

# SOME PROPERTIES OF THE BROCARD POINTS OF A CYCLIC QUADRILATERAL

DIMITAR BELEV

ABSTRACT. In this article we have constructed the Brocard points of a cyclic quadrilateral, we have found some of their properties and using these properties we have proved the problem of A. A. Zaslavsky.

## 1. THE PROBLEM

*Alexey Zaslavsky, Brocard's points in quadrilateral [4].*

Given a convex quadrilateral  $ABCD$ . It is easy to prove that there exists a unique point  $P$  such that  $\angle PAB = \angle PBC = \angle PCD$ . We will call this point *Brocard point* ( $Br(ABCD)$ ) and the respective angle *Brocard angle* ( $\phi(ABCD)$ ) of broken line  $ABCD$ . Note some properties of Brocard's points and angles:

- $\phi(ABCD) = \phi(DCBA)$  if  $ABCD$  is cyclic;
- if  $ABCD$  is harmonic then  $\phi(ABCD) = \phi(BCDA)$ . Thus there exist two points  $P, Q$  such that  $\angle PAB = \angle PBC = \angle PCD = \angle PDA = \angle QBA = \angle QCB = \angle QDC = \angle QAD$ . These points lie on the circle with diameter  $OL$  where  $O$  is the circumcenter of  $ABCD$ ,  $L$  is the common point of its diagonals and  $\angle POL = \angle QOL = \phi(ABCD)$ .

**Problem.** *Let  $ABCD$  be a cyclic quadrilateral,  $P_1 = Br(ABCD)$ ,  $P_2 = Br(BCDA)$ ,  $P_3 = Br(CDAB)$ ,  $P_4 = Br(DABC)$ ,  $Q_1 = Br(DCBA)$ ,  $Q_2 = Br(ADCB)$ ,  $Q_3 = Br(BADC)$ ,  $Q_4 = Br(CBAD)$ . Then  $S_{P_1P_2P_3P_4} = S_{Q_1Q_2Q_3Q_4}$ .*

I first saw this problem 2-3 years ago in an article by Alexei Myakishev (see [3]). I did not consider solving it then, but my pupils built a structure very similar (seemingly) to the described. Thus Zaslavsky's problem served as a stimulus (accelerator) to develop a good article (related to isogonal conjugate points), which they presented on international events. I sincerely hope that this article will appear on the pages of "Kvant". Now they are university students, but again I saw the problem in the "Journal of Classical Geometry". At first I thought that using the developed article I would figure out the solution quickly, but I did not. A new construction came out.

## 2. SOME PROPERTIES OF BROCARD POINTS OF A TRIANGLE.

Let  $P$  and  $Q$  be the first and the second Brocard points of a  $\triangle ABC$ . We will prove that the Brocard points and the three vertices  $A$ ,  $B$  and  $C$  define three pairs of similar triangles, and three pairs of triangles with equal areas (Fig. 1).

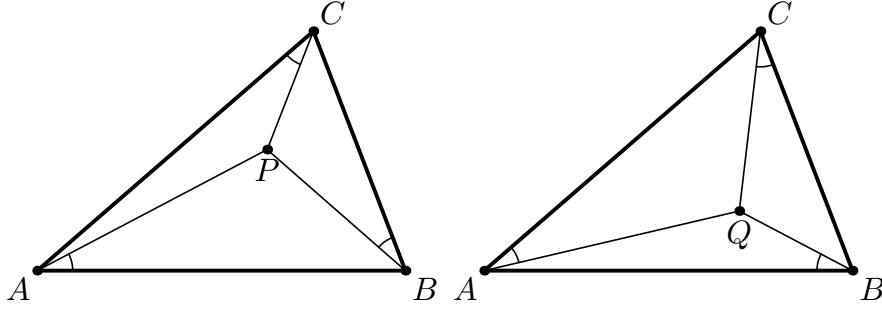


Fig. 1.

**Proposition 1.** *If  $P$  and  $Q$  are the first and the second Brocard points of a  $\triangle ABC$ , then:*

- (i)  $\triangle ABP \sim \triangle CBQ$ ,  $\triangle BCP \sim \triangle ACQ$  and  $\triangle CAP \sim \triangle BAQ$ ;
- (ii)  $S_{ABP} = S_{ACQ}$ ,  $S_{BCP} = S_{BAQ}$  and  $S_{CAP} = S_{CBQ}$ .

