A CURIOUS GEOMETRIC TRANSFORMATION

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ABSTRACT. An expansion is a little-known geometric transformation which maps directed circles to directed circles. We explore the applications of expansion to the solution of various problems in geometry by elementary means.

In his book *Geometricheskie Preobrazovaniya*, Isaac Moiseevich Yaglom describes one curious geometric transformation called *expansion*[1].

A few years after coming across this transformation, the author of the present paper was rather surprised to discover that it leads to quite appealing geometrical solutions of a number of problems otherwise very difficult to treat by elementary means. Most of these originated in Japanese *sangaku* – wooden votive tablets which Japanese mathematicians of the Edo period painted with the beautiful and extraordinary theorems they discovered and then hung in Buddhist temples and Shinto shrines.

Researching the available literature prior to compiling this paper brought another surprise: that expansion was not mentioned in almost any other books on geometry. Only two references were uncovered – one more book by I. M. Yaglom [7] and an encyclopedia article which he wrote for the five-volume *Entsiklopedia Elementarnoy Matematiki.*

Therefore, we begin our presentation with an informal introduction to expansion and the geometry of cycles and rays.

1. Introduction

We call an oriented circle a *cycle*. Whenever a circle k of radius R is oriented positively (i.e., counterclockwise), we label the resulting cycle k^+ and say that it is of radius R. Whenever k is oriented negatively (i.e., clockwise), we label the resulting cycle k^- and say that it is of radius $-R$. Therefore, the sign of a cycle's radius determines its orientation.

We say that two cycles touch if they touch as circles and are directed the same way at their contact point. Therefore, two cycles which touch externally need to be of opposite orientations and two cycles which touch internally need to be of the same orientation. Moreover, two cycles of radii r_1 and r_2 touch exactly when the distance between their centers equals $|r_1 - r_2|$.

[1]In the English translation of I. M. Yaglom's *Complex Numbers in Geometry* [7], the term *dilatation* is used. Here, we prefer to use "expansion" instead as, in English literature, "dilatation" and "dilation" usually mean "homothety".

We call an oriented line a *ray*. We imagine rays to be cycles of infinitely large radius; we also imagine points to be cycles of zero radius[2].

The definition of tangency naturally extends to cycles and rays (Fig. 1). It also extends to cycles and points: a cycle c and a point P, regarded as a zero-radius cycle, are tangent exactly when P lies on c.

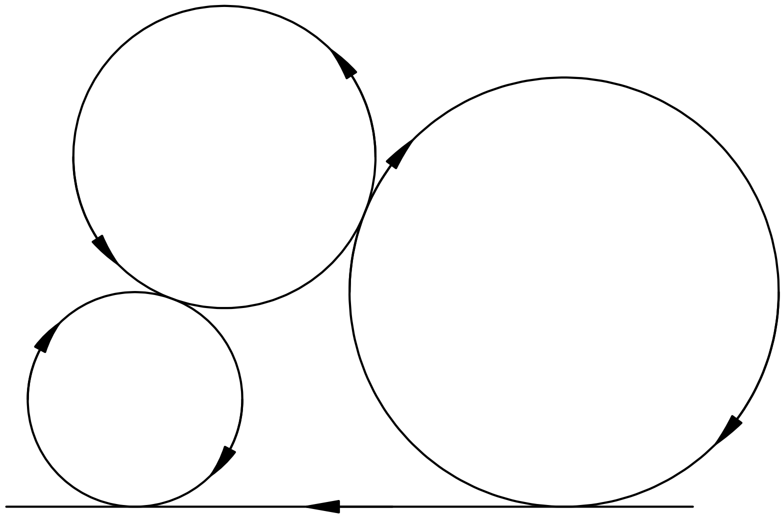

Fig. 1.

This view at tangency greatly simplifies many definitions and results in circle geometry. For instance, every two cycles have a single homothetic center; and the homothetic centers of every three cycles, taken by pairs, are collinear. Also, two cycles c_1 and c_2 always have at most two common tangents; and these two are symmetric – apart from orientation – in the line through the cycles' centers.

The length of the segment connecting a common tangent's contact points with c_1 and c_2 is called the *tangential distance* between c_1 and c_2 (Fig. 2). When c_1 and c_2 touch, the two contact points merge and the tangential distance equals zero.

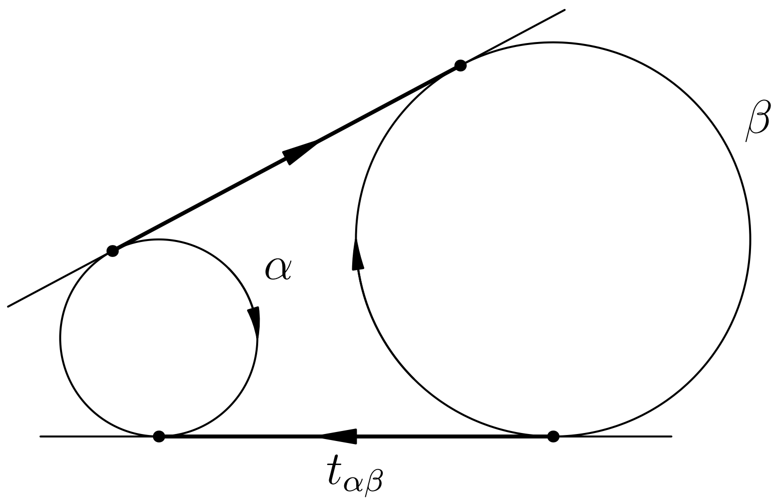

Fig. 2.

[2]Points, therefore, do not possess orientation.

The geometry of cycles and rays is well-known and there are many good introductions to it: two examples are section II.5 of I. M. Yaglom's [6] and J. F. Rigby's paper [4].

Notice that (as Fig. 3 hints) increasing or decreasing the radii of all cycles by the same amount r preserves tangency as well as tangential distance in general. Thus we arrive at a

Definition. We call an *expansion* of radius r (r being an arbitrary real number) the geometric transformation which (a) to every *cycle* of radius R maps a cycle of the same center and radius $R + r$, and (b) to every *ray* maps its translation by a distance of r "to the right" with respect to the ray's direction.

