to $A'A$ wrt $\angle A'PA_1$. But the angle-bisector of $\angle APA_1$ is parallel to the angle-bisector of $\angle B'A'C'$. So $A'A$ and $A'A_1$ are isogonal conjugates wrt $\angle B'A'C'$, and similar for others. But from Property 1, if P lies on the Neuberg locus of $\triangle ABC$ iff so does P^*.

So, $A'A$, $B'B$, $C'C$ concur, and $A'A_1$, $B'B_1$, $C'C_1$ concur. So P' lies on the Neuberg locus of $\triangle A_1B_1C_1$.

Now note that, if $A_2B_2C_2$ is the pedal triangle of P wrt ABC and Q is the isogonal conjugate of P wrt $A_2B_2C_2$, then the configuration $A_2B_2C_2Q$ is homothetic to the configuration $A_1B_1C_1P'$. So Q lies on the Neuberg locus of $\triangle A_2B_2C_2$. So again using Property 1, we get that P lies on the Neuberg locus of $\triangle A_2B_2C_2$.

$\square$

Corollary 1. *Given a triangle ABC and a point P on its Neuberg locus, suppose, O_a, O_b, O_c are the circumcenters of $\triangle BPC$, $\triangle CPA$, $\triangle APB$. Prove that, AO_a, BO_b, CO_c are concurrent. Call that concurrency point as Q. If P^* is the isogonal conjugate of P wrt ABC, then define Q^* for P^* similarly. Then Q and Q^* are isogonal conjugates wrt ABC. Furthermore, $Q = OP \cap HP^*$ and $Q^* = OP^* \cap HP$ where H is the orthocenter of ABC.*

Proof. From the proof of Property 2, we can directly observe(just replace ABC by $A'B'C'$) that Q and Q^* are isogonal conjugates wrt ABC. So using Lemma 3, we get that $Q = OP \cap HP^*$ and $Q^* = OP^* \cap HP$. □

Corollary 2. *Let P, P^* be any two isogonal conjugates in a triangle ABC. Suppose that the circumcenters of BPC, CPA, APB and BP^*C, CP^*A, AP^*B are O_a, O_b, O_c, O'_a, O'_b, O'_c respectively. Then the line joining the circumcenters of $O_aO_bO_c$ and $O'_aO'_bO'_c$ is parallel to PP^*. Furthermore, the circles $\odot O_aO_bO_c$, $\odot O'_aO'_bO'_c$, $\odot ABC$ are coaxial.*

