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Index

SOME PROPERTIES OF THE BROCARD POINTS OF A CYCLIC QUADRILATERAL

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Abstract. In this article we have constructed the Brocard points of a cyclic quadrilateral, we have found some of their properties and using these properties we have proved the problem of A. A. Zaslavsky.

1. The Problem

Alexey Zaslavsky, **Brocard's points in quadrilateral [4].**

Given a convex quadrilateral $ABCD$. It is easy to prove that there exists a unique point P such that $\angle PAB = \angle PBC = \angle PCD$. We will call this point *Brocard point* ($Br(ABCD)$) and the respective angle *Brocard angle* ($\phi(ABCD)$) of broken line $ABCD$. Note some properties of Brocard's points and angles:

– $\phi(ABCD) = \phi(DCBA)$ if $ABCD$ is cyclic;

– if $ABCD$ is harmonic then $\phi(ABCD) = \phi(BCDA)$. Thus there exist two points P, Q such that $\angle PAB = \angle PBC = \angle PCD = \angle PDA = \angle QBA = \angle QCB = \angle QDC = \angle QAD$. These points lie on the circle with diameter OL where O is the circumcenter of $ABCD$, L is the common point of its diagonals and $\angle POL = \angle QOL = \phi(ABCD)$.

Problem. *Let $ABCD$ be a cyclic quadrilateral, $P_1 = Br(ABCD)$, $P_2 = Br(BCDA)$, $P_3 = Br(CDAB)$, $P_4 = Br(DABC)$, $Q_1 = Br(DCBA)$, $Q_2 = Br(ADCB)$, $Q_3 = Br(BADC)$, $Q_4 = Br(CBAD)$. Then $S_{P_1P_2P_3P_4} = S_{Q_1Q_2Q_3Q_4}$.*

I first saw this problem 2-3 years ago in an article by Alexei Myakishev (see [3]). I did not consider solving it then, but my pupils built a structure very similar (seemingly) to the described. Thus Zaslavsky's problem served as a stimulus (accelerator) to develop a good article (related to isogonal conjugate points), which they presented on international events. I sincerely hope that this article will appear on the pages of "Kvant". Now they are university students, but again I saw the problem in the "Journal of Classical Geometry". At first I thought that using the developed article I would figure out the solution quickly, but I did not. A new construction came out.

2. Some Properties Of Brocard Points Of a Triangle.

Let P and Q be the first and the second Brocard points of a $\triangle ABC$. We will prove that the Brocard points and the three vertices A, B and C define three pairs of similar triangles, and three pairs of triangles with equal areas (Fig. 1).

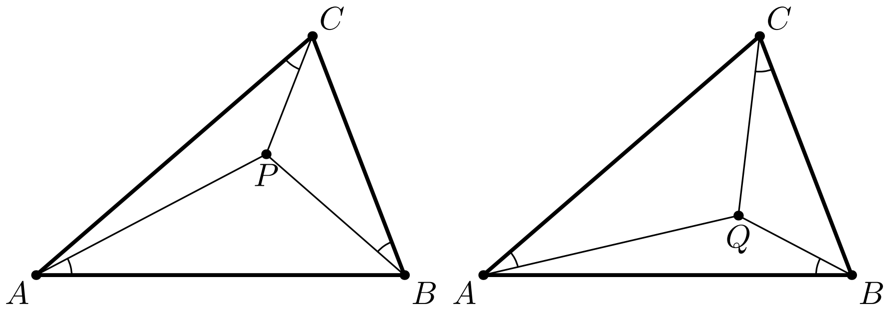

Fig. 1.

Proposition 1. *If P and Q are the first and the second Brocard points of a $\triangle ABC$, then:*

(i) $\triangle ABP \sim \triangle CBQ$, $\triangle BCP \sim \triangle ACQ$ and $\triangle CAP \sim \triangle BAQ$;

(ii) $S_{ABP} = S_{ACQ}$, $S_{BCP} = S_{BAQ}$ and $S_{CAP} = S_{CBQ}$.

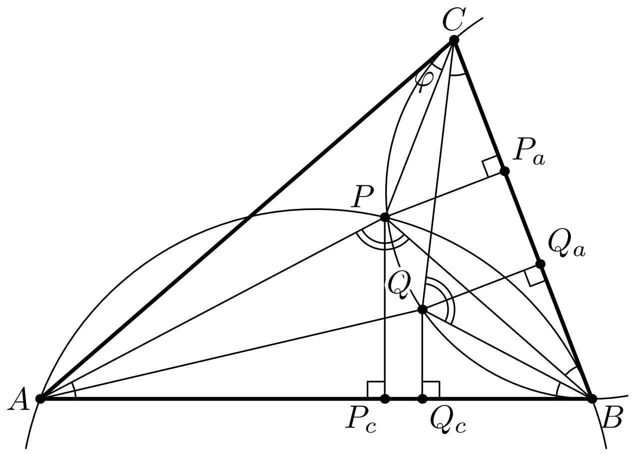

Fig. 2.

Proof. If we denote with φ the Brocard angle (Fig. 2), then $\angle BAP = \angle BCQ = \varphi$, and from the construction ([1]) of the Brocard points of $\triangle ABC$ we have $\angle APB = \angle CQB = 180^\circ - \angle B$. Hence $\triangle ABP \sim \triangle CBQ$.

We construct the altitudes PP_c, PP_a, QQ_c and QQ_a. Then, having $\triangle ABP \sim \triangle CBQ$, $\triangle P_cBP \sim \triangle Q_aBQ$ and $\triangle Q_cBP \sim \triangle P_aBQ$, we have the following equations:

$$\frac{S_{BPC}}{S_{BAQ}} = \frac{BC \cdot PP_a}{AB \cdot QQ_c} = \frac{QQ_a}{PP_c} \cdot \frac{PP_a}{QQ_c} = \frac{BQ}{BP} \cdot \frac{BP}{BQ} = 1.$$

For the other pairs of triangles, the proof is analogous. □

We are going to find, similar to these properties, for the Brocard points of a cyclic quadrilateral.

3. Construction

Let the quadrilateral $ABCD$ be a cyclic, $AB \cap CD = E$ and let for exactitude B is between A and E, $AD \cap BC = F$ and D is between A and F. We denote $\angle BAD = \alpha$ and $\angle ABC = \beta$.

In order to construct the Brocard points of the quadrilateral $ABCD$, we firstly construct the points M_1, M_2, M_3, M_4 and N_1, N_2, N_3, N_4, as shown in Table 1:

M_1	$M_1 \in AD$ and $BM_1 \parallel CD$	N_1	$N_1 \in AD$ and $CN_1 \parallel BA$
M_2	$M_2 \in AB$ and $CM_2 \parallel DA$	N_2	$N_2 \in AB$ and $DN_2 \parallel CB$
M_3	$M_3 \in BC$ and $DM_3 \parallel AB$	N_3	$N_3 \in BC$ and $AN_3 \parallel DC$
M_4	$M_4 \in CD$ and $AM_4 \parallel BC$	N_4	$N_4 \in CD$ and $BN_4 \parallel AD$

Table 1

Now we construct the intersection points of the pairs of circumcircles of the triangles, as shown in Table 2:

	$\triangle BAM_1$	$\triangle DCM_3$
$\triangle BCM_2$	P_1	P_2
$\triangle DAM_4$	P_4	P_3

	$\triangle DCN_1$	$\triangle BAN_3$
$\triangle BCN_4$	Q_1	Q_4
$\triangle ADN_2$	Q_2	Q_3

Table 2

Proposition 2. *The points P_1, P_2, P_3, P_4 and Q_1, Q_2, Q_3, Q_4 are the Brocard points of the quadrangle $ABCD$.*

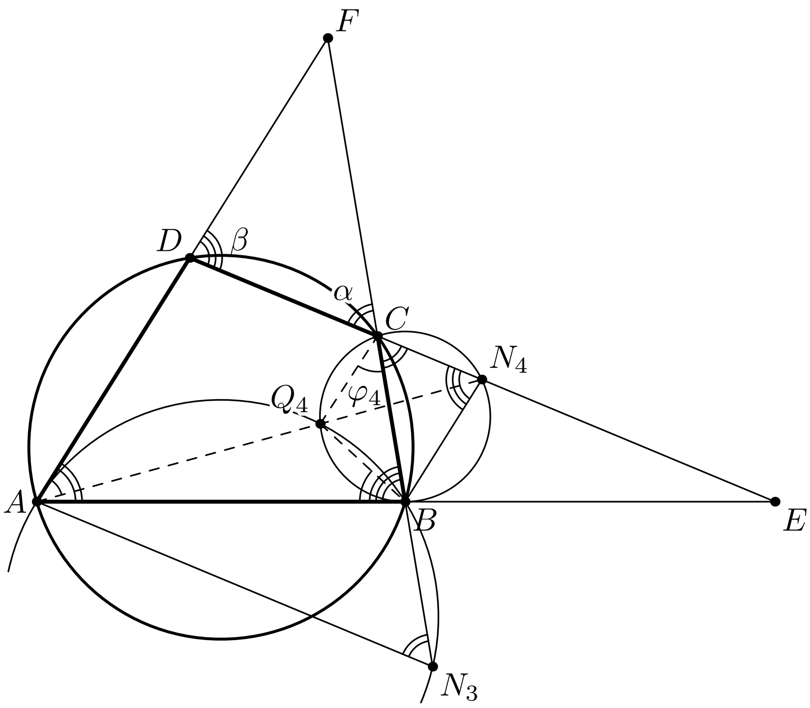

Fig. 3.

Proof. Let Q_4 be the intersection point of the circumcircles of triangles BAN_3 and BCN_4 (Table 2). We denote $\angle Q_4CB = \varphi_4$ (Fig. 3). Since $BN_4 \parallel AD$ and $AN_3 \parallel DC$ (Table 1), then $\angle CN_4B = \beta$ and $\angle BN_3A = \alpha$. So:

– $\angle ABQ_4 = \beta - \angle CBQ_4 = \beta - \angle CN_4Q_4 = \beta - (\beta - \varphi_4) = \varphi_4$,

– $\angle DAQ_4 = \alpha - \angle BAQ_4 = \alpha - (180° - \angle AQ_4B - \varphi_4) = \alpha - (180° - (180° - \alpha) - \varphi_4) = \varphi_4$ □

Hence $Q_4 = Br(CBAD)$ is a Brocard point and $\varphi_4 = \phi(CBAD)$ is a Brocard angle in the quadrilateral $ABCD$. As $ABCD$ is cyclic quadrilateral, hence $\varphi_4 =$

$\phi(DABC)$. For the other points, the proof is analogous. If we denote the Brocard angles in $ABCD$ with φ_1, φ_2, φ_3 and φ_4 then we have:

$$\begin{aligned}\varphi_1 &= \phi(ABCD) = \phi(DCBA) = \angle P_1AB = \angle P_1BC = \\ &= \angle P_1CD = \angle Q_1DC = \angle Q_1CB = \angle Q_1BA, \\ \varphi_2 &= \phi(BCDA) = \phi(ADCB) = \angle P_2BC = \angle P_2CD = \\ &= \angle P_2DA = \angle Q_2AD = \angle Q_2DC = \angle Q_2CB, \\ \varphi_3 &= \phi(CDAB) = \phi(BADC) = \angle P_3CD = \angle P_3DA = \\ &= \angle P_3AB = \angle Q_3BA = \angle Q_3AD = \angle Q_3DC, \\ \varphi_4 &= \phi(DABC) = \phi(CBAD) = \angle P_4DA = \angle P_4AB = \\ &= \angle P_4BC = \angle Q_4CB = \angle Q_4BA = \angle Q_4AD.\end{aligned} \tag{1}$$

4. Some properties of the Brocard points of a cyclic quadrilateral

Proposition 3. *The triads of points C, P_1, M_1; D, P_2, M_2; A, P_3, M_3; B, P_4, M_4; B, Q_1, N_1; C, Q_2, N_2; D, Q_3, N_3 and A, Q_4, N_4 are collinear.*

Proof. $\angle AQ_4B + \angle BQ_4N_4 = (\alpha - 180°) + \alpha = 180°$ (Fig. 3). For the other triads of points, the proof is analogous. □

Proposition 4. *The fours of lines CM_1, DM_2, AM_3, BM_4 and BN_1, CN_2, DN_3, AN_4 are concurrent.*

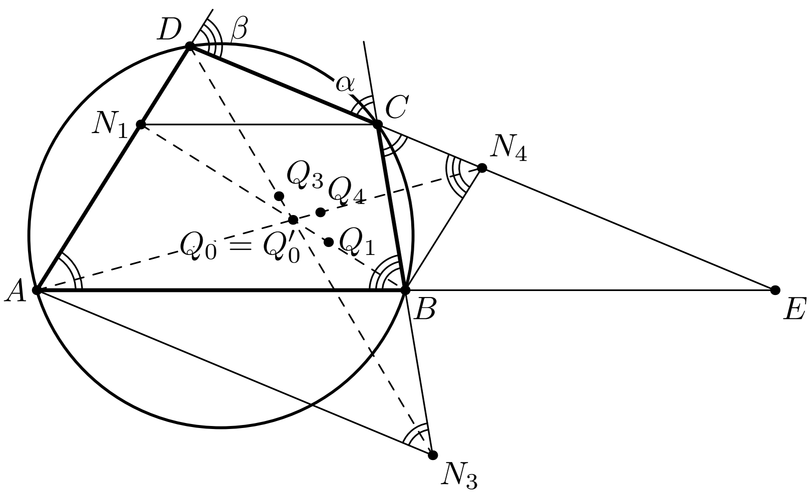

Fig. 4.