The extension of the idea of "increasing the radius by r" to rays is rather natural: in every positively oriented cycle c, the center is situated "to the right" with respect to all rays tangent to c. Notice also that every expansion maps rays to rays, points to cycles (moreover, cycles of radius r) and some cycles to points (namely, all cycles of radius $-r$). Finally, expansion alters the orientation of some cycles – namely, all cycles of radius R such that R and $R+r$ have different signs.

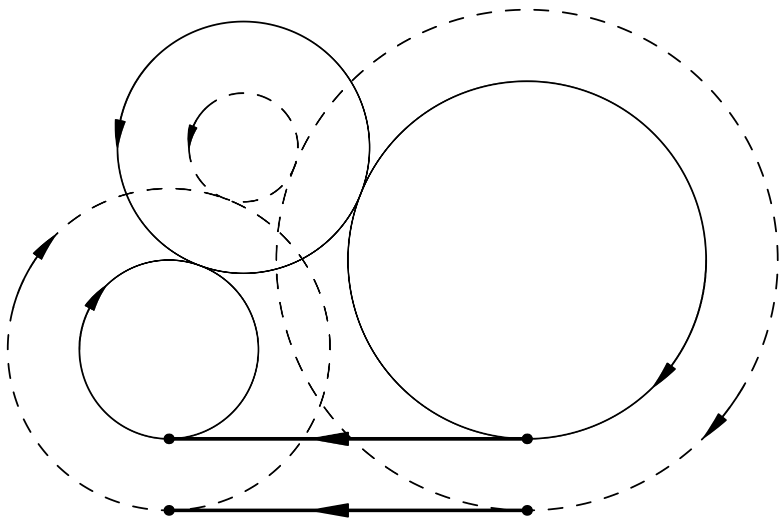

Fig. 3.

The most important property of expansion is that, as noted above, it preserves tangency between cycles and rays and tangential distance in general. This is a very intuitive result and we do not give a rigorous proof here.

2. Classical Problems

In *Geometricheskie Preobrazovaniya*, I. M. Yaglom demonstrates how expansion can be applied to some classical problems in circle geometry, yielding very short and beautiful solutions.

Problem 1. *Given two circles k_1 and k_2, construct (using straightedge and compass) their common tangents.*

Solution. We will show how to construct all common tangents (if any exist) to two *cycles* c_1 and c_2. The solution to the original problem then follows if we repeat the construction once for every possible choice of orientations for k_1 and k_2; there are only two substantially different such choices.

Let c_1 and c_2 be two cycles of centers O_1 and O_2. Apply an expansion f which maps c_1 to the point O_1 and c_2 to some cycle c'_2.

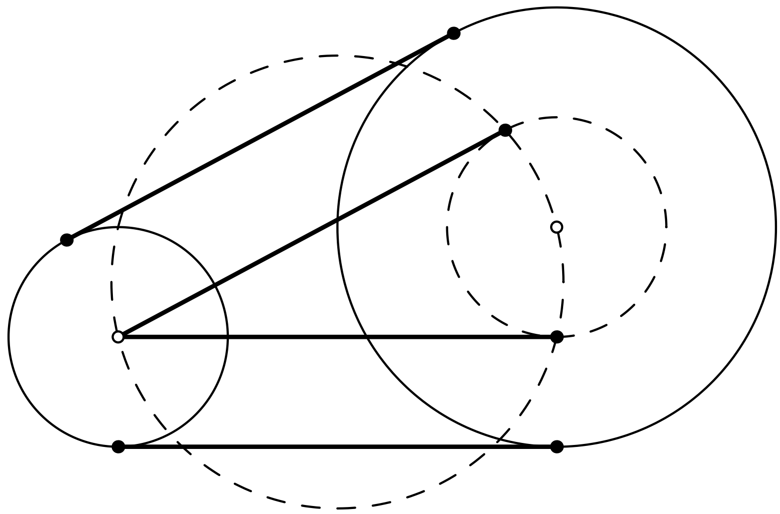

Fig. 4.

Let P and Q be the intersection points of the circle of diameter O_1O_2 and the cycle c'_2, if any exist (Fig. 4). Then the lines O_1P and O_1Q, properly oriented, constitute all rays tangent to both O_1 and c'_2. The inverse expansion f^{-1} maps these lines to the common tangents of c_1 and c_2, and we are done. □

Problem 2 (Apollonius' problem)**.** *Construct (using straightedge and compass) all circles tangent to three given circles.*

This famous problem is due to Apollonius of Perga and dates to about 200 BC. B. A. Rozenfeld suggests in [5] that perhaps the following is very close to Appolonius' original solution.

Solution. Once again, we will show how to construct all *cycles* tangent to three given cycles. The solution to the original problem is then obtained as above; only this time there are four substantially different choices for the given circles' orientations.

Let c_1, c_2, c_3 be the three given cycles. Apply an expansion f which maps c_1 to its center O and c_2 and c_3 to some cycles c'_2 and c'_3. Apply, then, inversion g of center O which maps c'_2 and c'_3 to some cycles c''_2 and c''_3 (inversion maps cycles and rays onto cycles and rays and preserves tangency).

Let c be any cycle tangent to c_1, c_2, c_3. Then f maps c to a cycle c' through O, tangent to c'_2 and c'_3, and g maps c' to some ray c'' tangent to c''_2 and c''_3.

But we already know (from the solution to Problem 1) how to construct all rays tangent to a given pair of cycles c''_2 and c''_3! Applying the inverse inversion $g^{-1} \equiv g$ and the inverse expansion f^{-1} maps these rays to all cycles tangent to c_1, c_2, c_3, and we are done. □

Since between zero and two solutions are possible for each choice of orientations for the original circles, the original problem always has between zero and eight solutions. It is not difficult to see that both extremities occur.

3. Modern Problems

Problem 3 (The equal incircles theorem)**.** *Two points D and E lie on the side BC of $\triangle ABC$ in such a way that that the incircles of $\triangle ABD$ and $\triangle ACE$ are equal. Show that the incircles of $\triangle ABE$ and $\triangle ACD$ are also equal (Fig. 5).*

Even though no explicit statement of this theorem has been found in *sangaku*, many closely related problems were treated there. In particular, the Japanese mathematicians knew how to expresses the inradius of $\triangle ABC$ in terms of the inradii of $\triangle ABD$ and $\triangle ADC$ and the altitude through A; the existence of such an expression yields the statement of the theorem.