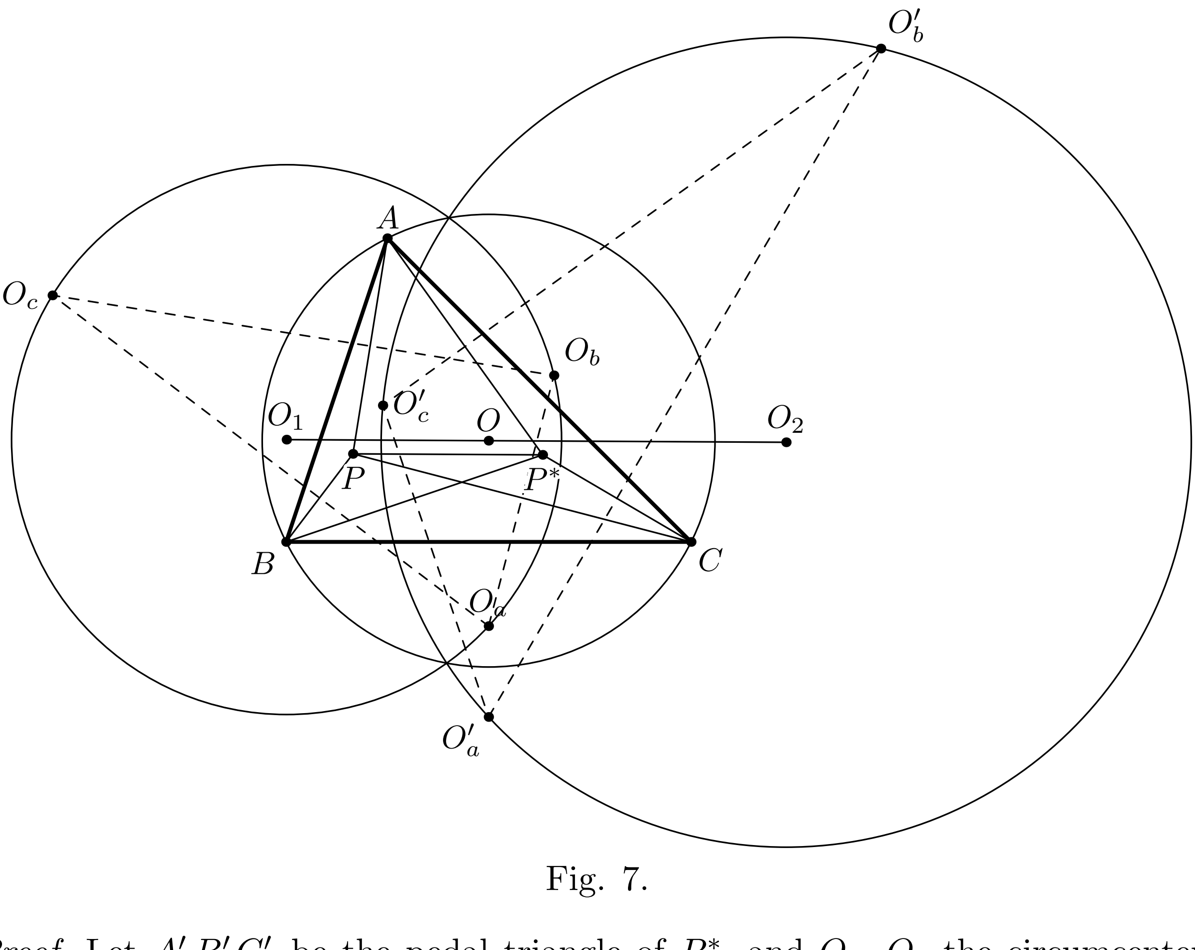

Fig. 7.

Proof. Let $A'_1B'_1C'_1$ be the pedal triangle of P^*, and O_1, O_2 the circumcenters of $O_aO_bO_c$ and $O'_aO'_bO'_c$. Then $O_bO_c \perp AP$ and $AP \perp B'_1C'_1$. So, note that the homothety that maps $A'_1B'_1C'_1$ to $O_aO_bO_c$ also sends the midpoint of PP^* to O_1 and P^* to O, where O is the circumcenter of ABC. So if J is the midpoint of PP^*, then we must have $P^*J \parallel OO_1$, and similarly $PJ \parallel OO_2$, so O_1, O_2, O are collinear and $O_1O_2 \parallel PP^*$.

For the extension, note that we have $OO_aO'_a \perp BC$, and also $\angle(O_cO'_c, O_cO_b) = \angle(AB, AP) = \angle(AC, AQ) = \angle(O'_bO'_c, O_bO'_b)$, implying $O_bO_cO'_bO'_c$ is cyclic. So, inversion about O with radius OA takes O_a to O'_a, O_b to O'_b and O_c to O'_c. Under this inversion, $\odot O_aO_bO_c$ maps to $\odot O'_aO'_bO'_c$, and their points of intersection remain fixed. So, they lie on $\odot ABC$, and the rest follows. $\square$

Property 3 (An Equivalence)**.** *Given a triangle ABC and a point P, prove that P lies on the Neuberg locus(or circumcircle or line at infinity) of ABC iff the Euler lines of $\triangle BPC$, $\triangle CPA$, $\triangle APB$ are concurrent.*

Proof. We may ignore the case when P lies on circumcircle and line at infinity. Let G_a, G_b, G_c be the centroids of the triangles BPC, CPA, APB. and O_a, O_b, O_c are the their circumcenters. Then we have $G_bG_c \parallel BC$, and $BG_c \cap CG_b$ is the midpoint of AP).