Proof. Let $AN_4 \cap BN_1 = Q_0$ (Fig. 4). Then using $\triangle AQ_0N_1 \sim \triangle N_4Q_0B$, $\triangle AED \sim \triangle BEN_4$ and $\triangle N_1CD \sim \triangle BEN_4$ (see Table 1) we have the following equations:

$$\frac{AQ_0}{Q_0N_4} = \frac{AN_1}{BN_4} = \frac{AD}{BN_4} - \frac{DN_1}{BN_4} = \frac{DE}{EN_4} - \frac{DC}{EN_4} = \frac{CE}{EN_4}.$$

Now, let $AD \cap DN_3 = Q'_0$. From the similarity of $\triangle AN_3Q'_0 \sim \triangle N_4DQ'_0$, $\triangle AN_3B \sim \triangle ECB$ and also because $AD \parallel BN_4$ we have consecutively the following equations:

$$\frac{AQ'_0}{Q'_0N_4} = \frac{AN_3}{DN_4} = \frac{CE.AB}{BE.DN_4} = \frac{CE}{BE} \cdot \frac{BE}{EN_4} = \frac{CE}{EN_4}.$$

Then $\dfrac{AQ_0}{Q_0N_4} = \dfrac{AQ'_0}{Q'_0N_4}$ and $Q_0 \equiv Q'_0$. In the same way $Q_0 \in CN_2$. So the lines BN_1, CN_2, DN_3, AN_4 intersect in a point Q_0, and similarly the lines CM_1, DM_2, AM_3, BM_4 have a common point P_0. □

Because $\triangle BEN_4 \sim \triangle CEB$ and the Law of Sines used for $\triangle CEB$, we attain:

$$\frac{AQ_0}{Q_0N_4} = \frac{CE}{EN_4} = \frac{CE^2}{BE^2} = \frac{sin^2\beta}{sin^2\alpha}.$$

In the same way we have the following equations:

$$(2) \qquad \begin{array}{c} \dfrac{AQ_0}{Q_0N_4} = \dfrac{CQ_0}{Q_0N_2} = \dfrac{AP_0}{P_0M_3} = \dfrac{CP_0}{P_0M_1} = \dfrac{\sin^2\beta}{\sin^2\alpha} \\ \text{and} \\ \dfrac{BQ_0}{Q_0N_1} = \dfrac{DQ_0}{Q_0N_3} = \dfrac{BP_0}{P_0M_4} = \dfrac{DP_0}{P_0M_2} = \dfrac{\sin^2\alpha}{\sin^2\beta} \end{array}$$

Now we will show that the points P_0, Q_0 and the vertices A, B, C and D define four pairs of triangles with equal areas (Fig. 5).

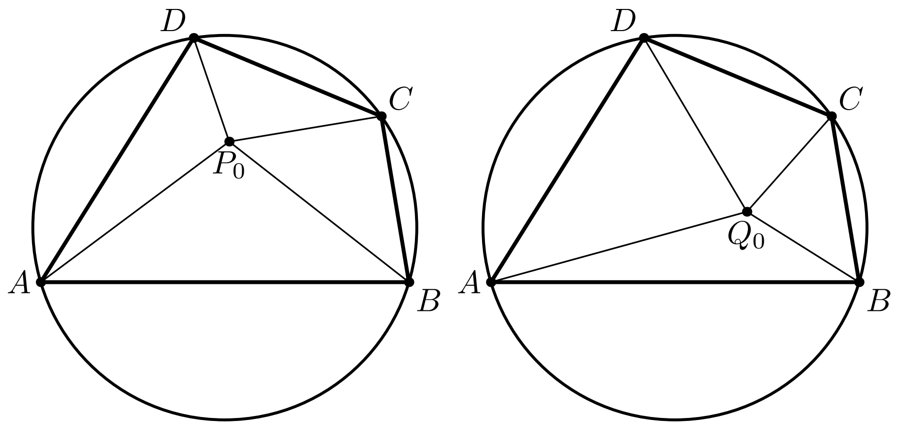

Fig. 5.

Proposition 5. *If P_0 and Q_0 are the intersection points of the fours of lines CM_1, DM_2, AM_3, BM_4 and BN_1, CN_2, DN_3, AN_4 (see Proposition 4), then:*

$$S_{ABP_0} = S_{DAQ_0},\ S_{BCP_0} = S_{ABQ_0},\ S_{CDP_0} = S_{BCQ_0} \text{ and } S_{DAP_0} = S_{CDQ_0}.$$

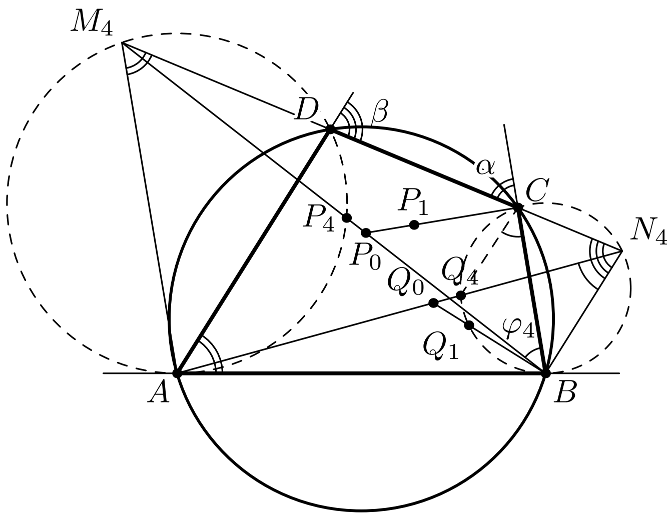

Fig. 6.

Taking into account that the fours of points B, P_0, P_4, M_4 and A, Q_0, Q_4, N_4 (Fig. 6) are collinear (Proposition 3 and Proposition 4) and equations (2) we attain:

$$S_{BCP_0} = \frac{BP_0}{BM_4} S_{BCM_4} = \frac{\sin^2\alpha}{\sin^2\alpha + \sin^2\beta} S_{BCM_4}, \tag{3}$$

$$S_{ABQ_0} = \frac{AQ_0}{AN_4} S_{BN_4A} = \frac{\sin^2\beta}{\sin^2\alpha + \sin^2\beta} S_{BN_4A},$$

Since $\angle BCM_4 = \angle ABN_4 = 180^\circ - \alpha$ (see Table 1) and $\angle CBM_4 = \angle BN_4A = \varphi_4$ (see (1)), then $\triangle BCM_4 \sim \triangle BN_4A$ and

$$\frac{S_{BCM_4}}{S_{BN_4A}} = \frac{BC^2}{BN_4^2} = \frac{\sin^2\beta}{\sin^2\alpha} \text{ (Law of Sines for } \triangle CBN_4) \tag{4}$$

Using equations (3) and (4) we calculate

$$\frac{S_{BCP_0}}{S_{ABQ_0}} = \frac{\sin^2\alpha}{\sin^2\beta} \cdot \frac{S_{BCM_4}}{S_{BN_4A}} = 1.$$

The proof, of the areas equality, of the other triangle pairs is similar.

Now we can consider the quadrangles $P_1P_2P_3P_4$ and $Q_1Q_2Q_3Q_4$. We will prove that the points P_0, Q_0 and the vertices of quadrangles $P_1P_2P_3P_4$ and $Q_1Q_2Q_3Q_4$ define four pairs of triangles with equal areas (Fig. 7).

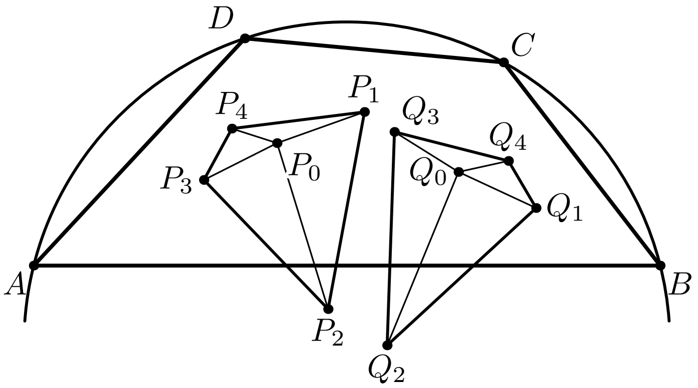

Fig. 7.

Proposition 6. *If P_0 and Q_0 are the intersection points of the fours of lines CM_1, DM_2, AM_3, BM_4 and BN_1, CN_2, DN_3, AN_4 (see Proposition 4), then:*

$$S_{P_1P_2P_0} = S_{Q_1Q_2Q_0}, \quad S_{P_2P_3P_0} = S_{Q_2Q_3Q_0},$$
$$S_{P_3P_4P_0} = S_{Q_3Q_4Q_0}, \quad S_{P_4P_1P_0} = S_{Q_4Q_1Q_0}.$$

Proof. Firstly we will show that the areas of the pairs of triangles CDP_1, BCQ_1 and CDP_2, BCQ_2 are equal (Fig. 8). The proof is similar to that of Proposition 1. Since $\angle P_1BC = \angle Q_1DC = \varphi_1$ and $\angle P_1CB = \angle Q_1CD$ (see equations (1)), hence $\triangle BCP_1 \sim \triangle DCQ_1$. We construct the altitudes P_1P_b, P_1P_c, Q_1Q_b and Q_1Q_c.

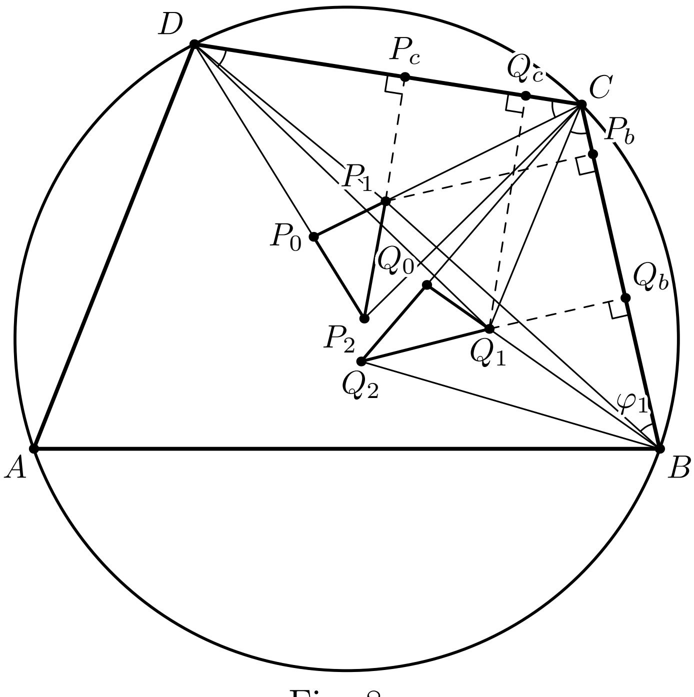

Fig. 8.

Then using $\triangle BCP_1 \sim \triangle DCQ_1$, $\triangle P_1P_bC \sim \triangle Q_1Q_cC$ and $\triangle P_1P_cC \sim \triangle Q_1Q_bC$, we have the following equations:

$$\frac{S_{CDP_1}}{S_{BCQ_1}} = \frac{DC.P_1P_c}{BC.Q_1Q_b} = \frac{Q_1Q_c}{P_1P_b} \cdot \frac{P_1P_c}{Q_1Q_b} = \frac{Q_1C}{P_1C} \cdot \frac{P_1C}{Q_1C} = 1.$$

So we attain: $S_{CDP_1} = S_{BCQ_1}$ and $S_{CDP_2} = S_{BCQ_2}$ (analogous).

Since the triads of points C, P_1, P_0; D, P_0, P_2; B, Q_1, Q_0 and C, Q_0, Q_2 are collinear (Propositions 3 and 4) and $S_{CDP_0} = S_{BCQ_0}$ (Proposition 5), hence

$$S_{CP_0P_2} = S_{BQ_0Q_2} \text{ and } S_{DP_0P_1} = S_{CQ_0Q_1}.$$

We calculate $S_{P_1P_2P_0} = \dfrac{P_0P_1}{P_0C} \cdot S_{CP_0P_2} = \dfrac{S_{DP_0P_1}}{S_{CDP_0}} \cdot S_{CP_0P_2}$ and $S_{Q_1Q_2Q_0} = \dfrac{Q_0Q_1}{Q_0B} \cdot S_{BQ_0Q_2} = \dfrac{S_{CQ_0Q_1}}{S_{BCQ_0}} \cdot S_{BQ_0Q_2}$, therefore $S_{P_1P_2P_0} = S_{Q_1Q_2Q_0}$.

The proof, of the areas equality, of the other pairs of triangles is similar. □

From Proposition 6 immediately follows

Consequence $S_{P_1P_2P_3P_4} = S_{Q_1Q_2Q_3Q_4}$.

We should note that another solution is given by *Chandan Banerjee* in his blog [2].

5. Two additional properties

Above we have proved that $\triangle BCP_1 \sim \triangle DCQ_1$ and $S_{CDP_1} = S_{BCQ_1}$. In fact, we have also $\triangle ABP_1 \sim \triangle CBQ_1$ and $S_{BCP_1} = S_{ABQ_1}$. In general, each pair of Brocard points P_i and Q_i, in a cyclic quadrilateral $ABCD$ and its corresponding vertices, define two pairs of triangles with equal areas (to compare see Proposition 1 and Fig. 1).

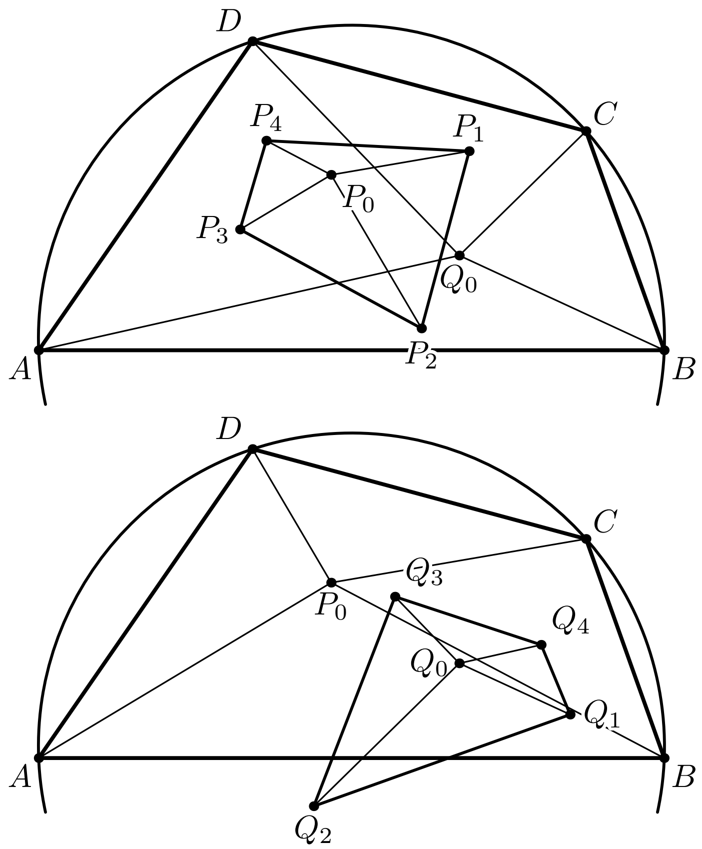

Fig. 9.

On the other hand, the points P_0 and Q_0 not only "stole" the pairs of triangles with equal areas related to vertices of the quadrilateral $ABCD$ (see Proposition 5 and Fig. 5), but they have and those in quadrilaterals $P_1P_2P_3P_4$ and $Q_1Q_2Q_3Q_4$ (see Proposition 6 and Fig. 7). Naturally arise the question, where are the pairs of similar triangles associated with P_0, Q_0 and vertices of quadrilaterals $ABCD$, $P_1P_2P_3P_4$ and $Q_1Q_2Q_3Q_4$. It turns out that they appear in a very curious way.