In the West, the result has been rediscovered several times; one reference is H. Demir and C. Tezer's paper [1].

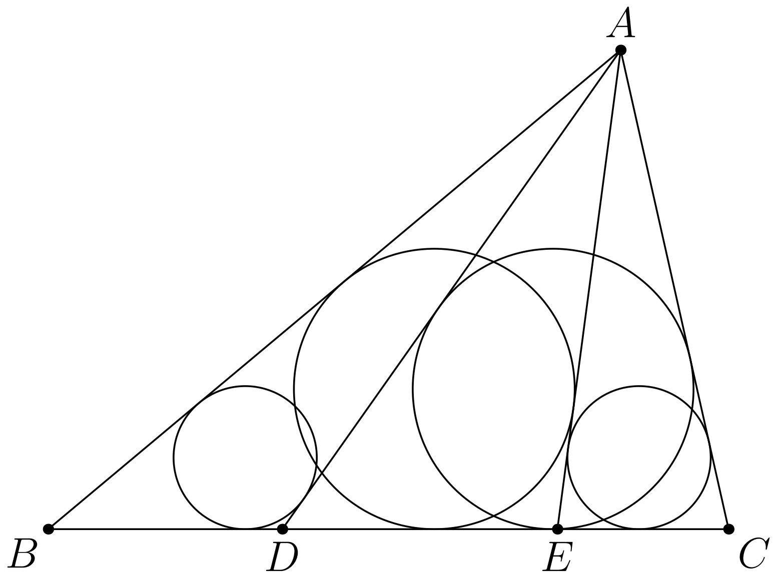

Fig. 5.

Solution. Suppose that $\triangle ABC$ is positively oriented and let ω_δ denote the incircle of the triangle δ.

Apply an expansion f which maps ω^+_{ABD} and ω^+_{ACE} to their centers I and J.

Let $f(CB^\to)$ meet $f(BA^\to)$, $f(DA^\to)$, $f(AE^\to)$, $f(AC^\to)$ in the points B', D', E', C', respectively, and let $P = f(DA^\to) \cap f(AC^\to)$ and $Q = f(BA^\to) \cap f(AE^\to)$ (Fig. 6). Let also ∞_{AB} and ∞_{AC} be the corresponding points at infinity.

The hexagon $PI\infty_{AB}QJ\infty_{AC}$ is circumscribed about the cycle $f(A)$. By Brianchon's theorem, its main diagonals PQ, IJ and $\infty_{AB}\infty_{AC}$ are concurrent – the latter meaning simply that $PQ \| IJ$.

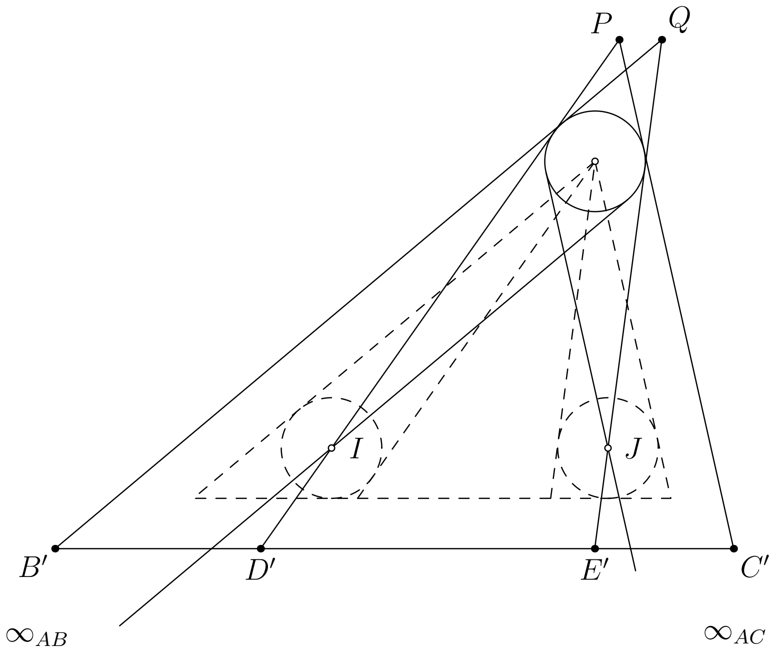

Fig. 6.

Let the tangent to $f(A)$ which is closer to $B'C'$ cut from $\triangle PD'C'$ and $\triangle QB'E'$ two smaller triangles, δ_p and δ_q, similar to the large ones. Since $PQ \| IJ \| B'C'$, the two ratios of similitude are equal; and, since the incircles of δ_p and δ_q are equal ($f(A)$ being their common incircle), the incircles of $\triangle PD'C'$ and $\triangle QB'E'$ must be equal as well.

Those two incircles, however, are in fact the images under f of ω_{ADC} and ω_{ABE}. From this we conclude that the incircles of $\triangle ADC$ and $\triangle ABE$ are equal, and the proof is complete. □

Problem 4. *Given is a right-angled $\triangle ABC$ with $\angle BAC = 90°$. The circle ω_a touches the segments AB and AC and its center lies on the segment BC. Two more circles ω_b and ω_c are constructed analogously. A fourth circle ω touches ω_a, ω_b, ω_c internally (Fig. 7). Show that the radius of ω equals $\frac{3}{2}r_a$, where r_a is the radius of ω_a.*

This problem is from an 1837 tablet found in Miyagi Prefecture [2].

Solution. Suppose that $\triangle ABC$ is positively oriented and apply expansion f of radius $-\frac{3}{2}r_a$. Let the images of ω_a^+, ω_b^+, ω_c^+ under f be ω_a', ω_b', ω_c', respectively. Our objective then becomes to show that ω_a', ω_b', ω_c' have a common point.