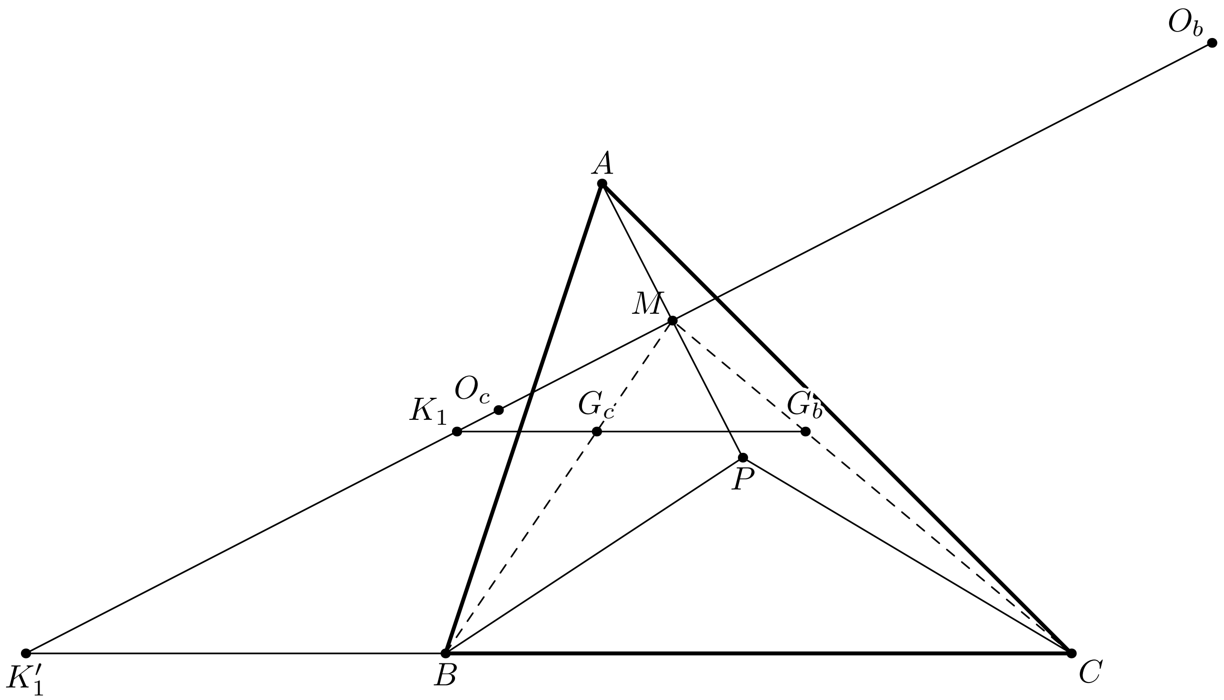

Fig. 8.

So, the homothety that maps G_bG_c to BC maps $O_bO_c \cap G_bG_c \equiv K_1$ to $O_bO_c \cap BC \equiv K'_1$. Thus, $\dfrac{G_cK_1}{G_bK_1} = \dfrac{BK'_1}{CK'_1}$. Since the triangles $G_aG_bG_c$ and $O_aO_bO_c$ are perspective, so we get $\displaystyle\prod_{cyc} \frac{G_cK_1}{G_bK_1} = -1$, which leads to $\displaystyle\prod_{cyc} \frac{BK'_1}{CK'_1} = -1$, so that ABC and $O_aO_bO_c$ are perspective. So using the corollary 1 of Property 2 we get that P lies on the Neuberg locus of $\triangle ABC$. $\square$

Property 4 (Euler lines concurrent on OH)**.** *Given a triangle ABC and a point P, if the Euler lines of BPC, CPA, APB are concurrent, then they concur on the Euler line of ABC.*

Proof. Suppose, O_a, O_b, O_c are the circumcenters of $\triangle BPC$, $\triangle CPA$, $\triangle APB$. G_a, G_b, G_c be the centroids of $\triangle BPC$,$\triangle CPA$,$\triangle APB$. Suppose, O_aG_a, O_bG_b, O_cG_c concur at some point X. Note that, $\triangle O_aO_bO_c$ and $\triangle G_aG_bG_c$ are perspective. Also note that, the perpendiculars from O_a, O_b, O_c to the sides of $\triangle G_aG_bG_c$ concur at the circumcenter of ABC and the perpendiculars from G_a, G_b, G_c to the sides of $O_aO_bO_c$ concur at the centroid of $\triangle ABC$. So by Sondat's Theorem (Lemma 2) O_aG_a, O_bG_b, O_cG_c concur on the Euler line of ABC.