In Fig. 9 we show that the triangles defined by the point P_0 and quadrilateral $P_1P_2P_3P_4$ are similar to the triangles determined by a point Q_0 and quadrilateral $ABCD$. Conversely, the triangles defined by the point Q_0 and quadrilateral $Q_1Q_2Q_3Q_4$ are similar to the triangles determined by a point P_0 and quadrilateral $ABCD$ (Fig. 9).

Proposition 7. *If P_0 and Q_0 are the intersection points of the fours of lines CM_1, DM_2, AM_3, BM_4 and BN_1, CN_2, DN_3, AN_4 (see Proposition 4), then:*

$$\triangle P_1P_2P_0 \sim \triangle CBQ_0, \quad \triangle P_2P_3P_0 \sim \triangle DCQ_0,$$
$$\triangle P_3P_4P_0 \sim \triangle ADQ_0, \quad \triangle P_4P_1P_0 \sim \triangle BAQ_0$$

and

$$\triangle Q_1Q_2Q_0 \sim \triangle DCP_0, \quad \triangle Q_2Q_3Q_0 \sim \triangle ADP_0,$$
$$\triangle Q_3Q_4Q_0 \sim \triangle BAP_0, \quad \triangle Q_4Q_1Q_0 \sim \triangle CBP_0.$$

Proof. We will prove that $\triangle Q_1Q_2Q_0 \sim \triangle DCP_0$ (Fig. 8). The points Q_1 and Q_2 lie on the circumcircle of the triangle $\triangle DCN_1$ (Table 2) therefore $\angle Q_1Q_2C = \angle Q_1DC = \varphi_4$. But $\angle Q_2Q_0Q_1 = \angle Q_0CB + \angle Q_1BC = \beta - \varphi_1 + \varphi_2$ (see equations (1)) and $\angle CP_0D = 180^\circ - (\angle P_0DC + \angle P_1CD) = 180^\circ - (180^\circ - \beta - \varphi_2 + \varphi_1) = \beta - \varphi_1 + \varphi_2$, namely $\angle Q_2Q_0Q_1 = \angle CP_0D$. Hence $\triangle Q_1Q_2Q_0 \sim \triangle DCP_0$. The similarity of the other pairs of triangles we prove similarly. □

From Proposition 7 follows, that $\angle P_1 + \angle P_3 = \angle Q_2 + \angle Q_4 = \angle A + \angle C = 180^\circ$ (see Fig. 9), which means that quadrilaterals $P_1P_2P_3P_4$ and $Q_1Q_2Q_3Q_4$ are cyclic.

6. Appendix

I apply the problems of my students.
Anton Belev, Kaloyan Bucovsky,

Problem 1. *If in the quadrangle $ABCD$ is inscribed ellipse with foci F_1 and F_2, and the isogonal conjugated points of F_1 and F_2 with respect to $\triangle DAB$, $\triangle ABC$, $\triangle BCD$, $\triangle CDA$ are A_1, B_1, C_1, D_1 and A_2, B_2, C_2, D_2, then the quadrangles $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ are congruent.*

Problem 2. *In the quadrangle $ABCD$ is inscribed a circle with center O. If the isogonal conjugated points of O with respect to $\triangle DAB$, $\triangle ABC$, $\triangle BCD$, $\triangle CDA$ are A_1, B_1, C_1 and D_1, then $A_1B_1C_1D_1$ is a parallelogram. If $AC \perp BD$, then $A_1B_1C_1D_1$ is a rhombus. If around the quadrangle $ABCD$ is described a circle, then $A_1B_1C_1D_1$ is a rectangle.*

The construction, which reveals how quadrangles $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ merged into a parallelogram (when F_1 get near to F_2) is really fascinating. We did it using GeoGebra.

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A CURIOUS GEOMETRIC TRANSFORMATION

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ABSTRACT. An expansion is a little-known geometric transformation which maps directed circles to directed circles. We explore the applications of expansion to the solution of various problems in geometry by elementary means.

In his book *Geometricheskie Preobrazovaniya*, Isaac Moiseevich Yaglom describes one curious geometric transformation called *expansion*[1].

A few years after coming across this transformation, the author of the present paper was rather surprised to discover that it leads to quite appealing geometrical solutions of a number of problems otherwise very difficult to treat by elementary means. Most of these originated in Japanese *sangaku* – wooden votive tablets which Japanese mathematicians of the Edo period painted with the beautiful and extraordinary theorems they discovered and then hung in Buddhist temples and Shinto shrines.

Researching the available literature prior to compiling this paper brought another surprise: that expansion was not mentioned in almost any other books on geometry. Only two references were uncovered – one more book by I. M. Yaglom [7] and an encyclopedia article which he wrote for the five-volume *Entsiklopedia Elementarnoy Matematiki.*

Therefore, we begin our presentation with an informal introduction to expansion and the geometry of cycles and rays.

1. INTRODUCTION

We call an oriented circle a *cycle*. Whenever a circle k of radius R is oriented positively (i.e., counterclockwise), we label the resulting cycle k^+ and say that it is of radius R. Whenever k is oriented negatively (i.e., clockwise), we label the resulting cycle k^- and say that it is of radius $-R$. Therefore, the sign of a cycle's radius determines its orientation.

We say that two cycles touch if they touch as circles and are directed the same way at their contact point. Therefore, two cycles which touch externally need to be of opposite orientations and two cycles which touch internally need to be of the same orientation. Moreover, two cycles of radii r_1 and r_2 touch exactly when the distance between their centers equals $|r_1 - r_2|$.

[1]In the English translation of I. M. Yaglom's *Complex Numbers in Geometry* [7], the term *dilatation* is used. Here, we prefer to use "expansion" instead as, in English literature, "dilatation" and "dilation" usually mean "homothety".

We call an oriented line a *ray*. We imagine rays to be cycles of infinitely large radius; we also imagine points to be cycles of zero radius[2].

The definition of tangency naturally extends to cycles and rays (Fig. 1). It also extends to cycles and points: a cycle c and a point P, regarded as a zero-radius cycle, are tangent exactly when P lies on c.

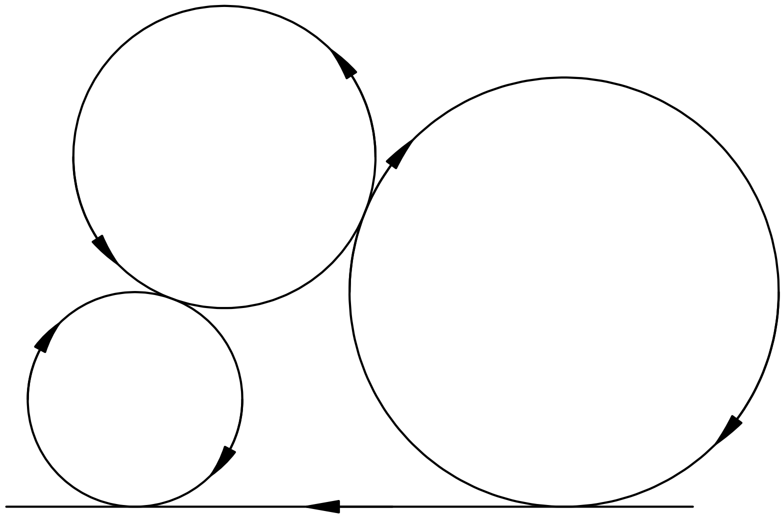

Fig. 1.

This view at tangency greatly simplifies many definitions and results in circle geometry. For instance, every two cycles have a single homothetic center; and the homothetic centers of every three cycles, taken by pairs, are collinear. Also, two cycles c_1 and c_2 always have at most two common tangents; and these two are symmetric – apart from orientation – in the line through the cycles' centers.

The length of the segment connecting a common tangent's contact points with c_1 and c_2 is called the *tangential distance* between c_1 and c_2 (Fig. 2). When c_1 and c_2 touch, the two contact points merge and the tangential distance equals zero.

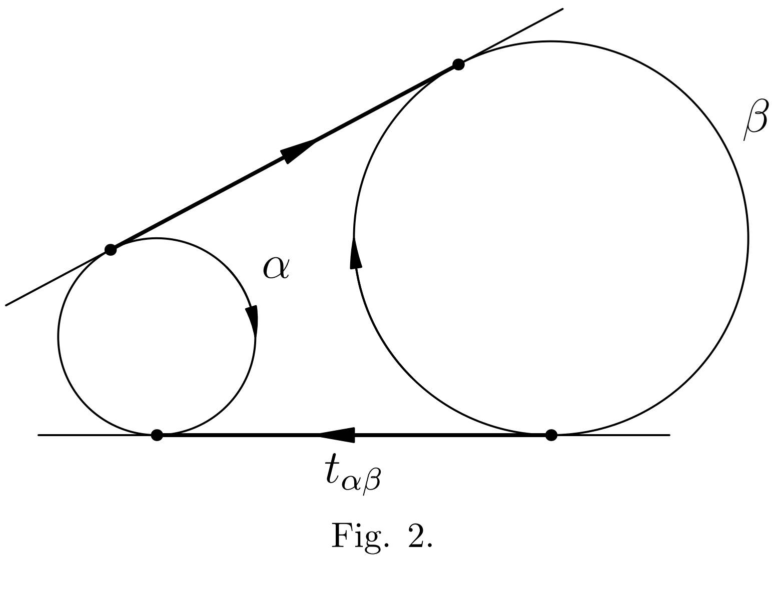

Fig. 2.

[2]Points, therefore, do not possess orientation.

The geometry of cycles and rays is well-known and there are many good introductions to it: two examples are section II.5 of I. M. Yaglom's [6] and J. F. Rigby's paper [4].

Notice that (as Fig. 3 hints) increasing or decreasing the radii of all cycles by the same amount r preserves tangency as well as tangential distance in general. Thus we arrive at a

Definition. We call an *expansion* of radius r (r being an arbitrary real number) the geometric transformation which (a) to every *cycle* of radius R maps a cycle of the same center and radius $R + r$, and (b) to every *ray* maps its translation by a distance of r "to the right" with respect to the ray's direction.

The extension of the idea of "increasing the radius by r" to rays is rather natural: in every positively oriented cycle c, the center is situated "to the right" with respect to all rays tangent to c. Notice also that every expansion maps rays to rays, points to cycles (moreover, cycles of radius r) and some cycles to points (namely, all cycles of radius $-r$). Finally, expansion alters the orientation of some cycles – namely, all cycles of radius R such that R and $R+r$ have different signs.

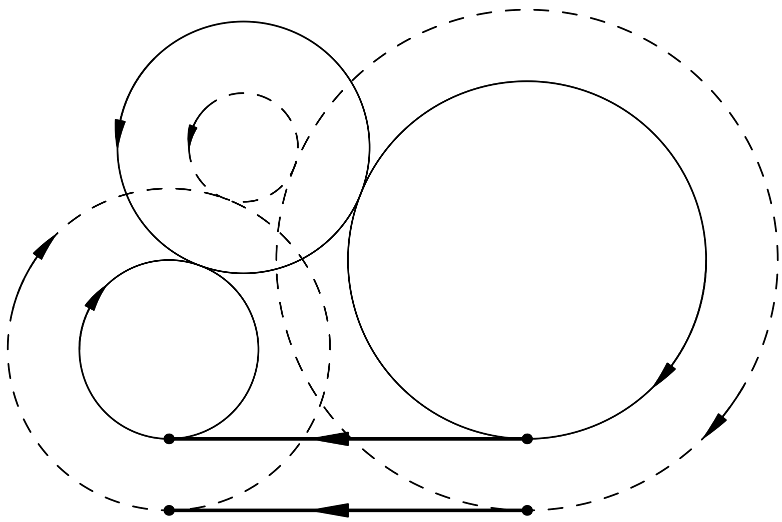

Fig. 3.

The most important property of expansion is that, as noted above, it preserves tangency between cycles and rays and tangential distance in general. This is a very intuitive result and we do not give a rigorous proof here.

2. Classical Problems

In *Geometricheskie Preobrazovaniya*, I. M. Yaglom demonstrates how expansion can be applied to some classical problems in circle geometry, yielding very short and beautiful solutions.

Problem 1. *Given two circles k_1 and k_2, construct (using straightedge and compass) their common tangents.*

Solution. We will show how to construct all common tangents (if any exist) to two *cycles* c_1 and c_2. The solution to the original problem then follows if we repeat the construction once for every possible choice of orientations for k_1 and k_2; there are only two substantially different such choices.

Let c_1 and c_2 be two cycles of centers O_1 and O_2. Apply an expansion f which maps c_1 to the point O_1 and c_2 to some cycle c_2'.

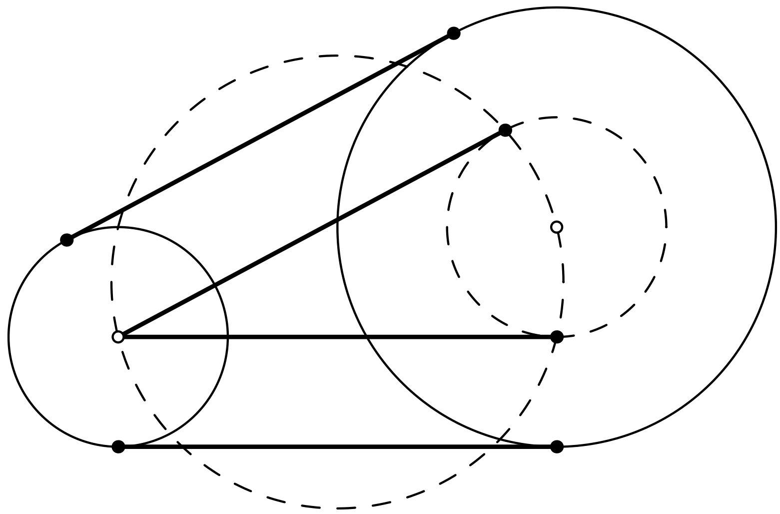

Fig. 4.

Let P and Q be the intersection points of the circle of diameter O_1O_2 and the cycle c_2', if any exist (Fig. 4). Then the lines O_1P and O_1Q, properly oriented, constitute all rays tangent to both O_1 and c_2'. The inverse expansion f^{-1} maps these lines to the common tangents of c_1 and c_2, and we are done. □

Problem 2 (Apollonius' problem)**.** *Construct (using straightedge and compass) all circles tangent to three given circles.*

This famous problem is due to Apollonius of Perga and dates to about 200 BC. B. A. Rozenfeld suggests in [5] that perhaps the following is very close to Appolonius' original solution.

Solution. Once again, we will show how to construct all *cycles* tangent to three given cycles. The solution to the original problem is then obtained as above; only this time there are four substantially different choices for the given circles' orientations.