To this end, we introduce the following

Lemma. *Let k be a fixed circle and l be a fixed line which does not meet k. Let P be that point of k which is closest to l, and let c be a circle through P which is tangent to l. Then the length of the common external tangent of c and k remains constant when c varies.*

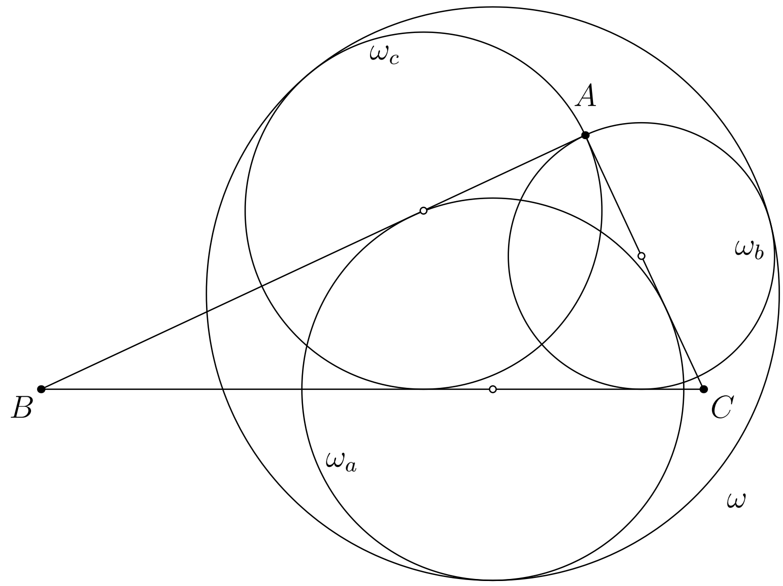

Fig. 7.

Proof. Let g be an inversion of center P which maps k to l and vice versa[3].

Continuity considerations show that there exists a circle which is concentric with k, contains k, and is preserved by g. Let s be such a circle.

Clearly, all tangents to k meet s at some fixed angle α. Conversly, all lines which meet s at an angle α are tangent to k.

Let c meet s at the point Q. Since g maps c to a line tangent to k, and preserves s, the angle between c and s equals α.

It follows that the tangent to c at Q is also a tangent to k. But the lengths of the tangents to k from a point on s are all equal (since k and s are concentric), and we are done. □

Back to Problem 4, let $k \equiv \omega'_a$ and $l \equiv f(BC^{\rightarrow})$ (ω'_a and $f(BC^{\rightarrow})$ regarded, respectively, as a circle and a line; see Fig. 8). Let P be that point of ω'_a which is closest to l. Then the radius of ω'_a equals $\frac{1}{2}r_a$ and the distance from P to l equals r_a.

If we allow c to be the circle of diameter PS, S being the projection of P onto l, we see that in this case the constant common external tangent length from the lemma equals r_a.

This, however, is also the tangential distance from ω_a to ω_b and ω_c in the original configuration, and, consequently, from ω'_a to ω'_b and ω'_c in the transformed one. Since ω'_b and ω'_c both touch l, we conclude from the lemma that they both pass through P. With this, the solution is complete. □

[3]The radius of g will then be an imaginary number. In case that we wish to consider real-radius inversions only, we can regard g as the composition of inversion of center P and two-fold rotational symmetry of center P.

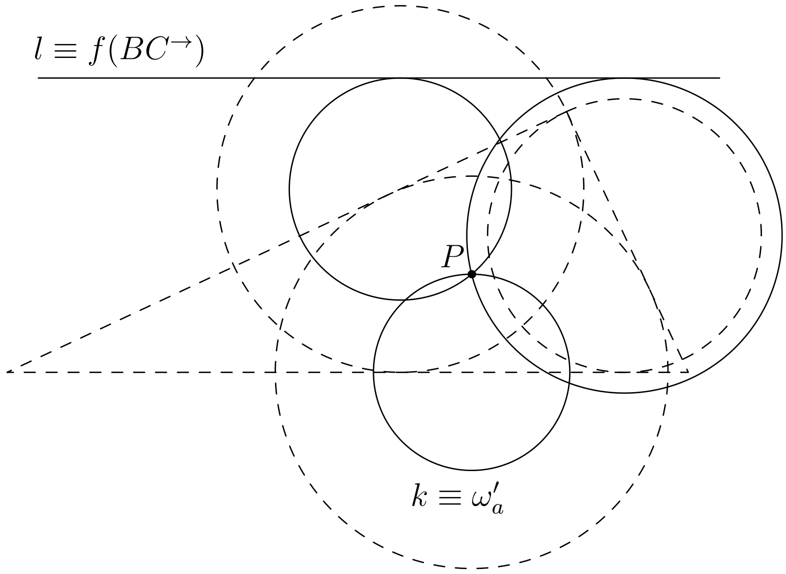

Fig. 8.