□

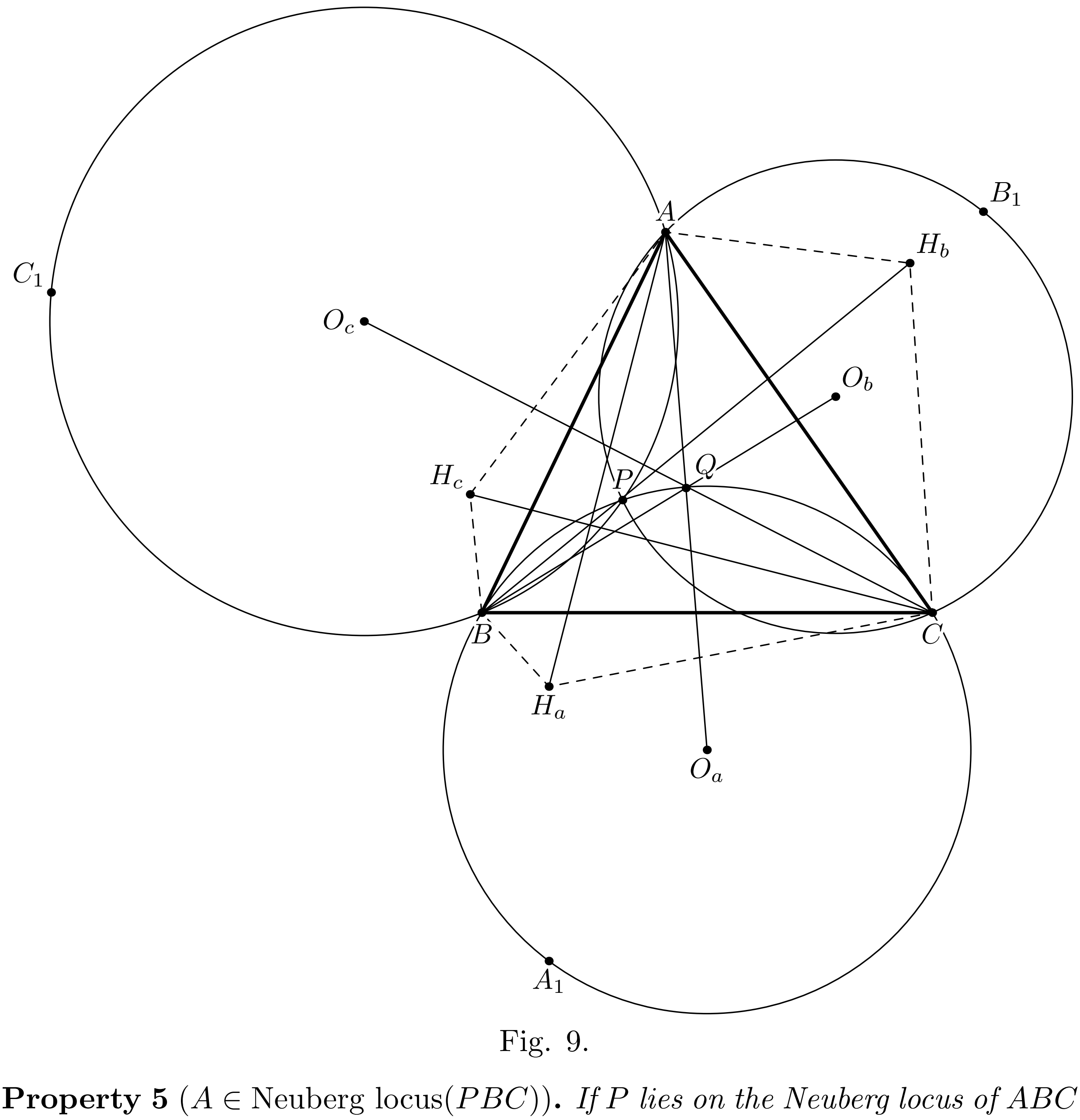

Fig. 9.