Let c_1, c_2, c_3 be the three given cycles. Apply an expansion f which maps c_1 to its center O and c_2 and c_3 to some cycles c_2' and c_3'. Apply, then, inversion g of center O which maps c_2' and c_3' to some cycles c_2'' and c_3'' (inversion maps cycles and rays onto cycles and rays and preserves tangency).

Let c be any cycle tangent to c_1, c_2, c_3. Then f maps c to a cycle c' through O, tangent to c_2' and c_3', and g maps c' to some ray c'' tangent to c_2'' and c_3''.

But we already know (from the solution to Problem 1) how to construct all rays tangent to a given pair of cycles c_2'' and c_3''! Applying the inverse inversion $g^{-1} \equiv g$ and the inverse expansion f^{-1} maps these rays to all cycles tangent to c_1, c_2, c_3, and we are done. □

Since between zero and two solutions are possible for each choice of orientations for the original circles, the original problem always has between zero and eight solutions. It is not difficult to see that both extremities occur.

3. Modern Problems

Problem 3 (The equal incircles theorem)**.** *Two points D and E lie on the side BC of $\triangle ABC$ in such a way that that the incircles of $\triangle ABD$ and $\triangle ACE$ are equal. Show that the incircles of $\triangle ABE$ and $\triangle ACD$ are also equal (Fig. 5).*

Even though no explicit statement of this theorem has been found in *sangaku*, many closely related problems were treated there. In particular, the Japanese mathematicians knew how to expresses the inradius of $\triangle ABC$ in terms of the inradii of $\triangle ABD$ and $\triangle ADC$ and the altitude through A; the existence of such an expression yields the statement of the theorem.

In the West, the result has been rediscovered several times; one reference is H. Demir and C. Tezer's paper [1].

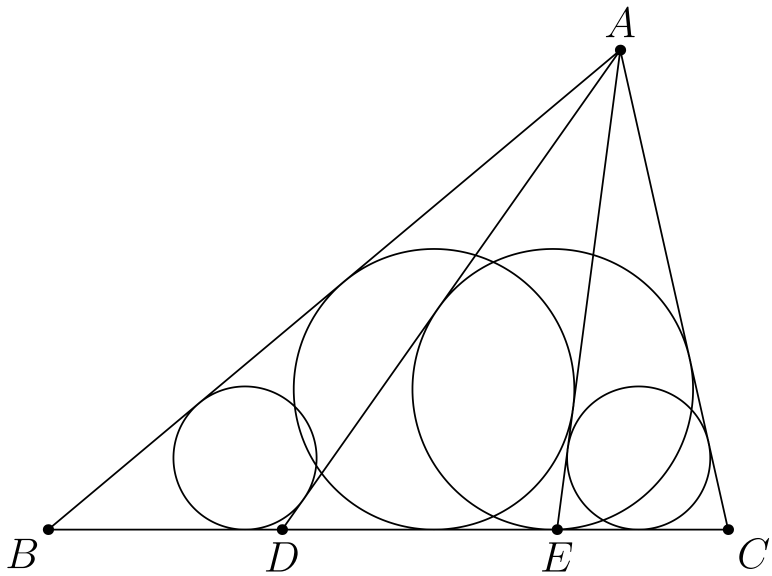

Fig. 5.

Solution. Suppose that $\triangle ABC$ is positively oriented and let ω_δ denote the incircle of the triangle δ.

Apply an expansion f which maps ω^+_{ABD} and ω^+_{ACE} to their centers I and J.

Let $f(CB^{\rightarrow})$ meet $f(BA^{\rightarrow})$, $f(DA^{\rightarrow})$, $f(AE^{\rightarrow})$, $f(AC^{\rightarrow})$ in the points B', D', E', C', respectively, and let $P = f(DA^{\rightarrow}) \cap f(AC^{\rightarrow})$ and $Q = f(BA^{\rightarrow}) \cap f(AE^{\rightarrow})$ (Fig. 6). Let also ∞_{AB} and ∞_{AC} be the corresponding points at infinity.

The hexagon $PI\infty_{AB}QJ\infty_{AC}$ is circumscribed about the cycle $f(A)$. By Brianchon's theorem, its main diagonals PQ, IJ and $\infty_{AB}\infty_{AC}$ are concurrent – the latter meaning simply that $PQ \| IJ$.

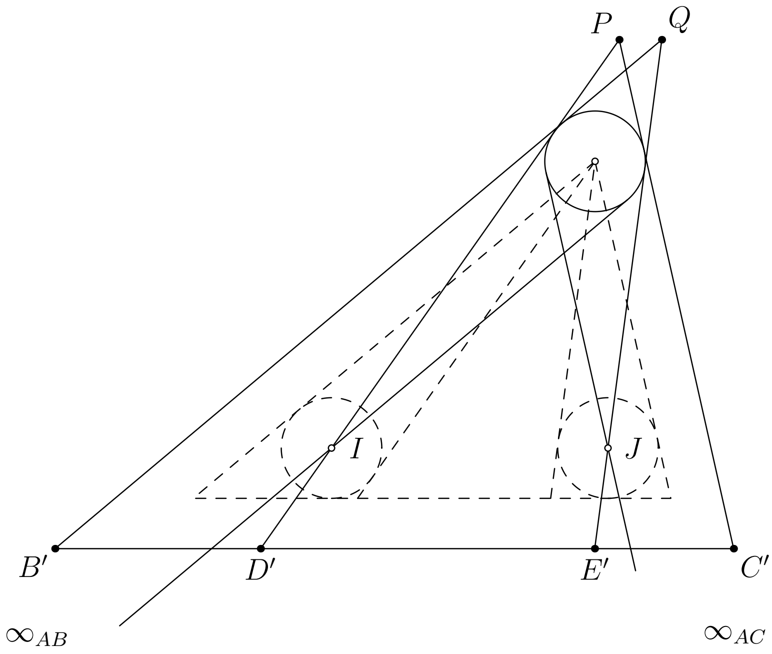

Fig. 6.

Let the tangent to $f(A)$ which is closer to $B'C'$ cut from $\triangle PD'C'$ and $\triangle QB'E'$ two smaller triangles, δ_p and δ_q, similar to the large ones. Since $PQ\|IJ\|B'C'$, the two ratios of similitude are equal; and, since the incircles of δ_p and δ_q are equal ($f(A)$ being their common incircle), the incircles of $\triangle PD'C'$ and $\triangle QB'E'$ must be equal as well.

Those two incircles, however, are in fact the images under f of ω_{ADC} and ω_{ABE}. From this we conclude that the incircles of $\triangle ADC$ and $\triangle ABE$ are equal, and the proof is complete. □

Problem 4. *Given is a right-angled $\triangle ABC$ with $\angle BAC = 90°$. The circle ω_a touches the segments AB and AC and its center lies on the segment BC. Two more circles ω_b and ω_c are constructed analogously. A fourth circle ω touches ω_a, ω_b, ω_c internally (Fig. 7). Show that the radius of ω equals $\frac{3}{2}r_a$, where r_a is the radius of ω_a.*

This problem is from an 1837 tablet found in Miyagi Prefecture [2].

Solution. Suppose that $\triangle ABC$ is positively oriented and apply expansion f of radius $-\frac{3}{2}r_a$. Let the images of ω_a^+, ω_b^+, ω_c^+ under f be ω_a', ω_b', ω_c', respectively. Our objective then becomes to show that ω_a', ω_b', ω_c' have a common point.

To this end, we introduce the following

Lemma. *Let k be a fixed circle and l be a fixed line which does not meet k. Let P be that point of k which is closest to l, and let c be a circle through P which is tangent to l. Then the length of the common external tangent of c and k remains constant when c varies.*

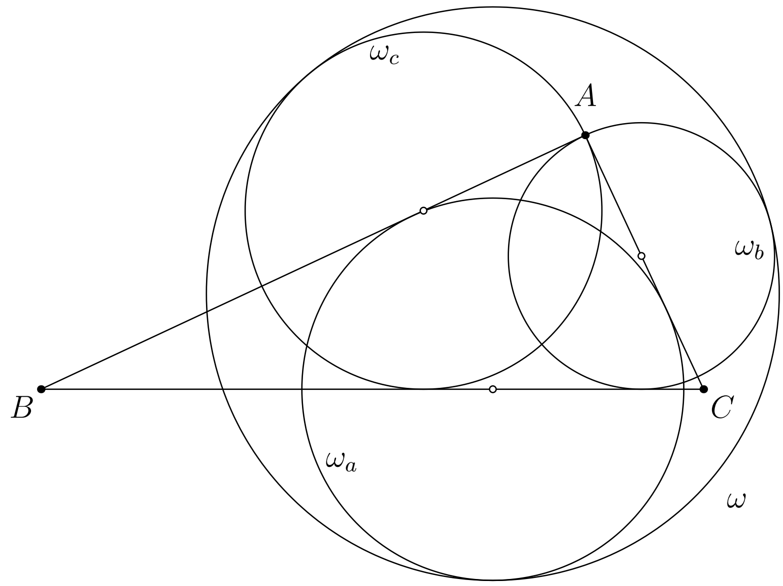

Fig. 7.

Proof. Let g be an inversion of center P which maps k to l and vice versa[3].

Continuity considerations show that there exists a circle which is concentric with k, contains k, and is preserved by g. Let s be such a circle.

Clearly, all tangents to k meet s at some fixed angle α. Conversly, all lines which meet s at an angle α are tangent to k.

Let c meet s at the point Q. Since g maps c to a line tangent to k, and preserves s, the angle between c and s equals α.

It follows that the tangent to c at Q is also a tangent to k. But the lengths of the tangents to k from a point on s are all equal (since k and s are concentric), and we are done. □

Back to Problem 4, let $k \equiv \omega'_a$ and $l \equiv f(BC^{\rightarrow})$ (ω'_a and $f(BC^{\rightarrow})$ regarded, respectively, as a circle and a line; see Fig. 8). Let P be that point of ω'_a which is closest to l. Then the radius of ω'_a equals $\frac{1}{2}r_a$ and the distance from P to l equals r_a.

If we allow c to be the circle of diameter PS, S being the projection of P onto l, we see that in this case the constant common external tangent length from the lemma equals r_a.

This, however, is also the tangential distance from ω_a to ω_b and ω_c in the original configuration, and, consequently, from ω'_a to ω'_b and ω'_c in the transformed one. Since ω'_b and ω'_c both touch l, we conclude from the lemma that they both pass through P. With this, the solution is complete. □

[3]The radius of g will then be an imaginary number. In case that we wish to consider real-radius inversions only, we can regard g as the composition of inversion of center P and two-fold rotational symmetry of center P.

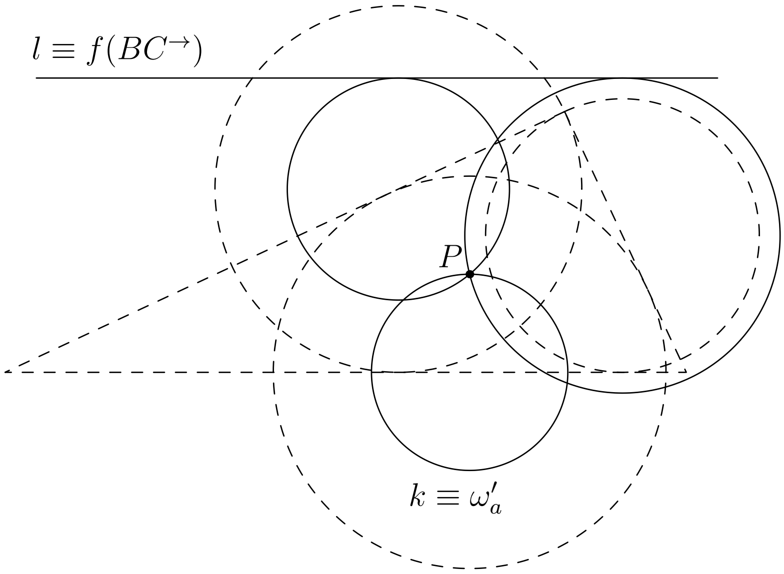

Fig. 8.

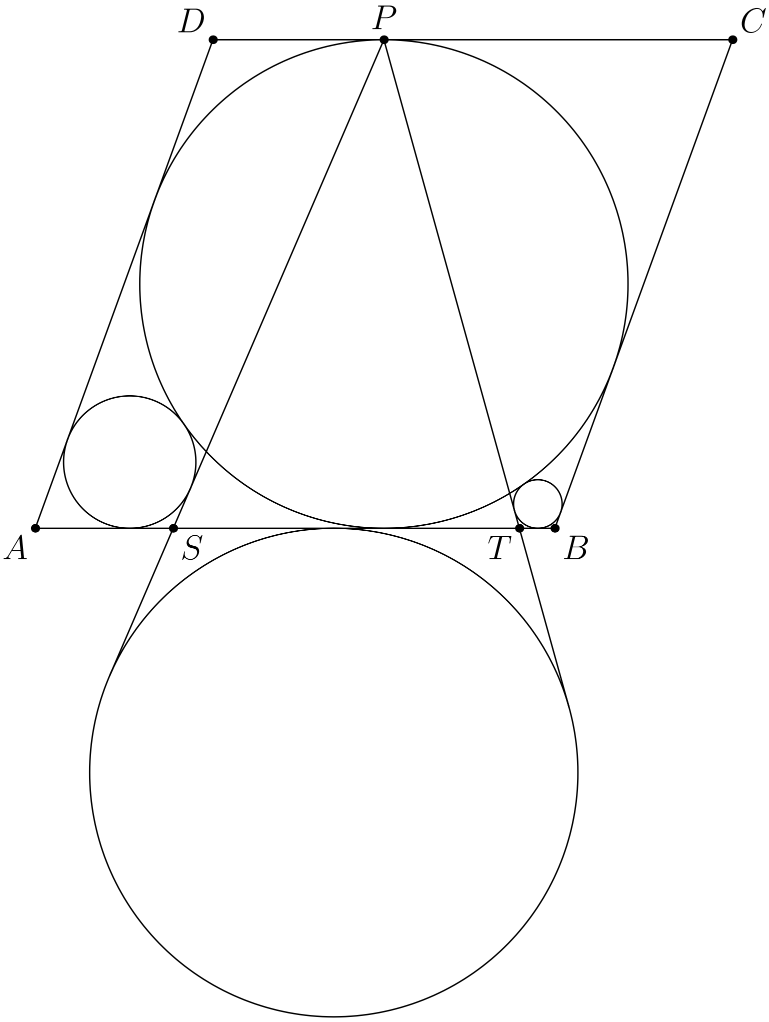

Fig. 9.