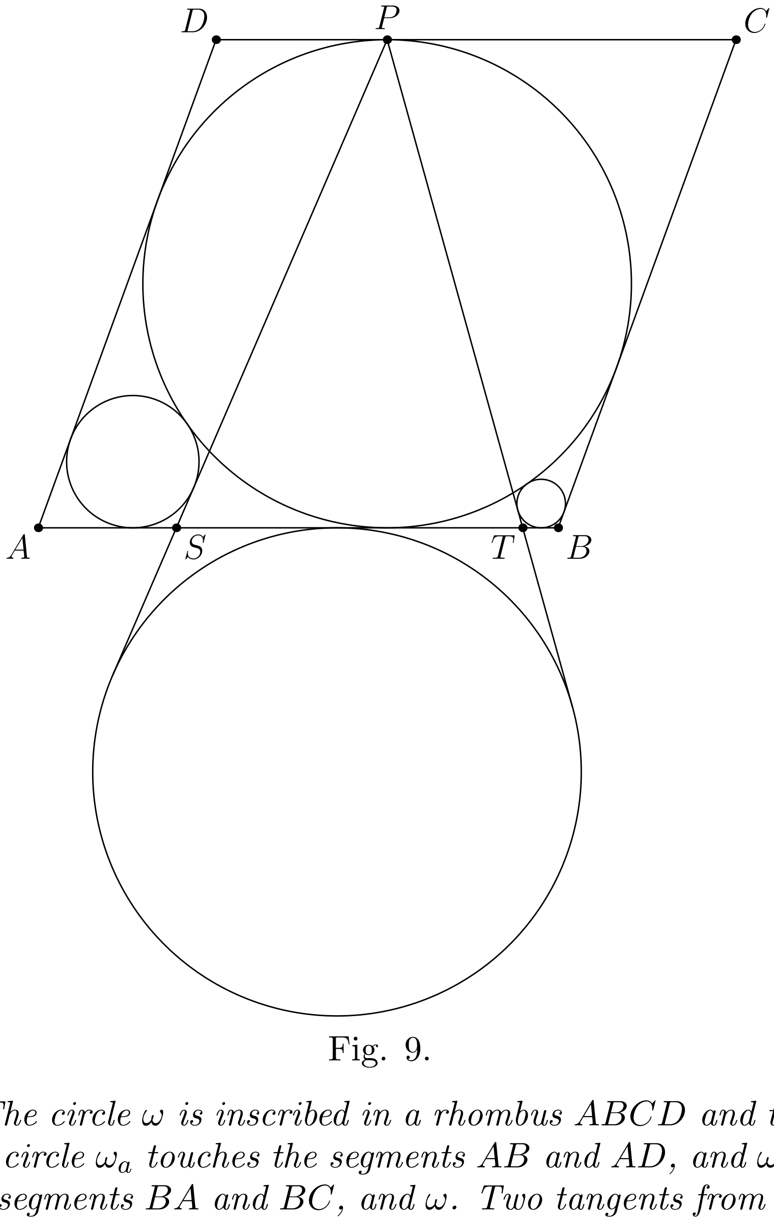

Fig. 9.

Problem 5. *The circle ω is inscribed in a rhombus $ABCD$ and touches the side CD at P. The circle ω_a touches the segments AB and AD, and ω, and the circle ω_b touches the segments BA and BC, and ω. Two tangents from P to ω_a and ω_b*

meet the segment AB in S and T, respectively. Show that the excircle of $\triangle PST$ *opposite P is equal to* ω *(Fig. 9).*

This beautiful problem is due to H. Okumura and E. Nakajima [3][4]. It was composed as a generalization of a 1966 problem by A. Hirayama and M. Matsuoka in which $ABCD$ is a square.

Solution. Suppose that $ABCD$ is positively oriented and apply an expansion f which maps ω^+ to its center O. Let $f(w_a^-) = k_a$, $f(w_b^-) = k_b$, $f(BA^{\rightarrow}) = l$, and $f(P) = k$; notice that O lies on k, k_a and k_b meet in O, and k_a and k_b are orthogonal (their radii at O being the diagonals of $ABCD$ and therefore perpendicular).

Our objective has become to show that the appropriate common external tangents of k and k_a, and k and k_b, meet on l (Fig. 10).

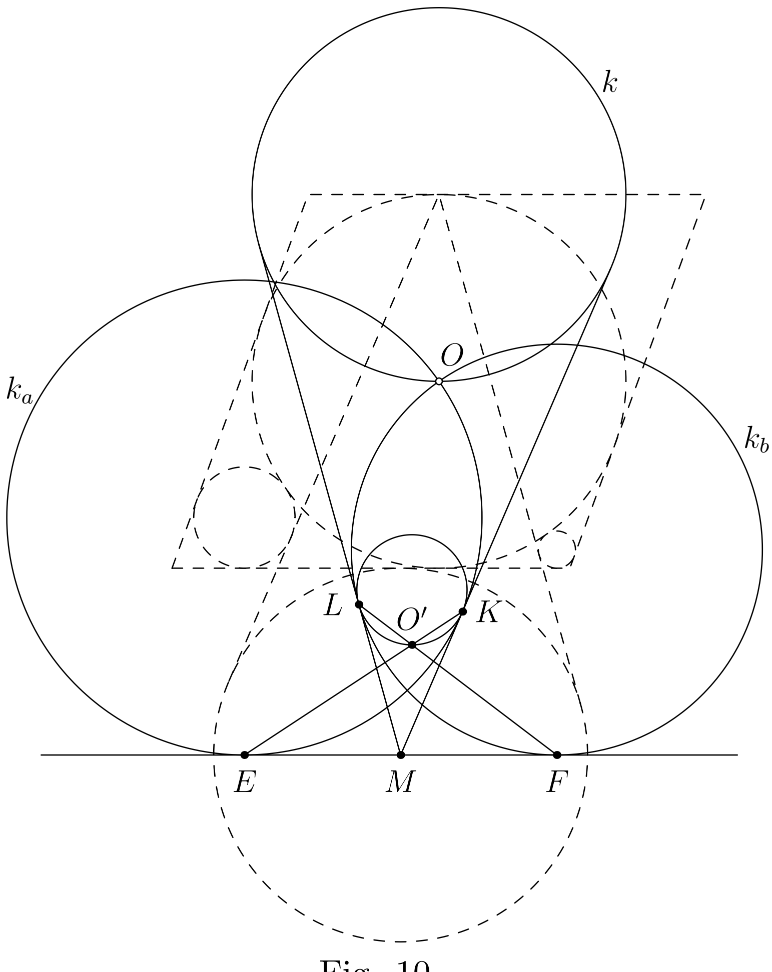

Fig. 10.

In the argument to follow, we may "forget" about orientation and simply speak of all cycles involved as of circles.

Let k_a and k_b touch l at E and F, respectively, and let M be the midpoint of EF. Let the second tangents to k_a and k_b from M meet k_a and k_b at K and L.

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[4]In [3]'s original formulation, the problem asks for a proof that the inradius of $\triangle PST$ equals one half the radius of ω. It is easy to see that the two formulations are equivalent.

Since $MK = ME = MF = ML$, there exists a circle k' tangent to MK and ML at K and L. From this, it follows that k' is also tangent to k_a and k_b.