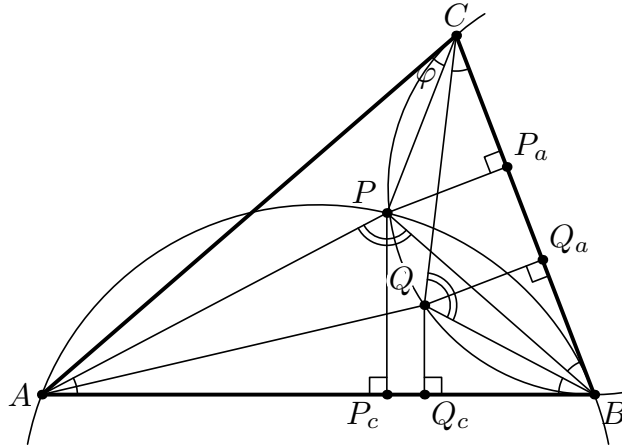


Fig. 2.

*Proof.* If we denote with  $\varphi$  the Brocard angle (Fig. 2), then  $\angle BAP = \angle BCQ = \varphi$ , and from the construction ([1]) of the Brocard points of  $\triangle ABC$  we have  $\angle APB = \angle CQB = 180^\circ - \angle B$ . Hence  $\triangle ABP \sim \triangle CBQ$ .

We construct the altitudes  $PP_c, PP_a, QQ_c$  and  $QQ_a$ . Then, having  $\triangle ABP \sim \triangle CBQ$ ,  $\triangle P_cBP \sim \triangle Q_aBQ$  and  $\triangle Q_cBP \sim \triangle P_aBQ$ , we have the following equations:

$$\frac{S_{BPC}}{S_{BAQ}} = \frac{BC \cdot PP_a}{AB \cdot QQ_c} = \frac{QQ_a}{PP_c} \cdot \frac{PP_a}{QQ_c} = \frac{BQ}{BP} \cdot \frac{BP}{BQ} = 1.$$

For the other pairs of triangles, the proof is analogous.  $\square$

We are going to find, similar to these properties, for the Brocard points of a cyclic quadrilateral.

### 3. CONSTRUCTION

Let the quadrilateral  $ABCD$  be a cyclic,  $AB \cap CD = E$  and let for exactitude  $B$  is between  $A$  and  $E$ ,  $AD \cap BC = F$  and  $D$  is between  $A$  and  $F$ . We denote  $\angle BAD = \alpha$  and  $\angle ABC = \beta$ .

In order to construct the Brocard points of the quadrilateral  $ABCD$ , we firstly construct the points  $M_1, M_2, M_3, M_4$  and  $N_1, N_2, N_3, N_4$ , as shown in Table 1:

$M_1$	$M_1 \in AD$ and $BM_1 \parallel CD$	$N_1$	$N_1 \in AD$ and $CN_1 \parallel BA$
$M_2$	$M_2 \in AB$ and $CM_2 \parallel DA$	$N_2$	$N_2 \in AB$ and $DN_2 \parallel CB$
$M_3$	$M_3 \in BC$ and $DM_3 \parallel AB$	$N_3$	$N_3 \in BC$ and $AN_3 \parallel DC$
$M_4$	$M_4 \in CD$ and $AM_4 \parallel BC$	$N_4$	$N_4 \in CD$ and $BN_4 \parallel AD$

Table 1

Now we construct the intersection points of the pairs of circumcircles of the triangles, as shown in Table 2:

	$\triangle BAM_1$	$\triangle DCM_3$		$\triangle DCN_1$	$\triangle BAN_3$
$\triangle BCM_2$	$P_1$	$P_2$	$\triangle BCN_4$	$Q_1$	$Q_4$
$\triangle DAM_4$	$P_4$	$P_3$	$\triangle ADN_2$	$Q_2$	$Q_3$

Table 2

**Proposition 2.** *The points  $P_1, P_2, P_3, P_4$  and  $Q_1, Q_2, Q_3, Q_4$  are the Brocard points of the quadrangle  $ABCD$ .*