Problem 5. *The circle ω is inscribed in a rhombus $ABCD$ and touches the side CD at P. The circle ω_a touches the segments AB and AD, and ω, and the circle ω_b touches the segments BA and BC, and ω. Two tangents from P to ω_a and ω_b*

meet the segment AB in S and T, respectively. Show that the excircle of $\triangle PST$ opposite P is equal to ω (Fig. 9).

This beautiful problem is due to H. Okumura and E. Nakajima [3][4]. It was composed as a generalization of a 1966 problem by A. Hirayama and M. Matsuoka in which $ABCD$ is a square.

Solution. Suppose that $ABCD$ is positively oriented and apply an expansion f which maps ω^+ to its center O. Let $f(w_a^-) = k_a$, $f(w_b^-) = k_b$, $f(BA^{\rightarrow}) = l$, and $f(P) = k$; notice that O lies on k, k_a and k_b meet in O, and k_a and k_b are orthogonal (their radii at O being the diagonals of $ABCD$ and therefore perpendicular).

Our objective has become to show that the appropriate common external tangents of k and k_a, and k and k_b, meet on l (Fig. 10).

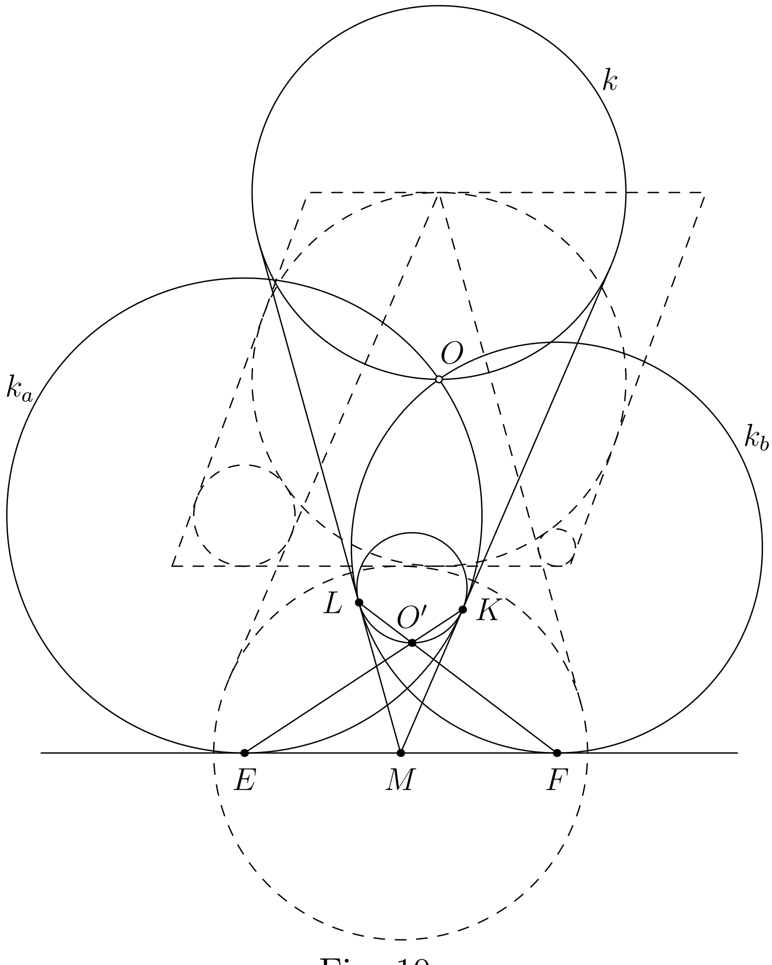

Fig. 10.

In the argument to follow, we may "forget" about orientation and simply speak of all cycles involved as of circles.

Let k_a and k_b touch l at E and F, respectively, and let M be the midpoint of EF. Let the second tangents to k_a and k_b from M meet k_a and k_b at K and L.

[4]In [3]'s original formulation, the problem asks for a proof that the inradius of $\triangle PST$ equals one half the radius of ω. It is easy to see that the two formulations are equivalent.

Since $MK = ME = MF = ML$, there exists a circle k' tangent to MK and ML at K and L. From this, it follows that k' is also tangent to k_a and k_b.

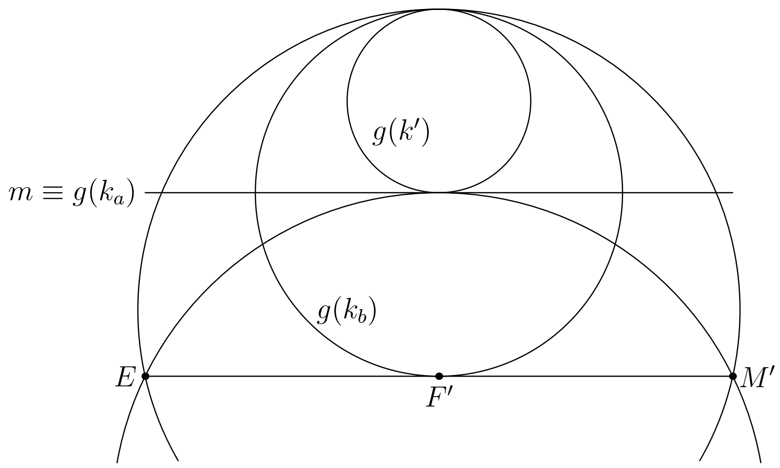

Fig. 11.

Let O' be that point of k' which is closest to l. Homothety of center K shows that the points K, O', E are collinear, and homothety of center L shows that the points L, O', F are collinear as well. Since the quadrilateral $ELKF$ is cyclic (being inscribed in a circle of center M), it follows that $EO'.O'K = FO'.O'L$ and O' lies on the radical axis of k_a and k_b.

The points M and O lie on this radical axis too, and we conclude that M, O', and O are collinear.

For any circle c, denote by $R(c)$ the ratio of its radius to the distance from its center to the line l. We know that $R(k) = r : 3r = 1 : 3$, where r is the radius of ω. We set out to calculate $R(k')$.

Let g be an inversion of center E. Under g, l is preserved, F and M are mapped to some F' and M' such that F' is the midpoint of EM', k_a is mapped to a line m parallel to l, k_b is mapped to a circle s which is tangent to l and whose center lies on m (k_a and k_b being orthogonal), and the lines MK and ML are mapped to two circles passing through E and M' (Fig. 11).

The whole configuration, then, becomes symmetric with respect to the perpendicular bisector of the segment EM'. It follows that the same must be true for $g(k')$.

If the radius of s equals r_s, this allows us to calculate $R(k') = R(g(k'))$ (as the center of inversion lies on l) $= \frac{1}{2}s : \frac{3}{2}s = 1 : 3$.

From the fact that $R(k') = R(k)$, and that M, O', O are collinear, it follows that k' and k are homothetic with center M. The appropriate common external tangents of k and k_a, and k and k_b, then, are the lines MK and ML – and they are indeed concurrent in a point M on l, as needed. □

Problem 6. *The circle ω of radius r is externally tangent to the circles k_1 and k_2, all three circles being tangent to the pair of lines t_1 and t_2. A fourth circle k is internally tangent to both of k_1 and k_2 (Fig. 12). Let ω_1 and ω_2 of radii r_1 and*

r_2 be the largest circles inscribed in the two segments which the lines t_1 and t_2 cut from k (and which do not contain ω). Show that $r_1 r_2 = r^2$.

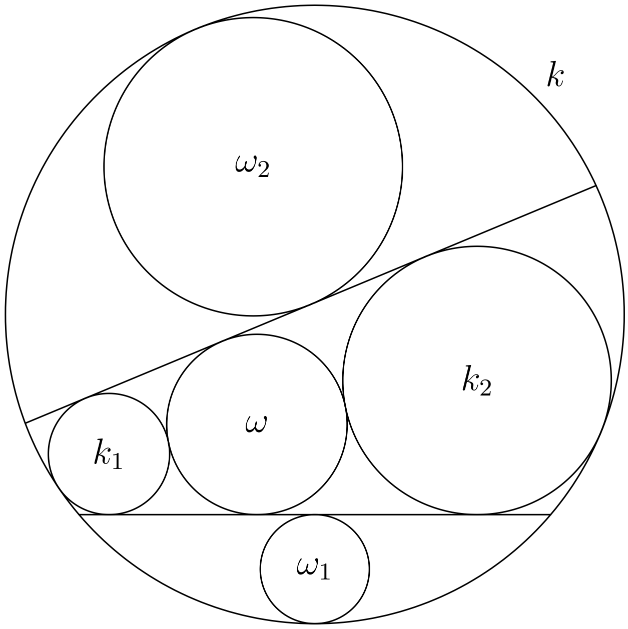

Fig. 12.

This problem is from an 1839 tablet found in Nagano Prefecture [2].

Solution. First we prove the following

Lemma. *Let t_1 and t_2 be the two common external tangents of a fixed pair of disjoint circles k_1 and k_2. Let k be an arbitrary circle internally tangent to both of k_1 and k_2, and let h_1 and h_2 be the altitudes of the two segments which t_1 and t_2 cut from k (and which do not contain k_1 and k_2). Then $h_1 h_2$ remains constant when k varies; furthermore, if d is the projection of the common external tangent of k_1 and k_2 onto their center line, then always $h_1 h_2 = \frac{1}{4}d^2$.*

Proof. Notice that h_1, h_2, and d are all invariant under expansion; therefore, it suffices to consider the case in which k_1 collapses into some point O (Fig. 13).

Let t_1 and t_2 meet k for the second time in A and B; let k_2 touch k, OA, and OB in T, U, and V, respectively; and let the midpoints of OA, OB, $\widehat{OA}$, and $\widehat{OB}$ be M, N, K, and L.

Since T is the homothetic center of k and k_2 and the tangents to these circles at K and U are parallel, the points T, U, K are collinear.

Since $\angle OTK = \widehat{OK} = \widehat{KA} = \angle UOK$, we have $\triangle OTK \sim \triangle UOK$ and therefore $KU.KT = KO^2$. Analogously, $LV.LT = LO^2$.

It follows that the line KL is the radical axis of O and k_2. Let O' be the center of k_2; then $KL \perp OO'$ and KL cuts OU and OV in their midpoints U' and V'. Finally, let KL cut OO' in H.

We have $\angle HLO = \widehat{KO} = \widehat{AK} = \angle MOK$. Analogously, $\angle NOL = \angle HKO$. It follows that $HOLN \sim MKOH$ and $h_1 h_2 = LN.MK = OH^2$. Since OH is the projection of $OU' = \frac{1}{2}OU$ onto OO', we have $OH = \frac{1}{2}d$ and $h_1 h_2 = \frac{1}{4}d^2$, as needed. □

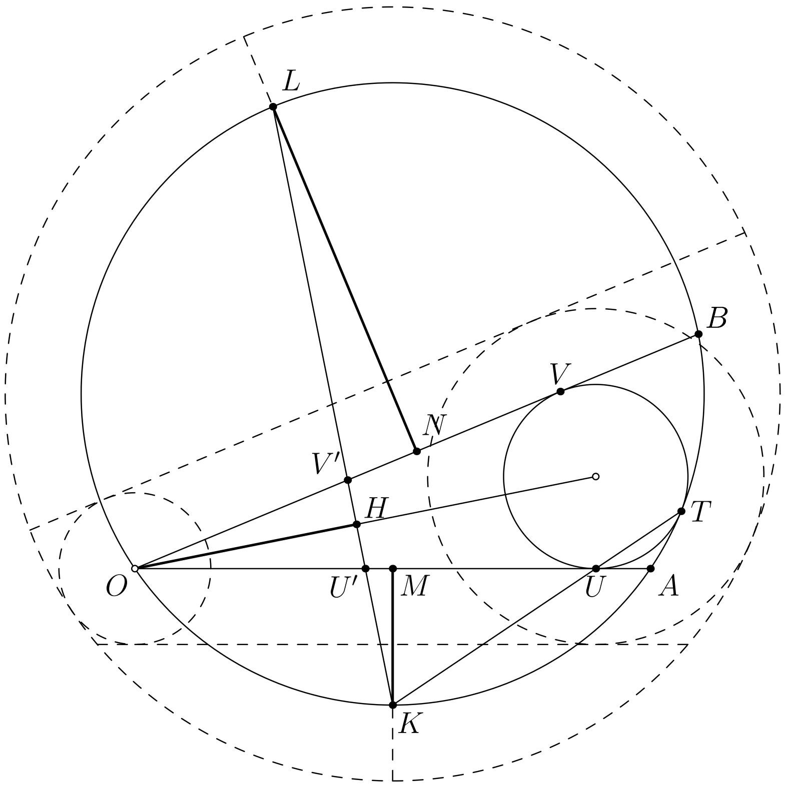

Fig. 13.

Back to the configuration of Problem 6, let the common internal tangents m_1 of ω and k_1, and m_2 of ω and k_2, cut from t_1 some segment s. Since m_1 bisects the common external tangent of ω and k_1, and m_2 bisects the common external tangent of ω and k_2, the length of s equals exactly one half the common external tangent of k_1 and k_2. Its projection onto their center line, on the other hand, is a diameter of ω; by the lemma, the desired equality follows. □

Notice that our lemma proves rather useful in a number of other *sangaku* problems as well. A problem from an 1823 tablet found in Fukusima Prefecture, for instance, asks for a proof that $h_1h_2 = 16r^2$ in the case when k_1 and k_2 are the end members of a chain of five successively tangent circles of radius r locked between t_1 and t_2 [2].

Problem 7. *Let ω be the largest circle inscribed in the smaller segment which the line l cuts from the circle k of radius r. The circle ω_2 of radius r_2 is equal to ω, touches l and is contained in the larger segment which l cuts from k. Two more circles ω_1 and ω_3 of radii r_1 and r_3 are tangent to l, externally tangent to ω_2 and internally tangent to k (Fig. 14). Show that $r = r_1 + r_2 + r_3$.*

Solution. Apply an expansion f which maps ω_2^- to its center O_2, and ω_1^+, ω_3^+, k^+ to some cycles ω_1', ω_3', k' of radii r_1', r_3', r', respectively (Fig. 15). Since the radii of the circles ω_1', ω_3', k' equal $r_1 + r_2$, $r_3 + r_2$, $r + r_2$, respectively, our aim has become to show that $r_1' + r_3' = r'$.