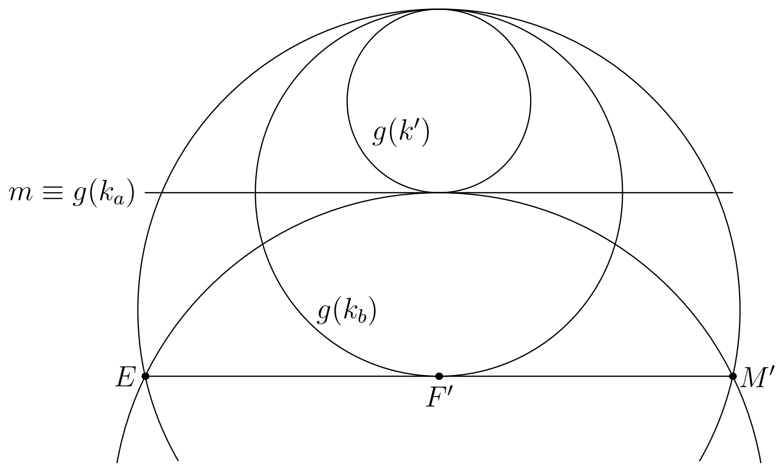

Fig. 11.

Let O' be that point of k' which is closest to l. Homothety of center K shows that the points K, O', E are collinear, and homothety of center L shows that the points L, O', F are collinear as well. Since the quadrilateral $ELKF$ is cyclic (being inscribed in a circle of center M), it follows that $EO'.O'K = FO'.O'L$ and O' lies on the radical axis of k_a and k_b.

The points M and O lie on this radical axis too, and we conclude that M, O', and O are collinear.

For any circle c, denote by $R(c)$ the ratio of its radius to the distance from its center to the line l. We know that $R(k) = r : 3r = 1 : 3$, where r is the radius of ω. We set out to calculate $R(k')$.

Let g be an inversion of center E. Under g, l is preserved, F and M are mapped to some F' and M' such that F' is the midpoint of EM', k_a is mapped to a line m parallel to l, k_b is mapped to a circle s which is tangent to l and whose center lies on m (k_a and k_b being orthogonal), and the lines MK and ML are mapped to two circles passing through E and M' (Fig. 11).

The whole configuration, then, becomes symmetric with respect to the perpendicular bisector of the segment EM'. It follows that the same must be true for $g(k')$.

If the radius of s equals r_s, this allows us to calculate $R(k') = R(g(k'))$ (as the center of inversion lies on l) $= \frac{1}{2}s : \frac{3}{2}s = 1 : 3$.

From the fact that $R(k') = R(k)$, and that M, O', O are collinear, it follows that k' and k are homothetic with center M. The appropriate common external tangents of k and k_a, and k and k_b, then, are the lines MK and ML – and they are indeed concurrent in a point M on l, as needed. □

Problem 6. *The circle ω of radius r is externally tangent to the circles k_1 and k_2, all three circles being tangent to the pair of lines t_1 and t_2. A fourth circle k is internally tangent to both of k_1 and k_2 (Fig. 12). Let ω_1 and ω_2 of radii r_1 and*

r_2 be the largest circles inscribed in the two segments which the lines t_1 and t_2 cut from k (and which do not contain ω). Show that $r_1r_2 = r^2$.

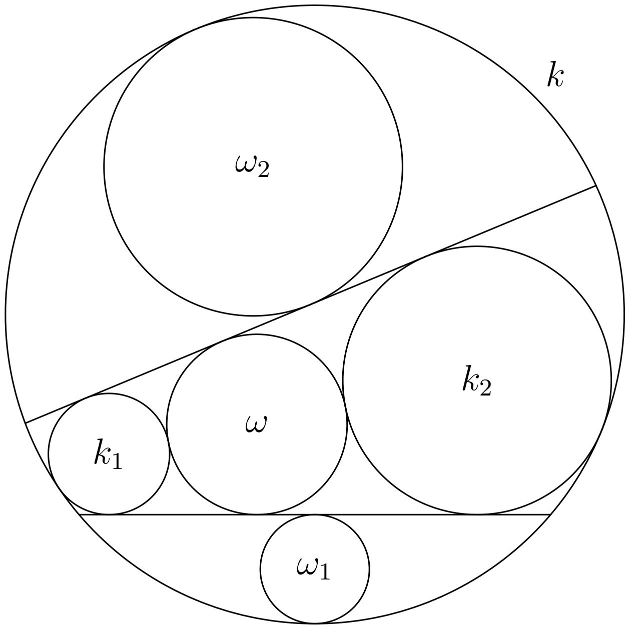

Fig. 12.

This problem is from an 1839 tablet found in Nagano Prefecture [2].

Solution. First we prove the following

Lemma. *Let t_1 and t_2 be the two common external tangents of a fixed pair of disjoint circles k_1 and k_2. Let k be an arbitrary circle internally tangent to both of k_1 and k_2, and let h_1 and h_2 be the altitudes of the two segments which t_1 and t_2 cut from k (and which do not contain k_1 and k_2). Then h_1h_2 remains constant when k varies; furthermore, if d is the projection of the common external tangent of k_1 and k_2 onto their center line, then always $h_1h_2 = \frac{1}{4}d^2$.*

Proof. Notice that h_1, h_2, and d are all invariant under expansion; therefore, it suffices to consider the case in which k_1 collapses into some point O (Fig. 13).

Let t_1 and t_2 meet k for the second time in A and B; let k_2 touch k, OA, and OB in T, U, and V, respectively; and let the midpoints of OA, OB, $\overset{\frown}{OA}$, and $\overset{\frown}{OB}$ be M, N, K, and L.

Since T is the homothetic center of k and k_2 and the tangents to these circles at K and U are parallel, the points T, U, K are collinear.

Since $\angle OTK = \overset{\frown}{OK} = \overset{\frown}{KA} = \angle UOK$, we have $\triangle OTK \sim \triangle UOK$ and therefore $KU.KT = KO^2$. Analogously, $LV.LT = LO^2$.

It follows that the line KL is the radical axis of O and k_2. Let O' be the center of k_2; then $KL \perp OO'$ and KL cuts OU and OV in their midpoints U' and V'. Finally, let KL cut OO' in H.