Property 5 ($A \in$ Neuberg locus(PBC))**.** *If P lies on the Neuberg locus of ABC, then A lies on the Neuberg locus of BPC and similarly.*

Proof. Note that, we have proved if the Euler lines of PBC, PCA, PAB are concurrent, then they concur on the Euler line of ABC. So using Property 3, we get that A lies on the Neuberg locus of PBC. □

Property 6 (Orthocenters of A_iBC)**.** *Given a triangle ABC and a point P, suppose that A_1, B_1, C_1 are the intersection points of AP, BP, CP with $\odot PBC$, $\odot PCA$, $\odot PAB$. H_a, H_b, H_c are the orthocenters of $\triangle A_1BC$, $\triangle B_1CA$, $\triangle C_1AB$. Prove that AH_a, BH_b, CH_c are concurrent iff P lies on the Neuberg locus of $\triangle ABC$ or the circles with diameter BC, CA, AB or the line at infinity.*

Proof. Suppose, O_a, O_b, O_c are the circumcenters of PBC, PCA, PAB. Note that, CO_a, CH_b are isogonal conjugates wrt $\angle ACB$ and similar for others. Suppose, $A' = BH_c \cap CH_b$. Similarly, define B', C'. Consider the case when P lies on the Neuberg locus of ABC. Clearly, AO_a, BO_b, CO_c concur at some point Q. Let Q^* be the isogonal conjugate of Q wrt ABC. Note that, AA', BB', CC' concur at Q^*.

So applying converse of Brianchon's theorem on $BA'CB'AC'$ we get that, there exists a conic touching BH_a, H_aC, CH_b, H_bA, AH_c, H_cB. So applying Brianchon's theorem on $AH_cBH_aCH_b$ we get that AH_a, BH_b, CH_c are concurrent.

The other two cases are just special cases which can be verified easily.

□

Property 7 (Concurrency of Brocard axes)**.** *Given a triangle ABC and point P, prove that Brocard axes of PBC, PCA, PAB are concurrent iff P lies on the Neuberg locus of ABC or the circumcircle of ABC or the line at infinity.*

Proof. Suppose, O_a, H_a, K_a are the circumcenter, orthocenter and symmedian point of PBC. Let O be the circumcenter of ABC. Note that, its enough to prove that,

$$\prod_{cyc}(O_aO_b, O_aO_c; O_aP, O_aK_a) = 1.$$

Suppose, $P'B'C'$ is the Lemoine axis of PBC. Then note that,

$$(O_aO_b, O_aO_c; O_aP, O_aK_a) = (PC, PB; PP', B'C') = \frac{C'P'}{B'P'}.$$

Let $P_1B_1C_1$ be the isotomic line of $P'B'C'$. A line through P parallel to BC intersects B_1C_1 at U. Using Lemma 5 we get that, $\dfrac{B_1U}{C_1U} = \dfrac{B'P'}{P'C'}$. Hence,

$$(PB, PC; PU, B_1C_1) = (PB, PC; PP', B'C').$$

However,

$$(PB, PC; PU, B_1C_1) = (O_aO_c, O_aO_b; O_aO, O_aH_a).$$

So,

$$(O_aO_c, O_aO_b; O_aO, O_aH_a) = (O_aO_c, O_aO_b; O_aP, O_aK_a).$$

But since the lines O_aH_a are concurrent, so

$$\prod_{cyc}(O_aO_c, O_aO_b; O_aO, O_aH_a) = 1,$$

and we are done. □

Property 8 (Quadrangles Involutifs [3]). *For a triangle ABC, a point P lies on its Neuberg locus if and only if*

$$\begin{vmatrix} 1 & BC^2+AP^2 & BC^2\cdot AP^2 \\ 1 & CA^2+BP^2 & CA^2\cdot BP^2 \\ 1 & AB^2+CP^2 & AB^2\cdot CP^2 \end{vmatrix} = 0;$$