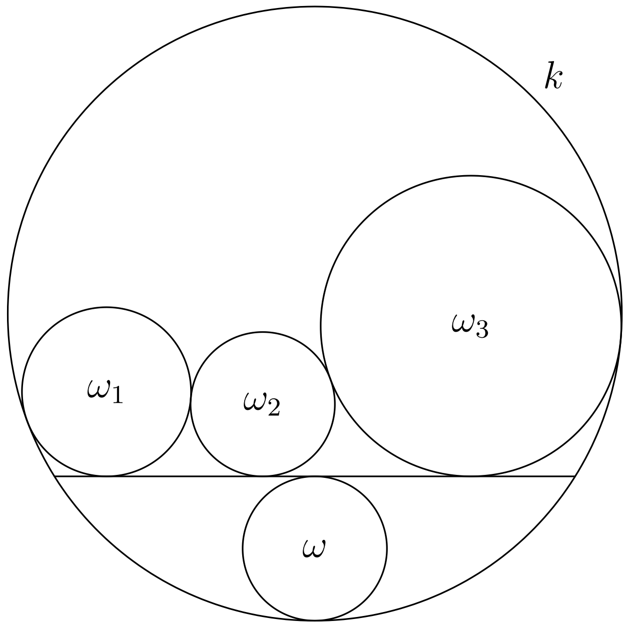

Fig. 14.

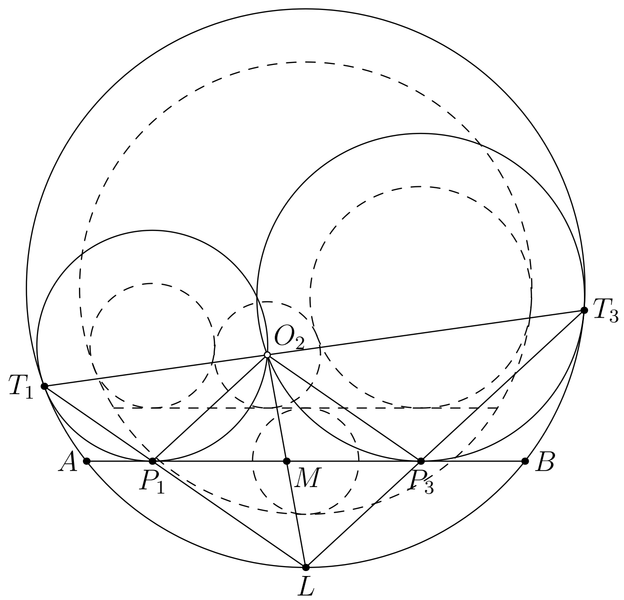

Fig. 15.

This problem is from an 1888 tablet found in Fukusima Prefecture [2].

Let the image l' of l under f (which is a common external tangent of ω_1' and ω_3') meet k' in A and B; let also ω_1' touch k' and AB at T_1 and P_1, respectively, and define T_3 and P_3 analogously for ω_3'. Finally, let L be the midpoint of $\widehat{AB}$.

As in the solution to the previous problem, we see that the points L, P_1, T_1 are collinear, the points L, P_3, T_3 are collinear, and $LP_1.LT_1 = LA^2 = LP_3.LT_3$. It follows that LO_2 is the radical axis of ω_1' and ω_3'. Therefore, LO_2 cuts their common tangent P_1P_3 in its midpoint M.

The distances from both L and O_2 to l' equal $2r_2$; therefore, M is the midpoint of LO_2 as well and $LP_1O_2P_3$ is a parallelogram.

Since the tangents to the circles ω_1', ω_3', k' at the points P_1, P_3, L are parallel and $LP_1T_1 \| P_3O_2$ and $LP_3T_3 \| P_1O_2$, the three triangles $\triangle P_1T_1O_2$, $\triangle P_3O_2T_3$, $\triangle LT_1T_3$ are similar and homothetic. It follows that the points T_1, O_2, T_3 are collinear.

Since the circumradii r_1', r_3', r' of these triangles are proportional to their corresponding sides T_1O_2, O_2T_3, T_1T_3, and $T_1O_2 + O_2T_3 = T_1T_3$, we also have $r_1' + r_3' = r'$, as needed. □

Finally, expansion may sometimes serve to produce new problems. Consider the following

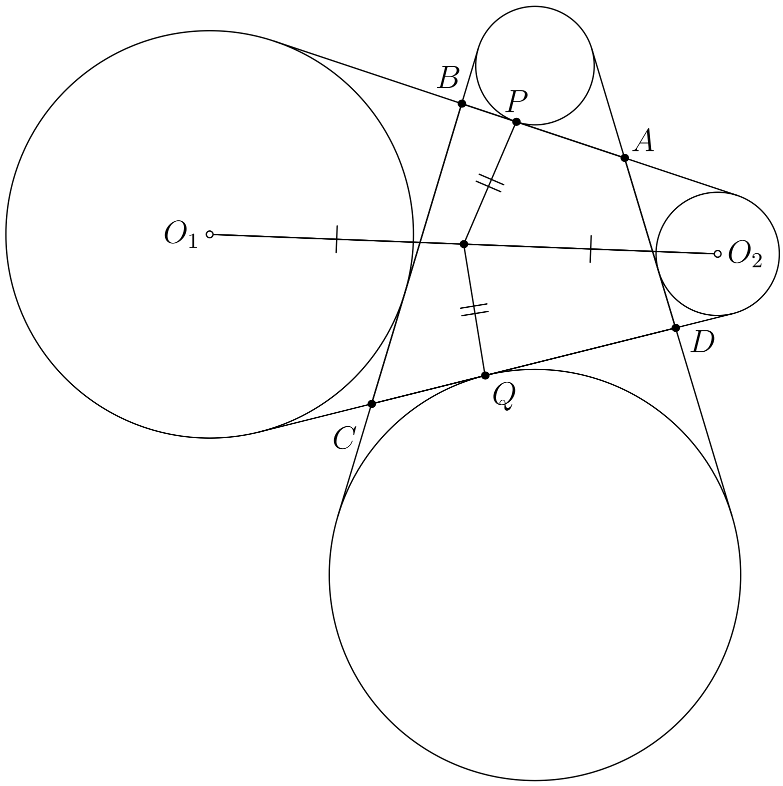

Fig. 16.

Problem 8a. *A point A lies on the common external tangent t_1 of two disjoint circles ω_1 and ω_2 of centers O_1 and O_2, respectively. The second tangents from A to ω_1 and ω_2 meet their second common external tangent t_2 in B and C. If the excircle of $\triangle ABC$ opposite A touches BC at D, show that the points A and D are equidistant from the midpoint of the segment O_1O_2.*

This problem was proposed for the 2005 St. Petersburg Mathematical Olympiad by D. Dzhukich and A. Smirnov. While working on this paper, it occured to the author that expansion would preserve tangency in the problem's configuration. It turned out to preserve the required property as well, and thus a pleasingly symmetric generalization was discovered.

Problem 8b. *Given is a convex quadrilateral $ABCD$. Four circles are constructed as follows: ω_a touches the segment AB and the extensions of DA and BC beyond A and B, and for the definitions of ω_b, ω_c, ω_d, the letters A, B, C, D*

are permuted cyclically (Fig. 16). Let ω_a and ω_c touch AB and CD at P and Q, and let O_b and O_d be the centers of ω_b and ω_d. Show that the points P and Q are equidistant from the midpoint of the segment O_bO_d.

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NEUBERG LOCUS AND ITS PROPERTIES

DEBDYUTI BANERJEE AND SAYAN MUKHERJEE

Abstract. In this article we discuss the famous Neuberg Locus. We also explore some special properties of the cubic, and provide purely synthetic proofs to them.

1. Introduction

In this article we will discuss the Neuberg Locus Problem. It is related with concurrency of some specific lines in the triangles PBC, PCA, PAB given any triangle ABC and a point P on its plane. The Neuberg cubic can be defined as the locus of all such points in the plane of ABC which satisfy the concurrency of the Euler lines, or the Brocard Axes.

We try to convince the reader that the problem, and all the properties are interrelated. So let us come to know about the actual problem:

The Neuberg Problem. *Given a triangle ABC and a point P, suppose, A', B', C' are the reflections of P on BC, CA, AB. Prove that AA', BB', CC' are concurrent iff PP^* is parallel to the Euler line of ABC where P^* is the isogonal conjugate of P wrt ABC. The locus of such points P is the Neuberg Cubic of ABC. But we will not use any property of cubics in our proof. So let us call the locus of P which satisfies the above concurrency fact as Neuberg locus (we are avoiding the term cubic).*

The proof that we will be discussing is synthetic. We also prove a lot of properties of the Neuberg locus which did not have well-known synthetic proofs.

2. Some Useful Lemmas

Before going into the proof of the main problem, we will mention some lemmas that will prove to be handy. These are very useful lemmas and properties which can turn out to be helpful in a lot of concurrency situations.

Lemma 1 (Definition of $\Gamma(ABC)$)**.** *Given two triangles ABC and $A'B'C'$(not homothetic), consider the set of all triangles(no two of them homothetic), so that both ABC and $A'B'C'$ are orthologic to them. Then the locus of the center of orthology of ABC and this set of triangles lie on a conic passing through A, B, C or the line at infinity.*

Proof. On the sides of BC, CA, AB take points C_1, A_1, B_1 so that $\triangle A_1B_1C_1$ is homothetic to $\triangle A'B'C'$.

Through C_1 draw parallel to AC to intersect AB at A_2. Cyclically define B_2, C_2 on BC, AC. Applying the converse of Pascal's theorem on the hexagon $A_1C_2B_1A_2C_1B_2$ we get that A_1, B_1, C_1, A_2, B_2, C_2 lie on a conic(call this conic $\Gamma(A_1B_1C_1)$). $A_3 = A_2C_1 \cap B_2A_1$, $B_3 = B_2A_1 \cap B_1C_2$, $C_3 = C_2B_1 \cap A_2C_1$. Clearly, ABC and A_3, B_3, C_3 are homothetic.

So AA_3, BB_3, CC_3 concur. Since A_2A_1 is the harmonic conjugate of AA_3 wrt AB, AC, so if we draw parallels to A_1A_2, B_1B_2, C_1C_2 through A, B, C, then they intersect BC, CA, AB at three collinear points. So there exists a conic passing through A, B, C so that the tangent at A is parallel to A_1A_2 and similar for B, C. Call this conic $\Gamma(ABC)$. Take any point X on $\Gamma(ABC)$.

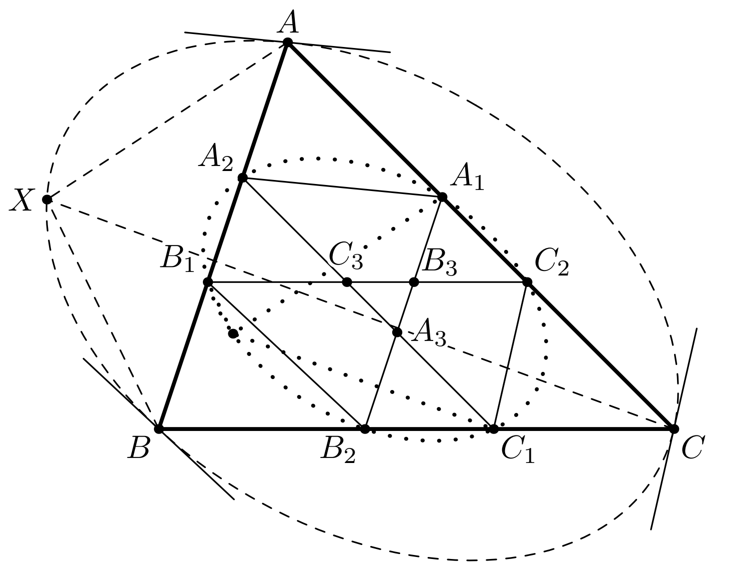

Fig. 1.

Through A_1 draw a line ℓ_a parallel to AX and similarly define ℓ_b, ℓ_c. Note that

$$(l_a, A_1A_2; A_1C_2, A_1B_2) = (AX, A_1A_2; AC, AB) =$$
$$= (BX, BA; BC, B_1B_2) = (l_b, B_1A_2; B_1C_2, B_1B_2);$$

So $\ell_a \cap \ell_b$ lies on $\Gamma(A_1B_1C_1)$ and similar for others. So ℓ_a, ℓ_b, ℓ_c concur on $\Gamma(A_1B_1C_1)$. Its easy to prove that its not true if X doesn't lie on $\Gamma(ABC)$.

In this article we will use the notation $\Gamma(ABC)$ wrt $A'B'C'$ for any two triangles ABC and $A'B'C'$ in this sense. □

Lemma 2 (Sondat's Theorem)**.** *Given two triangles ABC and $A'B'C'$, such that they are orthologic and perspective, prove that the two centers of orthology are collinear with the perspector. This is known as the Sondat's Theorem.*

Proof. Suppose, $AP \perp B'C'$, $BP \perp C'A'$, $CP \perp A'B'$. Similarly define P' such that $A'P' \perp BC$, $B'P' \perp CA$, $C'P' \perp AB$. Suppose that, AA', BB', CC' concur at some point Q.

Note that, if we define $\Gamma(ABC)$ wrt $\triangle A'B'C'$ same as Lemma 1, then we get that, its the rectangular hyperbola passing through A, B, C, P and the orthocenter of ABC. Now note that, BPC and $B'P'C'$ are orthologic and A, A' are the two centers of orthology.

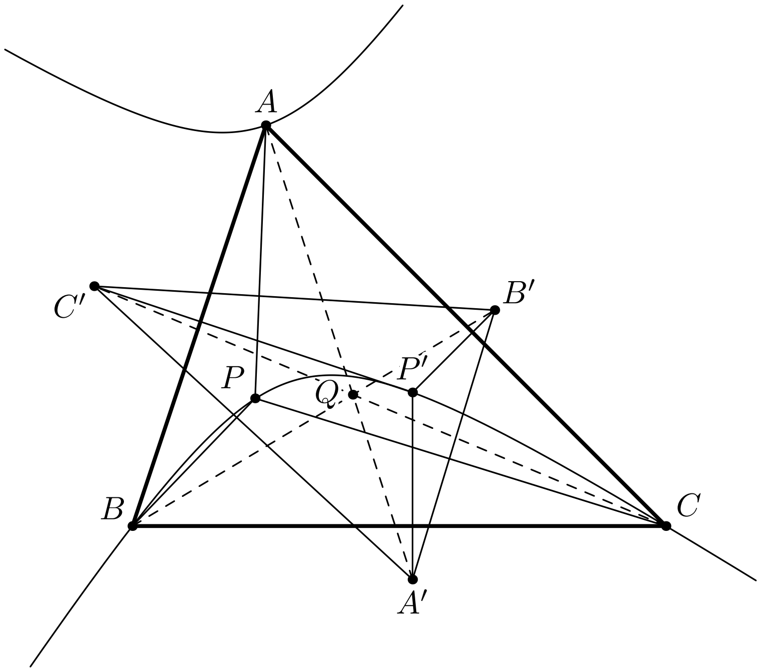

Fig. 2.

Note that, $\Gamma(ABC)$ passes through the orthocenter of $\triangle BPC$, but $\Gamma(BPC)$ wrt $\triangle B'P'C'$ is the rectangular hyperbola passing through B, C, P, A and the orthocenter of $\triangle BPC$.