We have $\angle HLO = \overset{\frown}{KO} = \overset{\frown}{AK} = \angle MOK$. Analogously, $\angle NOL = \angle HKO$. It follows that $HOLN \sim MKOH$ and $h_1h_2 = LN.MK = OH^2$. Since OH is the projection of $OU' = \frac{1}{2}OU$ onto OO', we have $OH = \frac{1}{2}d$ and $h_1h_2 = \frac{1}{4}d^2$, as needed. □

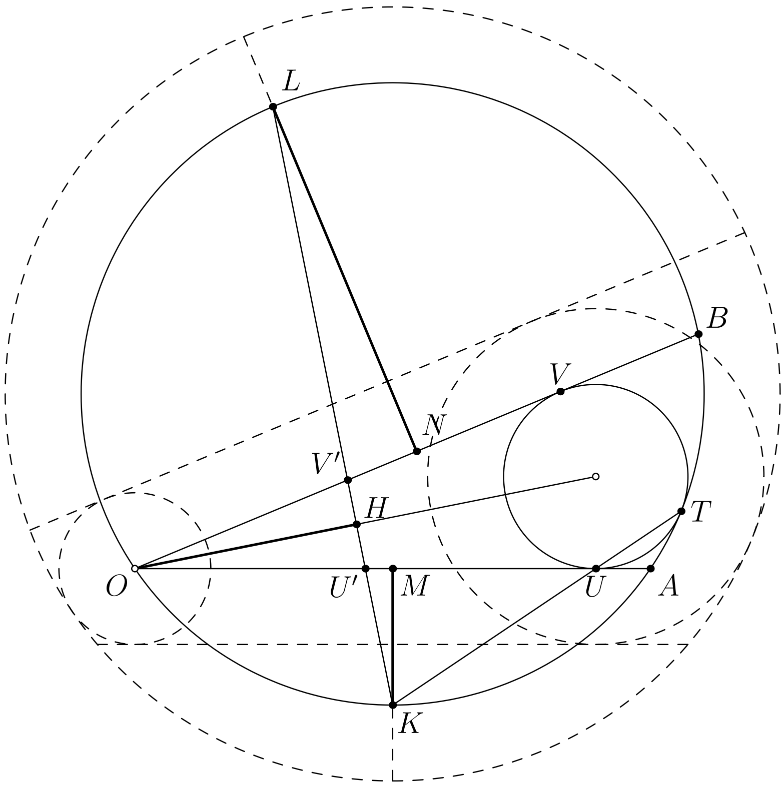

Fig. 13.

Back to the configuration of Problem 6, let the common internal tangents m_1 of ω and k_1, and m_2 of ω and k_2, cut from t_1 some segment s. Since m_1 bisects the common external tangent of ω and k_1, and m_2 bisects the common external tangent of ω and k_2, the length of s equals exactly one half the common external tangent of k_1 and k_2. Its projection onto their center line, on the other hand, is a diameter of ω; by the lemma, the desired equality follows. □

Notice that our lemma proves rather useful in a number of other *sangaku* problems as well. A problem from an 1823 tablet found in Fukusima Prefecture, for instance, asks for a proof that $h_1 h_2 = 16r^2$ in the case when k_1 and k_2 are the end members of a chain of five successively tangent circles of radius r locked between t_1 and t_2 [2].

Problem 7. *Let ω be the largest circle inscribed in the smaller segment which the line l cuts from the circle k of radius r. The circle ω_2 of radius r_2 is equal to ω, touches l and is contained in the larger segment which l cuts from k. Two more circles ω_1 and ω_3 of radii r_1 and r_3 are tangent to l, externally tangent to ω_2 and internally tangent to k (Fig. 14). Show that $r = r_1 + r_2 + r_3$.*

Solution. Apply an expansion f which maps ω_2^- to its center O_2, and ω_1^+, ω_3^+, k^+ to some cycles ω_1', ω_3', k' of radii r_1', r_3', r', respectively (Fig. 15). Since the radii of the circles ω_1', ω_3', k' equal $r_1 + r_2$, $r_3 + r_2$, $r + r_2$, respectively, our aim has become to show that $r_1' + r_3' = r'$.

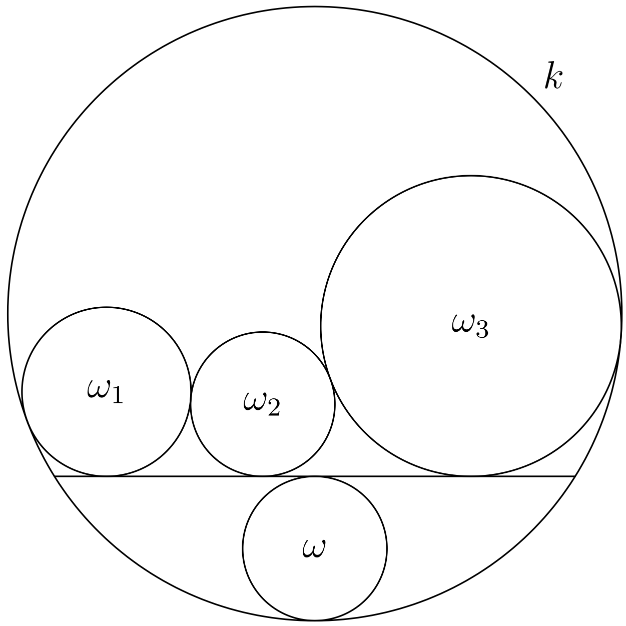

Fig. 14.

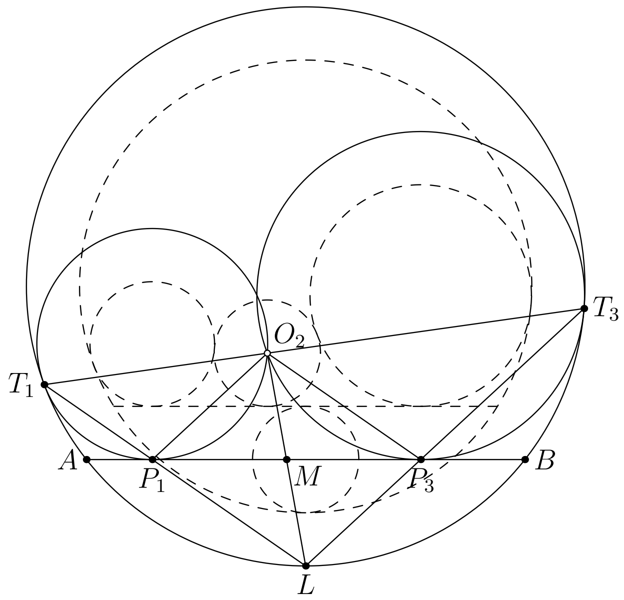

Fig. 15.