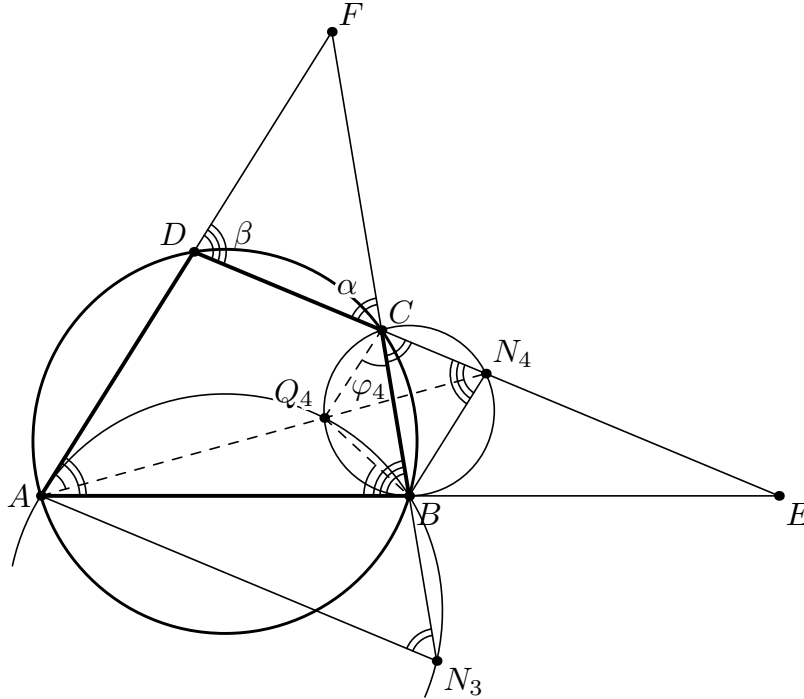


Fig. 3.

*Proof.* Let  $Q_4$  be the intersection point of the circumcircles of triangles  $BAN_3$  and  $BCN_4$  (Table 2). We denote  $\angle Q_4CB = \varphi_4$  (Fig. 3). Since  $BN_4 \parallel AD$  and  $AN_3 \parallel DC$  (Table 1), then  $\angle CN_4B = \beta$  and  $\angle BN_3A = \alpha$ . So:

$$\begin{aligned}
 & - \angle ABQ_4 = \beta - \angle CBQ_4 = \beta - \angle CN_4Q_4 = \beta - (\beta - \varphi_4) = \varphi_4, \\
 & - \angle DAQ_4 = \alpha - \angle BAQ_4 = \alpha - (180^\circ - \angle AQ_4B - \varphi_4) = \alpha - (180^\circ - (180^\circ - \alpha) - \varphi_4) = \varphi_4 \quad \square
 \end{aligned}$$

Hence  $Q_4 = Br(CBAD)$  is a Brocard point and  $\varphi_4 = \phi(CBAD)$  is a Brocard angle in the quadrilateral  $ABCD$ . As  $ABCD$  is cyclic quadrilateral, hence  $\varphi_4 =$

$\phi(DABC)$ . For the other points, the proof is analogous. If we denote the Brocard angles in  $ABCD$  with  $\varphi_1, \varphi_2, \varphi_3$  and  $\varphi_4$  then we have:

$$\begin{aligned}
 (1) \quad \varphi_1 &= \phi(ABCD) = \phi(DCBA) = \angle P_1AB = \angle P_1BC = \\
 &= \angle P_1CD = \angle Q_1DC = \angle Q_1CB = \angle Q_1BA, \\
 \varphi_2 &= \phi(BCDA) = \phi(ADCB) = \angle P_2BC = \angle P_2CD = \\
 &= \angle P_2DA = \angle Q_2AD = \angle Q_2DC = \angle Q_2CB, \\
 \varphi_3 &= \phi(CDAB) = \phi(BADC) = \angle P_3CD = \angle P_3DA = \\
 &= \angle P_3AB = \angle Q_3BA = \angle Q_3AD = \angle Q_3DC, \\
 \varphi_4 &= \phi(DABC) = \phi(CBAD) = \angle P_4DA = \angle P_4AB = \\
 &= \angle P_4BC = \angle Q_4CB = \angle Q_4BA = \angle Q_4AD.
 \end{aligned}$$

#### 4. SOME PROPERTIES OF THE BROCARD POINTS OF A CYCLIC QUADRILATERAL

**Proposition 3.** *The triads of points  $C, P_1, M_1; D, P_2, M_2; A, P_3, M_3; B, P_4, M_4; B, Q_1, N_1; C, Q_2, N_2; D, Q_3, N_3$  and  $A, Q_4, N_4$  are collinear.*

*Proof.*  $\angle AQ_4B + \angle BQ_4N_4 = (\alpha - 180^\circ) + \alpha = 180^\circ$  (Fig. 3). For the other triads of points, the proof is analogous.  $\square$