So $\Gamma(BPC)$ wrt $\triangle B'P'C'$ is same as $\Gamma(ABC)$ wrt $\triangle A'B'C'$. But note that, Q lies on $\Gamma(ABC)$. So Q lies on $\Gamma(BPC)$ wrt $B'P'C'$. So $PQ \parallel P'Q$. So P, P', Q are collinear. □

Lemma 3 (Complete quadrilateral isogonality). *Given two points P, Q and their isogonal conjugates wrt ABC be P', Q', then suppose, $X = PQ' \cap QP'$ and $Y = PQ \cap P'Q'$, then Y is the isogonal conjugate of X wrt ABC.*

Proof. The projective transformation that takes a point P to Q keeping the triangle ABC fixed, sends Q' to some point K. So

$$(AB, AC; AP, AQ') = (AB, AC; AQ, AK),$$

Since the transformation preserves cross-ratio.

Also $(AB, AC; AP, AQ') = (AC, AB; AP', AQ)$ since reflecting on the angle-bisector of $\angle BAC$ doesn't change the cross-ratio of the lines. And

$$(AC, AB; AP', AQ) = (AB, AC; AQ, AP').$$

So $AP' \equiv AK$. Similarly $BP' \equiv BK$. So $K \equiv P'$. So the transformation maps Q' to P'. Thence the projective transformation mapping P to Q' maps Q to P'.

Therefore $(P'A, P'B; P'C, P'Q') = (QA, QB; QC, QP)$, which leads to

$$(P'A, P'B; P'C, P'Y) = (QA, QB; QC, QY).$$

So A, B, C, Q, Y, P' lie on the same conic, which is the isogonal conjugate of the line PQ' wrt ABC.

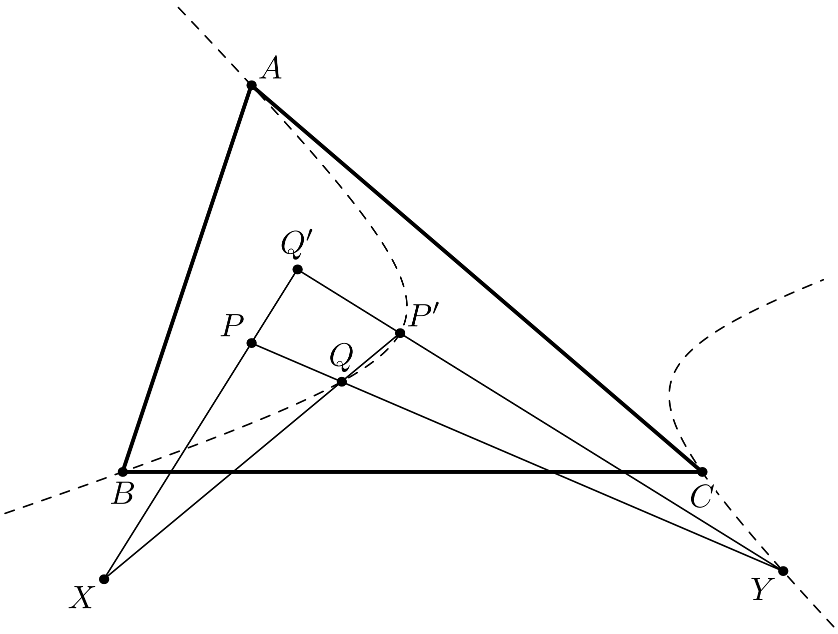

Fig. 3.

So isogonal conjugate of Y lies on PQ' and similarly it lies on QP'. So X is the isogonal conjugate of Y.

Also note that, if two points X, Y are on PQ' and PQ, such that X, Y are isogonal conjugates, then $X = PQ' \cap QP'$ and $Y = PQ \cap P'Q'$.

□

Lemma 4 (Isotomic of Lemoine Axis)**.** *Isotomic line of the Lemoine axis of a triangle ABC is perpendicular to the Euler line of ABC.*

Proof. Suppose, $A'B'C'$ is the Lemoine axis of ABC. Consider the circles with diameter AA', BB', CC'. Clearly, the orthocenter of ABC, say H, lies on their radical axis. Also the circumcenter of ABC (call it O) lies on their radical axis. So they are co-axial with the line OH as their radical axis.

However, using Gaussian line theorem we get that the line joining the midpoints of AA', BB', CC' is parallel to the isotomic line of $A'B'C'$ wrt ABC. So the isotomic line of $A'B'C'$ wrt ABC is perpendicular to OH. □

Lemma 5 (An isotomic property)**.** *Let ABC be a triangle, and let $\overleftrightarrow{A_1B_1C_1}$ and $\overleftrightarrow{A_2B_2C_2}$ be two isotomic lines such that $A_1, A_2 \in BC$ and similar for others. If L is the point where the line through A parallel to BC meets $\overleftrightarrow{A_1B_1C_1}$ then we have $\dfrac{B_1L}{C_1L} = \dfrac{A_2B_2}{A_2C_2}$.*

Proof. Note that, $\dfrac{B_1L}{C_1L} = \dfrac{AB_1}{AC_1}.\dfrac{AB}{AC}$; and also $\dfrac{B_2A_2}{A_2C_2} = \dfrac{B_2C}{C_2B}.\dfrac{AB}{AC}$. But since B_1, B_2 are isotomic points wrt A, so $AB_1 = CB_2$ and $AC_1 = BC_2$. So,

$$\frac{B_1L}{LC_1} = \frac{B_2A_2}{A_2C_2}.$$

□

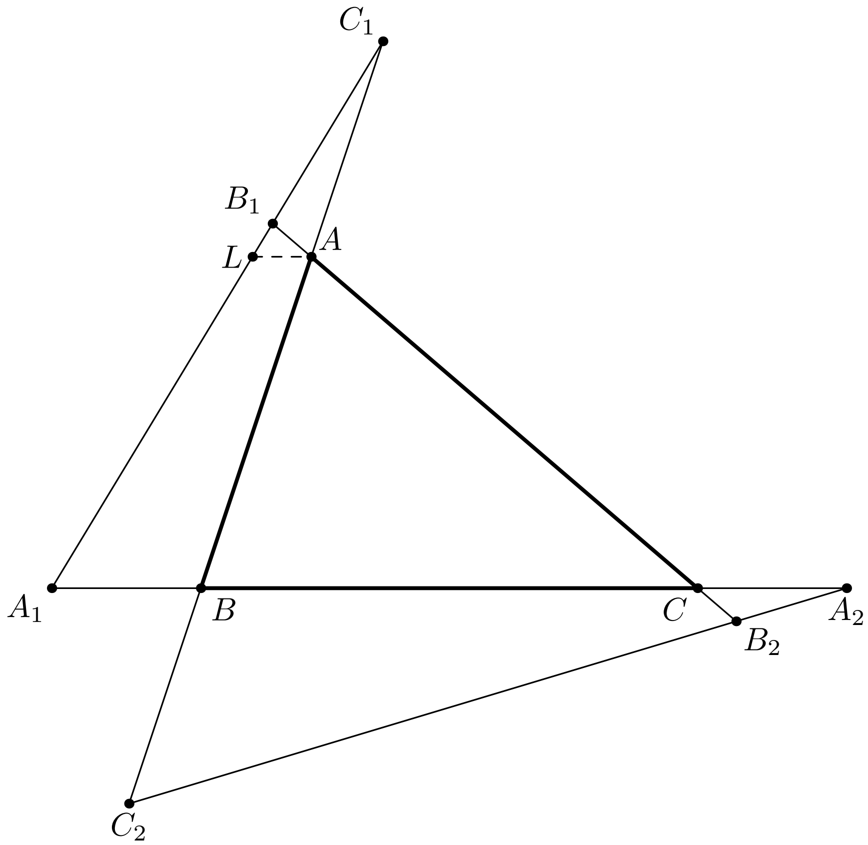

Fig. 4.

3. The Neuberg Problem

3.1. **Properties of the Neuberg locus.** At first we will prove some properties of the Neuberg locus. The properties are as follows.

Property 1 (P, P^* lie on Neuberg locus)**.** *Given a triangle ABC, if a point P lies on the Neuberg locus of ABC, then isogonal conjugate of P wrt ABC also lies on the Neuberg locus of ABC.*

Proof. Suppose, A', B', C' are the reflections of P on BC, CA, AB. Suppose, $A_1 = PA' \cap \odot A'B'C'$. Similarly define B_1, C_1. Under inversion wrt P with power $PA' \cdot PA_1$, A goes to the reflection of P on B_1C_1 and similar for others. Now note that, if A_2, B_2, C_2 are the reflections of P' on BC, CA, AB, then $\triangle A_2B_2C_2$ is homothetic to $\triangle A_1B_1C_1$. So if A_3, B_3, C_3 are the reflections of P' on B_2C_2, C_2A_2, A_2B_2, then $\odot P'A_2A_3$, $\odot P'B_2B_3$, $\odot P'C_2C_3$ are co-axial.

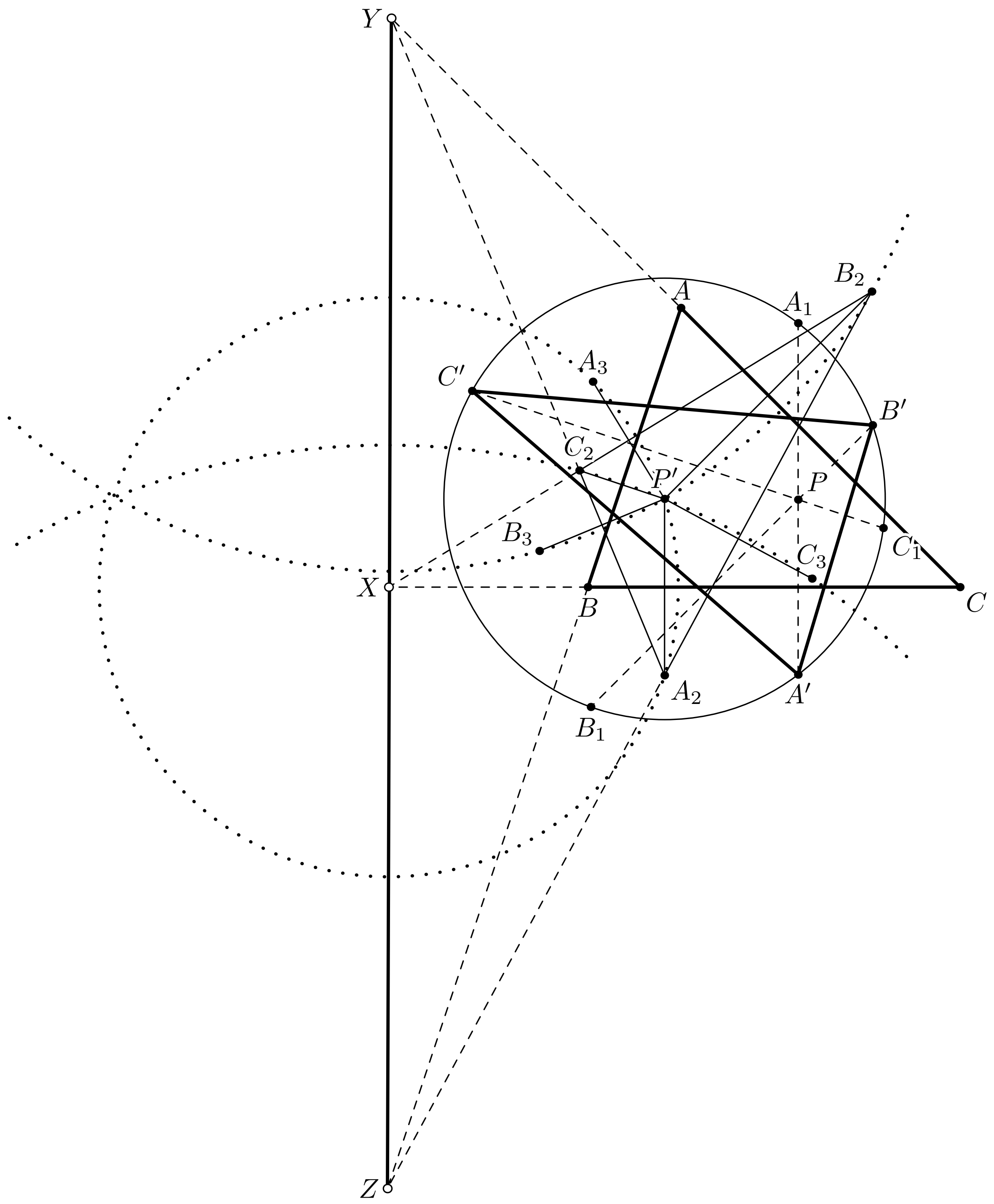

Fig. 5.

But, note that the center of $\odot P'A_2A_3$ is the intersection point of B_2C_2 and BC and similar for others. So $B_2C_2 \cap BC$, $C_2A_2 \cap CA$, $A_2B_2 \cap AB$ are collinear. So by Desargues' Theorem, $\triangle ABC$ and $\triangle A_2B_2C_2$ are perspective. So P' lies on the Neuberg locus of ABC. $\square$

Property 2 (Neuberg locus of pedal triangle)**.** *Given a triangle ABC and a point P, prove that P lies on the Neuberg locus of ABC iff P lies on the Neuberg locus of the pedal triangle of P wrt ABC.*

Proof. Suppose, P^* is the isogonal conjugate of P wrt ABC. A', B', C' be the reflections of P^* on BC, CA, AB. Note that, P is the circumcenter of $\triangle A'B'C'$.