This problem is from an 1888 tablet found in Fukusima Prefecture [2].

Let the image l' of l under f (which is a common external tangent of ω_1' and ω_3') meet k' in A and B; let also ω_1' touch k' and AB at T_1 and P_1, respectively, and define T_3 and P_3 analogously for ω_3'. Finally, let L be the midpoint of $\widehat{AB}$.

As in the solution to the previous problem, we see that the points L, P_1, T_1 are collinear, the points L, P_3, T_3 are collinear, and $LP_1.LT_1 = LA^2 = LP_3.LT_3$. It follows that LO_2 is the radical axis of ω_1' and ω_3'. Therefore, LO_2 cuts their common tangent P_1P_3 in its midpoint M.

The distances from both L and O_2 to l' equal $2r_2$; therefore, M is the midpoint of LO_2 as well and $LP_1O_2P_3$ is a parallelogram.

Since the tangents to the circles ω_1', ω_3', k' at the points P_1, P_3, L are parallel and $LP_1T_1 \| P_3O_2$ and $LP_3T_3 \| P_1O_2$, the three triangles $\triangle P_1T_1O_2$, $\triangle P_3O_2T_3$, $\triangle LT_1T_3$ are similar and homothetic. It follows that the points T_1, O_2, T_3 are collinear.

Since the circumradii r_1', r_3', r' of these triangles are proportional to their corresponding sides T_1O_2, O_2T_3, T_1T_3, and $T_1O_2 + O_2T_3 = T_1T_3$, we also have $r_1' + r_3' = r'$, as needed. □

Finally, expansion may sometimes serve to produce new problems. Consider the following

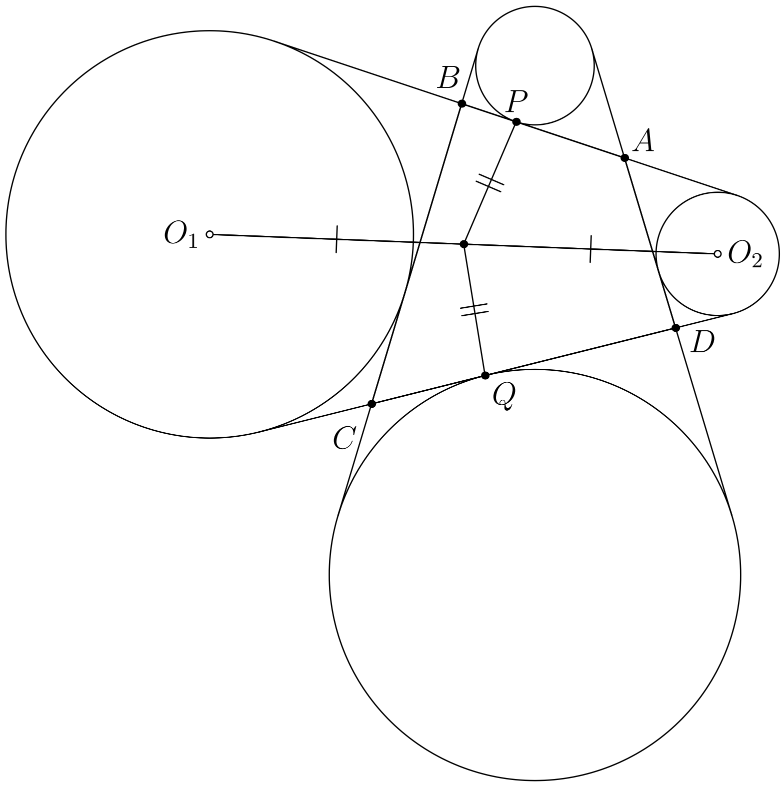

Fig. 16.

Problem 8a. *A point A lies on the common external tangent t_1 of two disjoint circles ω_1 and ω_2 of centers O_1 and O_2, respectively. The second tangents from A to ω_1 and ω_2 meet their second common external tangent t_2 in B and C. If the excircle of $\triangle ABC$ opposite A touches BC at D, show that the points A and D are equidistant from the midpoint of the segment O_1O_2.*

This problem was proposed for the 2005 St. Petersburg Mathematical Olympiad by D. Dzhukich and A. Smirnov. While working on this paper, it occured to the author that expansion would preserve tangency in the problem's configuration. It turned out to preserve the required property as well, and thus a pleasingly symmetric generalization was discovered.

Problem 8b. *Given is a convex quadrilateral $ABCD$. Four circles are constructed as follows: ω_a touches the segment AB and the extensions of DA and BC beyond A and B, and for the definitions of ω_b, ω_c, ω_d, the letters A, B, C, D*

are permuted cyclically (Fig. 16). Let ω_a and ω_c touch AB and CD at P and Q, and let O_b and O_d be the centers of ω_b and ω_d. Show that the points P and Q are equidistant from the midpoint of the segment O_bO_d.

References

[1] H. Demir and C. Tezer. More on incircles. *Mathematics magazine*, 62(2):107–114, 1989.

[2] H. Fukagawa and D. Pedoe. *Japanese temple geometry problems: san gaku.* Charles Babbage Research Center, 1989.

[3] H. Okumura and E. Nakajima. Incircles of rhombuses. *Mathematics and informatics quarterly 8*, (3):143–148, 1998.

[4] J. F. Rigby. Cycles and tangent rays. *Mathematics magazine*:155–167, 1991.

[5] B. A. Rozenfeld. *Appoloniy pergskiy.* M.:MCCME, 2004.

[6] I. M. Yaglom. *Geometric transformations.* Vol. 2. Mathematical Assn of Amer, 1968.

[7] I. M. Yaglom. *Complex numbers in geometry.* Academic Press New York, 1968.

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