**Proposition 4.** *The fours of lines  $CM_1, DM_2, AM_3, BM_4$  and  $BN_1, CN_2, DN_3, AN_4$  are concurrent.*

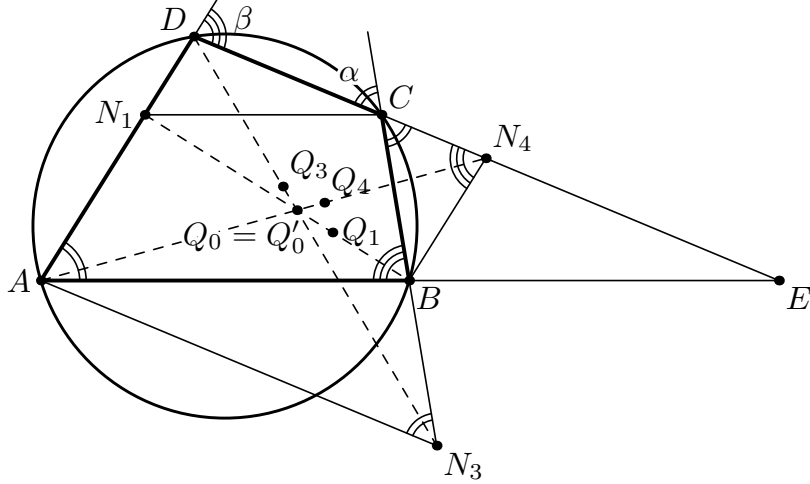


Fig. 4.

*Proof.* Let  $AN_4 \cap BN_1 = Q_0$  (Fig. 4). Then using  $\triangle AQ_0N_1 \sim \triangle N_4Q_0B$ ,  $\triangle AED \sim \triangle BEN_4$  and  $\triangle N_1CD \sim \triangle BEN_4$  (see Table 1) we have the following equations:

$$\frac{AQ_0}{Q_0N_4} = \frac{AN_1}{BN_4} = \frac{AD}{BN_4} - \frac{DN_1}{BN_4} = \frac{DE}{EN_4} - \frac{DC}{EN_4} = \frac{CE}{EN_4}.$$

Now, let  $AD \cap DN_3 = Q'_0$ . From the similarity of  $\triangle AN_3Q'_0 \sim \triangle N_4DQ'_0$ ,  $\triangle AN_3B \sim \triangle ECB$  and also because  $AD \parallel BN_4$  we have consecutively the following equations:

$$\frac{AQ'_0}{Q'_0N_4} = \frac{AN_3}{DN_4} = \frac{CE \cdot AB}{BE \cdot DN_4} = \frac{CE}{BE} \cdot \frac{BE}{EN_4} = \frac{CE}{EN_4}.$$

Then  $\frac{AQ_0}{Q_0N_4} = \frac{AQ'_0}{Q'_0N_4}$  and  $Q_0 \equiv Q'_0$ . In the same way  $Q_0 \in CN_2$ . So the lines  $BN_1, CN_2, DN_3, AN_4$  intersect in a point  $Q_0$ , and similarly the lines  $CM_1, DM_2, AM_3, BM_4$  have a common point  $P_0$ .  $\square$

Because  $\triangle BEN_4 \sim \triangle CEB$  and the Law of Sines used for  $\triangle CEB$ , we attain:

$$\frac{AQ_0}{Q_0N_4} = \frac{CE}{EN_4} = \frac{CE^2}{BE^2} = \frac{\sin^2 \beta}{\sin^2 \alpha}.$$

In the same way we have the following equations:

$$(2) \quad \begin{aligned} \frac{AQ_0}{Q_0N_4} = \frac{CQ_0}{Q_0N_2} = \frac{AP_0}{P_0M_3} = \frac{CP_0}{P_0M_1} = \frac{\sin^2 \beta}{\sin^2 \alpha} \\ \text{and} \\ \frac{BQ_0}{Q_0N_1} = \frac{DQ_0}{Q_0N_3} = \frac{BP_0}{P_0M_4} = \frac{DP_0}{P_0M_2} = \frac{\sin^2 \alpha}{\sin^2 \beta} \end{aligned}$$

Now we will show that the points  $P_0, Q_0$  and the vertices  $A, B, C$  and  $D$  define four pairs of triangles with equal areas (Fig. 5).