Which is equivalent with

$$(AP^2-AB^2)(BP^2-BC^2)(CP^2-CA^2) = (AP^2-AC^2)(BP^2-BA^2)(CP^2-CB^2).$$

Proof. From the proof of Property 6(we will use the same notations in this proof too), we get that

$$\prod_{cyc} \frac{C'P'}{P'B'} = 1.$$

However,

$$\frac{C'P'}{P'B'} = \frac{C'B}{BA}\cdot\frac{AC}{CB'} = \frac{BC^2}{BC^2-PB^2}\cdot\frac{BC^2-PC^2}{BC^2} = \frac{BC^2-PC^2}{BC^2-PB^2};$$

And our result follows. □

4. Proof of The Neuberg Problem.

Given a triangle ABC and a point P, suppose, P^* is the isogonal conjugate of P wrt ABC; P_a, P_b, P_c are the circumcenters of $\triangle BPC$, $\triangle CPA$, $\triangle APB$; P_a^*, P_b^*, P_c^* be the circumcenters of $\triangle BP^*C$, $\triangle CP^*A$, $\triangle AP^*B$, respectively.

Suppose, P lies on the Neuberg locus of $\triangle P_aP_bP_c$, so that AP_a, BP_b, CP_c are concurrent. Clearly, it implies AP_a^*, BP_b^*, CP_c^* are concurrent.

Suppose, O is the circumcenter of $\triangle ABC$. Note that, the triangle formed by the circumcenters of $\triangle OP_bP_c$, $\triangle OP_cP_a$, $\triangle OP_aP_b$ is homothetic to $\triangle ABC$. So if we draw parallels to the Euler lines of $\triangle OP_bP_c$, $\triangle OP_cP_a$, $\triangle OP_aP_b$ through A, B, C they will meet at some point Q. Similarly define Q^* for P^*.

Now note that, P_bP_c is anti-parallel to P_b^*, P_c^* wrt $\angle P_bOP_c$. So the Euler line of $\triangle OP_bP_c$ is anti-parallel to the Euler line of $\triangle OP_b^*P_c^*$ wrt $\angle P_bOP_c$, and similar for others. Therefore, Q^* is the isogonal conjugate of Q wrt ABC. Note that, the triangle formed by the centroids of $\triangle OP_bP_c$, $\triangle OP_cP_a$, $\triangle OP_aP_b$ is homothetic to $\triangle P_aP_bP_c$. So if we draw parallels to AQ, BQ, CQ through P_a, P_b, P_c, then they will concur. So using Lemma 1 we get that, Q lies on $\Gamma(ABC)$ wrt $\triangle P_aP_bP_c$ which is the rectangular hyperbola passing through A, B, C, H, P where H is the orthocenter of $\triangle ABC$.

So Q^* lies on OP^*. Similarly, Q lies on OP. Thence, using Lemma 3, we get that, $Q = OP \cap HP^*$ and $Q^* = OP^* \cap HP$.

Now note that, if R is the concurrency point of AP_a, BP_b, CP_c and R^* is the concurrency point of AP_a^*, BP_b^*, CP_c^*, then from Corollary 1 of Property 2, we have $R = OP \cap HP^*$ and $R^* = OP^* \cap HP$.

All of this leads to $R \equiv Q$ and $R^* \equiv Q^*$.

From Corollary 2 of Property 2, we get that the PQ is parallel to the Euler line of $\triangle P_aP_bP_c$. But now we have, Q lies on OP, and O is the isogonal conjugate of P wrt $P_aP_bP_c$.

So the line joining P and isogonal conjugate of P wrt $P_aP_bP_c$ is parallel to the Euler line of $P_aP_bP_c$. Hence proved.

References

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