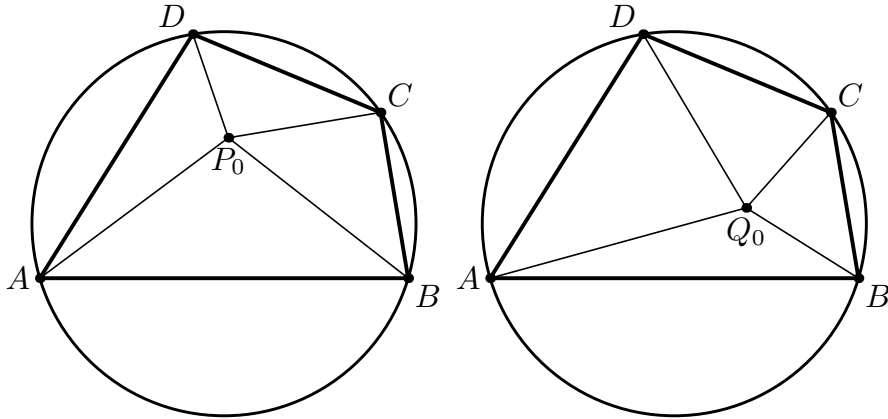


Fig. 5.

**Proposition 5.** *If  $P_0$  and  $Q_0$  are the intersection points of the fours of lines  $CM_1, DM_2, AM_3, BM_4$  and  $BN_1, CN_2, DN_3, AN_4$  (see Proposition 4), then:*

$$S_{ABP_0} = S_{DAQ_0}, S_{BCP_0} = S_{ABQ_0}, S_{CDP_0} = S_{BCQ_0} \text{ and } S_{DAP_0} = S_{CDQ_0}.$$

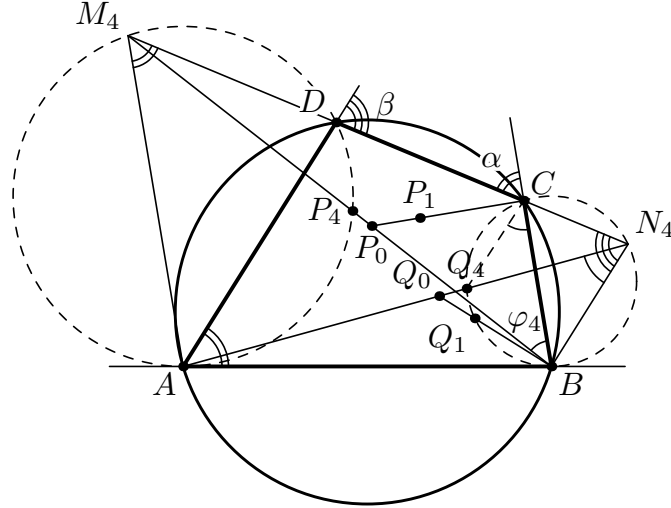


Fig. 6.

Taking into account that the fours of points  $B, P_0, P_4, M_4$  and  $A, Q_0, Q_4, N_4$  (Fig. 6) are collinear (Proposition 3 and Proposition 4) and equations (2) we attain:

$$(3) \quad S_{BCP_0} = \frac{BP_0}{BM_4} S_{BCM_4} = \frac{\sin^2 \alpha}{\sin^2 \alpha + \sin^2 \beta} S_{BCM_4},$$

$$S_{ABQ_0} = \frac{AQ_0}{AN_4} S_{BN_4A} = \frac{\sin^2 \beta}{\sin^2 \alpha + \sin^2 \beta} S_{BN_4A},$$

Since  $\angle BCM_4 = \angle ABN_4 = 180^\circ - \alpha$  (see Table 1) and  $\angle CBM_4 = \angle BN_4A = \varphi_4$  (see (1)), then  $\triangle BCM_4 \sim \triangle BN_4A$  and

$$(4) \quad \frac{S_{BCM_4}}{S_{BN_4A}} = \frac{BC^2}{BN_4^2} = \frac{\sin^2 \beta}{\sin^2 \alpha} \text{ (Law of Sines for } \triangle CBN_4)$$

Using equations (3) and (4) we calculate

$$\frac{S_{BCP_0}}{S_{ABQ_0}} = \frac{\sin^2 \alpha}{\sin^2 \beta} \cdot \frac{S_{BCM_4}}{S_{BN_4A}} = 1.$$

The proof, of the areas equality, of the other triangle pairs is similar.

Now we can consider the quadrangles  $P_1P_2P_3P_4$  and  $Q_1Q_2Q_3Q_4$ . We will prove that the points  $P_0, Q_0$  and the vertices of quadrangles  $P_1P_2P_3P_4$  and  $Q_1Q_2Q_3Q_4$  define four pairs of triangles with equal areas (Fig. 7).

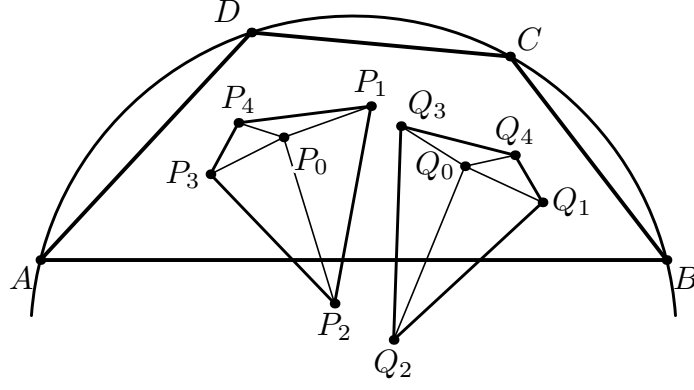


Fig. 7.

**Proposition 6.** *If  $P_0$  and  $Q_0$  are the intersection points of the fours of lines  $CM_1, DM_2, AM_3, BM_4$  and  $BN_1, CN_2, DN_3, AN_4$  (see Proposition 4), then:*

$$\begin{aligned} S_{P_1P_2P_0} &= S_{Q_1Q_2Q_0}, & S_{P_2P_3P_0} &= S_{Q_2Q_3Q_0}, \\ S_{P_3P_4P_0} &= S_{Q_3Q_4Q_0}, & S_{P_4P_1P_0} &= S_{Q_4Q_1Q_0}. \end{aligned}$$

*Proof.* Firstly we will show that the areas of the pairs of triangles  $CDP_1, BCQ_1$  and  $CDP_2, BCQ_2$  are equal (Fig. 8). The proof is similar to that of Proposition 1. Since  $\angle P_1BC = \angle Q_1DC = \varphi_1$  and  $\angle P_1CB = \angle Q_1CD$  (see equations (1)), hence  $\triangle BCP_1 \sim \triangle DCQ_1$ . We construct the altitudes  $P_1P_b, P_1P_c, Q_1Q_b$  and  $Q_1Q_c$ .

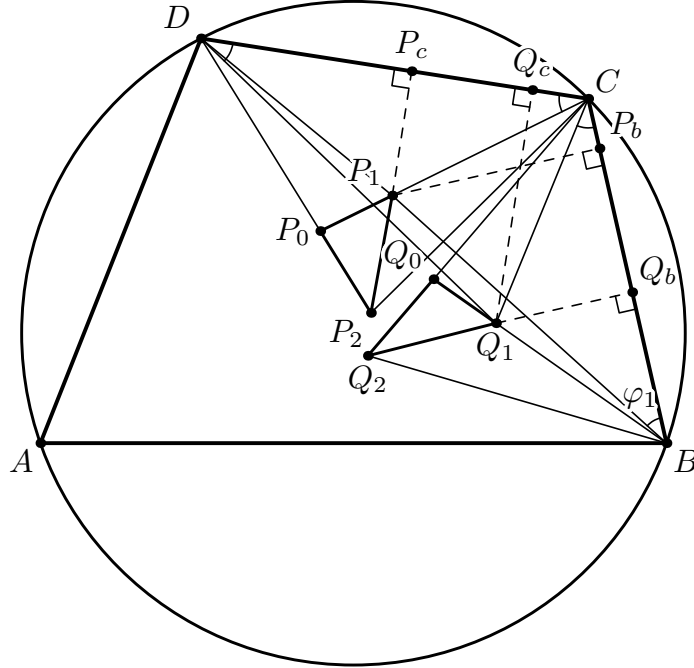


Fig. 8.

Then using  $\triangle BCP_1 \sim \triangle DCQ_1$ ,  $\triangle P_1P_bC \sim \triangle Q_1Q_cC$  and  $\triangle P_1P_cC \sim \triangle Q_1Q_bC$ , we have the following equations:

$$\frac{S_{CDP_1}}{S_{BCQ_1}} = \frac{DC \cdot P_1P_c}{BC \cdot Q_1Q_b} = \frac{Q_1Q_c}{P_1P_b} \cdot \frac{P_1P_c}{Q_1Q_b} = \frac{Q_1C}{P_1C} \cdot \frac{P_1C}{Q_1C} = 1.$$

So we attain:  $S_{CDP_1} = S_{BCQ_1}$  and  $S_{CDP_2} = S_{BCQ_2}$  (analogous).

Since the triads of points  $C, P_1, P_0$ ;  $D, P_0, P_2$ ;  $B, Q_1, Q_0$  and  $C, Q_0, Q_2$  are collinear (Propositions 3 and 4) and  $S_{CDP_0} = S_{BCQ_0}$  (Proposition 5), hence

$$S_{CP_0P_2} = S_{BQ_0Q_2} \text{ and } S_{DP_0P_1} = S_{CQ_0Q_1}.$$

We calculate  $S_{P_1P_2P_0} = \frac{P_0P_1}{P_0C} \cdot S_{CP_0P_2} = \frac{S_{DP_0P_1}}{S_{CDP_0}} \cdot S_{CP_0P_2}$  and  $S_{Q_1Q_2Q_0} = \frac{Q_0Q_1}{Q_0B} \cdot S_{BQ_0Q_2} = \frac{S_{CQ_0Q_1}}{S_{BCQ_0}} \cdot S_{BQ_0Q_2}$ , therefore  $S_{P_1P_2P_0} = S_{Q_1Q_2Q_0}$ .

The proof, of the areas equality, of the other pairs of triangles is similar.  $\square$

From Proposition 6 immediately follows

CONSEQUENCE  $S_{P_1P_2P_3P_4} = S_{Q_1Q_2Q_3Q_4}$ .

We should note that another solution is given by *Chandan Banerjee* in his blog [2].

## 5. TWO ADDITIONAL PROPERTIES

Above we have proved that  $\triangle BCP_1 \sim \triangle DCQ_1$  and  $S_{CDP_1} = S_{BCQ_1}$ . In fact, we have also  $\triangle ABP_1 \sim \triangle CBQ_1$  and  $S_{BCP_1} = S_{ABQ_1}$ . In general, each pair of Brocard points  $P_i$  and  $Q_i$ , in a cyclic quadrilateral  $ABCD$  and its corresponding vertices, define two pairs of triangles with equal areas (to compare see Proposition 1 and Fig. 1).

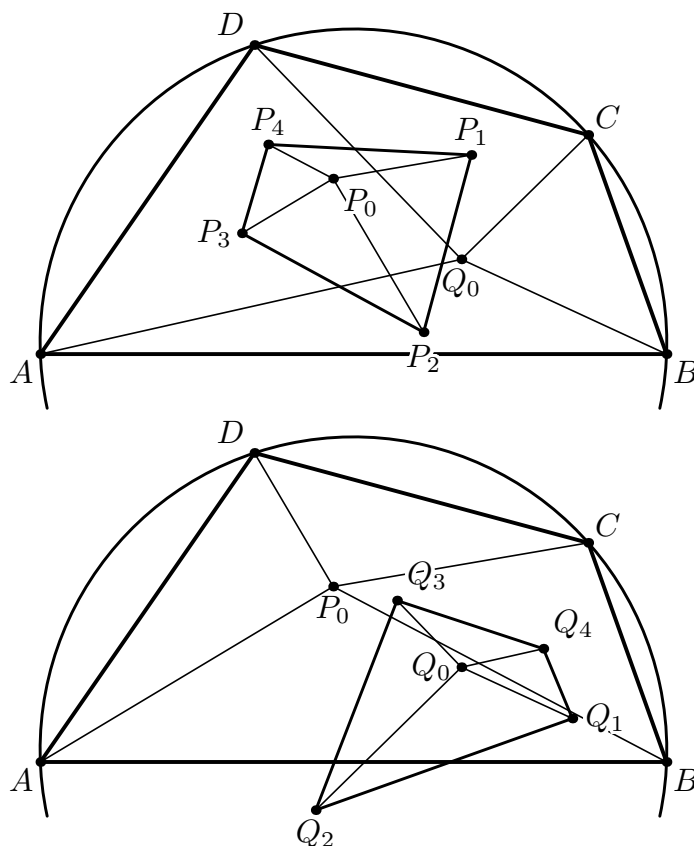


Fig. 9.



On the other hand, the points  $P_0$  and  $Q_0$  not only “stole” the pairs of triangles with equal areas related to vertices of the quadrilateral  $ABCD$  (see Proposition 5 and Fig. 5), but they have and those in quadrilaterals  $P_1P_2P_3P_4$  and  $Q_1Q_2Q_3Q_4$  (see Proposition 6 and Fig. 7). Naturally arise the question, where are the pairs of similar triangles associated with  $P_0$ ,  $Q_0$  and vertices of quadrilaterals  $ABCD$ ,  $P_1P_2P_3P_4$  and  $Q_1Q_2Q_3Q_4$ . It turns out that they appear in a very curious way.

In Fig. 9 we show that the triangles defined by the point  $P_0$  and quadrilateral  $P_1P_2P_3P_4$  are similar to the triangles determined by a point  $Q_0$  and quadrilateral  $ABCD$ . Conversely, the triangles defined by the point  $Q_0$  and quadrilateral  $Q_1Q_2Q_3Q_4$  are similar to the triangles determined by a point  $P_0$  and quadrilateral  $ABCD$  (Fig. 9).

**Proposition 7.** *If  $P_0$  and  $Q_0$  are the intersection points of the fours of lines  $CM_1$ ,  $DM_2$ ,  $AM_3$ ,  $BM_4$  and  $BN_1$ ,  $CN_2$ ,  $DN_3$ ,  $AN_4$  (see Proposition 4), then:*

$$\begin{aligned} \triangle P_1P_2P_0 &\sim \triangle CBQ_0, & \triangle P_2P_3P_0 &\sim \triangle DCQ_0, \\ \triangle P_3P_4P_0 &\sim \triangle ADQ_0, & \triangle P_4P_1P_0 &\sim \triangle BAQ_0 \end{aligned}$$

and

$$\begin{aligned} \triangle Q_1Q_2Q_0 &\sim \triangle DCP_0, & \triangle Q_2Q_3Q_0 &\sim \triangle ADP_0, \\ \triangle Q_3Q_4Q_0 &\sim \triangle BAP_0, & \triangle Q_4Q_1Q_0 &\sim \triangle CBP_0. \end{aligned}$$

*Proof.* We will prove that  $\triangle Q_1Q_2Q_0 \sim \triangle DCP_0$  (Fig. 8). The points  $Q_1$  and  $Q_2$  lie on the circumcircle of the triangle  $\triangle DCN_1$  (Table 2) therefore  $\angle Q_1Q_2C = \angle Q_1DC = \varphi_4$ . But  $\angle Q_2Q_0Q_1 = \angle Q_0CB + \angle Q_1BC = \beta - \varphi_1 + \varphi_2$  (see equations (1)) and  $\angle CP_0D = 180^\circ - (\angle P_0DC + \angle P_1CD) = 180^\circ - (180^\circ - \beta - \varphi_2 + \varphi_1) = \beta - \varphi_1 + \varphi_2$ , namely  $\angle Q_2Q_0Q_1 = \angle CP_0D$ . Hence  $\triangle Q_1Q_2Q_0 \sim \triangle DCP_0$ . The similarity of the other pairs of triangles we prove similarly.  $\square$

From Proposition 7 follows, that  $\angle P_1 + \angle P_3 = \angle Q_2 + \angle Q_4 = \angle A + \angle C = 180^\circ$  (see Fig. 9), which means that quadrilaterals  $P_1P_2P_3P_4$  and  $Q_1Q_2Q_3Q_4$  are cyclic.

## 6. APPENDIX

I apply the problems of my students.

*Anton Belev, Kaloyan Bucovsky,*

**Problem 1.** *If in the quadrangle  $ABCD$  is inscribed ellipse with foci  $F_1$  and  $F_2$ , and the isogonal conjugated points of  $F_1$  and  $F_2$  with respect to  $\triangle DAB$ ,  $\triangle ABC$ ,  $\triangle BCD$ ,  $\triangle CDA$  are  $A_1, B_1, C_1, D_1$  and  $A_2, B_2, C_2, D_2$ , then the quadrangles  $A_1B_1C_1D_1$  and  $A_2B_2C_2D_2$  are congruent.*

**Problem 2.** *In the quadrangle  $ABCD$  is inscribed a circle with center  $O$ . If the isogonal conjugated points of  $O$  with respect to  $\triangle DAB$ ,  $\triangle ABC$ ,  $\triangle BCD$ ,  $\triangle CDA$  are  $A_1, B_1, C_1$  and  $D_1$ , then  $A_1B_1C_1D_1$  is a parallelogram. If  $AC \perp BD$ , then  $A_1B_1C_1D_1$  is a rhombus. If around the quadrangle  $ABCD$  is described a circle, then  $A_1B_1C_1D_1$  is a rectangle.*

The construction, which reveals how quadrangles  $A_1B_1C_1D_1$  and  $A_2B_2C_2D_2$  merged into a parallelogram (when  $F_1$  get near to  $F_2$ ) is really fascinating. We did it using GeoGebra.

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SOFIA HIGH SCHOOL IN CONSTRUCTION, ARCHITECTURE AND GEODESY "HRISTO BOTEV".  
E-mail address: [dbelev@gbg.bg](mailto:dbelev@gbg.bg)