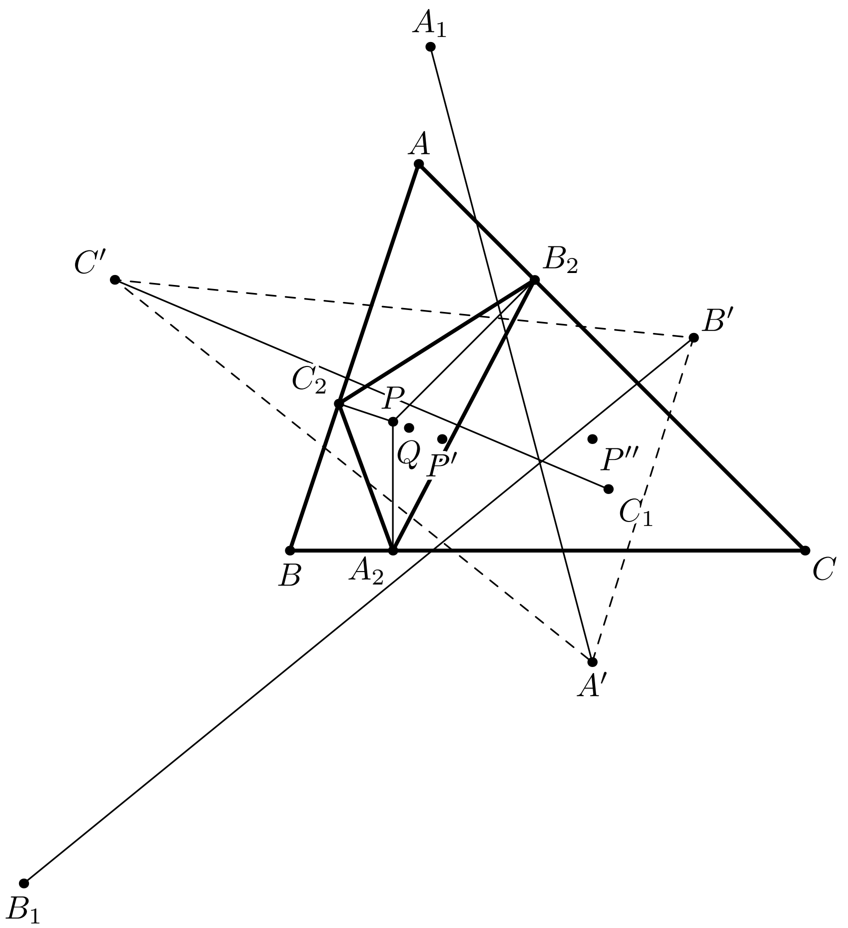

Fig. 6.

Let P' be the isogonal conjugate of P^* wrt $A'B'C'$. Let A_1, B_1, C_1 be the circumcenters of $\triangle B'P'C'$, $\triangle C'P'A'$, $\triangle A'P'B'$. Then,

$$\angle PAB' = 180^\circ - \angle B'P^*C' \text{and}$$

$$\angle PB'A_1 = \angle PB'C' + \angle C'B'A_1 = 90^\circ - \angle B'A'C' + \angle B'P'C' - 90^\circ.$$

So $\angle PB'A_1 = B'P'C' - B'A'C' = 180^\circ - \angle B'P^*C'$, leading to $PA \cdot PA_1 = PB'^2 = PA'^2$. Then, $A'A_1$ is anti-parallel to $A'A$ wrt $\angle A'PA_1$. But the angle-bisector of $\angle APA_1$ is parallel to the angle-bisector of $\angle B'A'C'$. So $A'A$ and $A'A_1$ are isogonal conjugates wrt $\angle B'A'C'$, and similar for others. But from Property 1, if P lies on the Neuberg locus of $\triangle ABC$ iff so does P^*.

So, $A'A$, $B'B$, $C'C$ concur, and $A'A_1$, $B'B_1$, $C'C_1$ concur. So P' lies on the Neuberg locus of $\triangle A_1B_1C_1$.

Now note that, if $A_2B_2C_2$ is the pedal triangle of P wrt ABC and Q is the isogonal conjugate of P wrt $A_2B_2C_2$, then the configuration $A_2B_2C_2Q$ is homothetic to the configuration $A_1B_1C_1P'$. So Q lies on the Neuberg locus of $\triangle A_2B_2C_2$. So again using Property 1, we get that P lies on the Neuberg locus of $\triangle A_2B_2C_2$.

□

Corollary 1. *Given a triangle ABC and a point P on its Neuberg locus, suppose, O_a, O_b, O_c are the circumcenters of $\triangle BPC$, $\triangle CPA$, $\triangle APB$. Prove that, AO_a, BO_b, CO_c are concurrent. Call that concurrency point as Q. If P^* is the isogonal conjugate of P wrt ABC, then define Q^* for P^* similarly. Then Q and Q^* are isogonal conjugates wrt ABC. Furthermore, $Q = OP \cap HP^*$ and $Q^* = OP^* \cap HP$ where H is the orthocenter of ABC.*

Proof. From the proof of Property 2, we can directly observe(just replace ABC by $A'B'C'$) that Q and Q^* are isogonal conjugates wrt ABC. So using Lemma 3, we get that $Q = OP \cap HP^*$ and $Q^* = OP^* \cap HP$. □

Corollary 2. *Let P, P^* be any two isogonal conjugates in a triangle ABC. Suppose that the circumcenters of BPC, CPA, APB and BP^*C, CP^*A, AP^*B are O_a, O_b, O_c, O'_a, O'_b, O'_c respectively. Then the line joining the circumcenters of $O_aO_bO_c$ and $O'_aO'_bO'_c$ is parallel to PP^*. Furthermore, the circles $\odot O_aO_bO_c$, $\odot O'_aO'_bO'_c$, $\odot ABC$ are coaxial.*

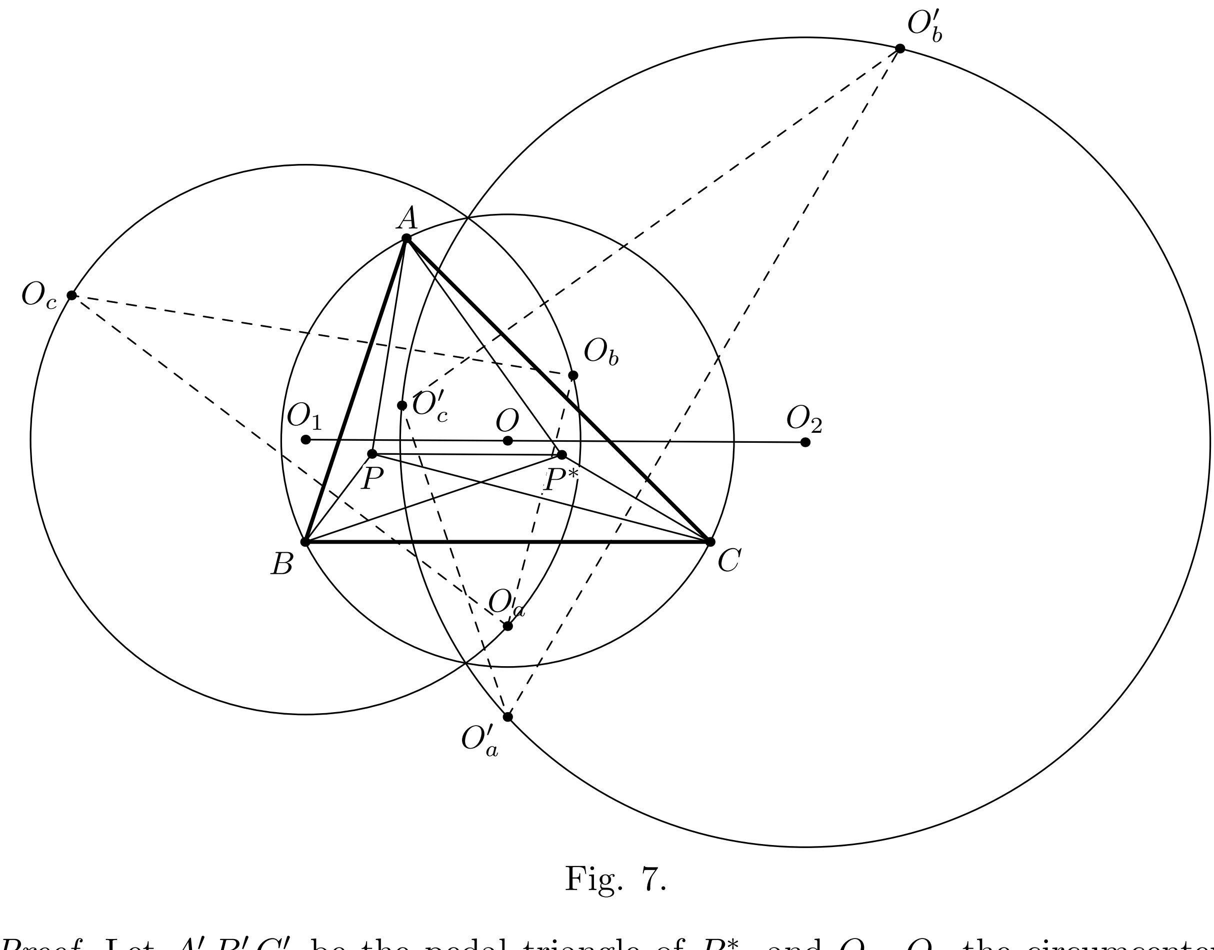

Fig. 7.

Proof. Let $A'_1B'_1C'_1$ be the pedal triangle of P^*, and O_1, O_2 the circumcenters of $O_aO_bO_c$ and $O'_aO'_bO'_c$. Then $O_bO_c \perp AP$ and $AP \perp B'_1C'_1$. So, note that the homothety that maps $A'_1B'_1C'_1$ to $O_aO_bO_c$ also sends the midpoint of PP^* to O_1 and P^* to O, where O is the circumcenter of ABC. So if J is the midpoint of PP^*, then we must have $P^*J \parallel OO_1$, and similarly $PJ \parallel OO_2$, so O_1, O_2, O are collinear and $O_1O_2 \parallel PP^*$.

For the extension, note that we have $OO_aO'_a \perp BC$, and also $\angle(O_cO'_c, O_cO_b) = \angle(AB, AP) = \angle(AC, AQ) = \angle(O'_bO'_c, O_bO'_b)$, implying $O_bO_cO'_bO'_c$ is cyclic. So, inversion about O with radius OA takes O_a to O'_a, O_b to O'_b and O_c to O'_c. Under this inversion, $\odot O_aO_bO_c$ maps to $\odot O'_aO'_bO'_c$, and their points of intersection remain fixed. So, they lie on $\odot ABC$, and the rest follows. □

Property 3 (An Equivalence)**.** *Given a triangle ABC and a point P, prove that P lies on the Neuberg locus(or circumcircle or line at infinity) of ABC iff the Euler lines of $\triangle BPC$, $\triangle CPA$, $\triangle APB$ are concurrent.*

Proof. We may ignore the case when P lies on circumcircle and line at infinity. Let G_a, G_b, G_c be the centroids of the triangles BPC, CPA, APB. and O_a, O_b, O_c are the their circumcenters. Then we have $G_bG_c \parallel BC$, and $BG_c \cap CG_b$ is the midpoint of AP).

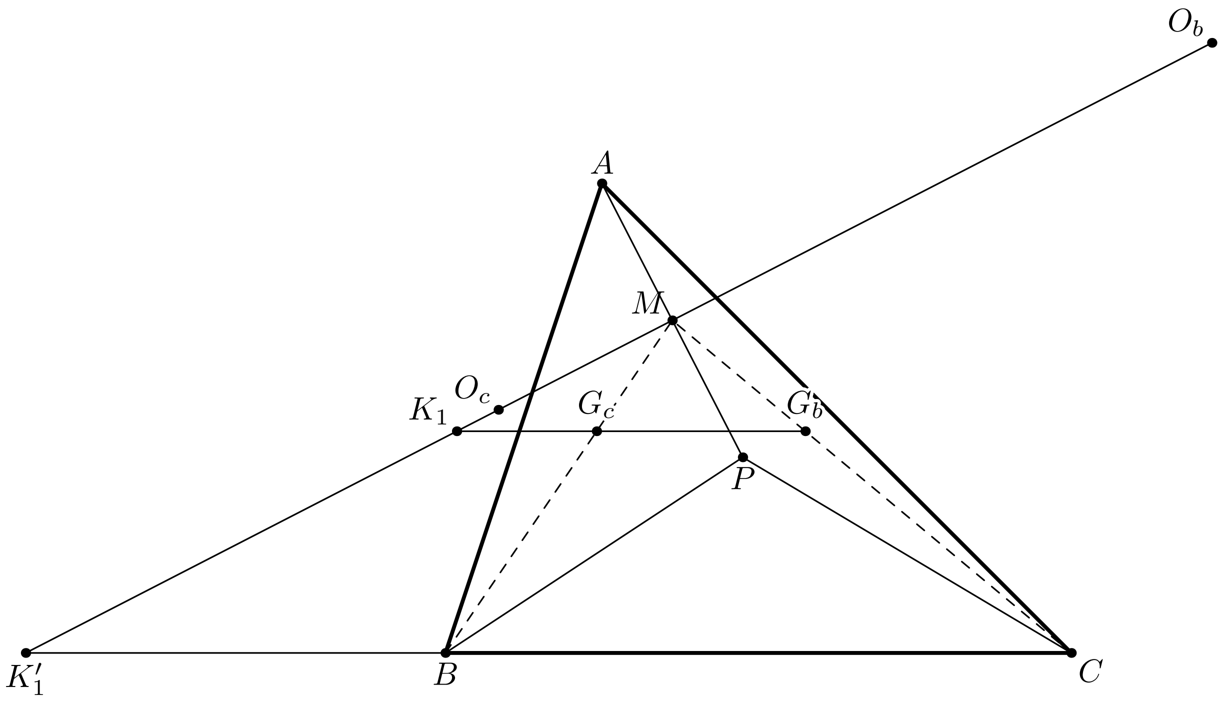

Fig. 8.

So, the homothety that maps G_bG_c to BC maps $O_bO_c \cap G_bG_c \equiv K_1$ to $O_bO_c \cap BC \equiv K'_1$. Thus, $\dfrac{G_cK_1}{G_bK_1} = \dfrac{BK'_1}{CK'_1}$. Since the triangles $G_aG_bG_c$ and $O_aO_bO_c$ are perspective, so we get $\prod_{cyc} \dfrac{G_cK_1}{G_bK_1} = -1$, which leads to $\prod_{cyc} \dfrac{BK'_1}{CK'_1} = -1$, so that ABC and $O_aO_bO_c$ are perspective. So using the corollary 1 of Property 2 we get that P lies on the Neuberg locus of $\triangle ABC$. □

Property 4 (Euler lines concurrent on OH)**.** *Given a triangle ABC and a point P, if the Euler lines of BPC, CPA, APB are concurrent, then they concur on the Euler line of ABC.*

Proof. Suppose, O_a, O_b, O_c are the circumcenters of $\triangle BPC$, $\triangle CPA$, $\triangle APB$. G_a, G_b, G_c be the centroids of $\triangle BPC$,$\triangle CPA$,$\triangle APB$. Suppose, O_aG_a, O_bG_b, O_cG_c concur at some point X. Note that, $\triangle O_aO_bO_c$ and $\triangle G_aG_bG_c$ are perspective. Also note that, the perpendiculars from O_a, O_b, O_c to the sides of $\triangle G_aG_bG_c$ concur at the circumcenter of ABC and the perpendiculars from G_a, G_b, G_c to the sides of $O_aO_bO_c$ concur at the centroid of $\triangle ABC$. So by Sondat's Theorem (Lemma 2) O_aG_a, O_bG_b, O_cG_c concur on the Euler line of ABC.

□

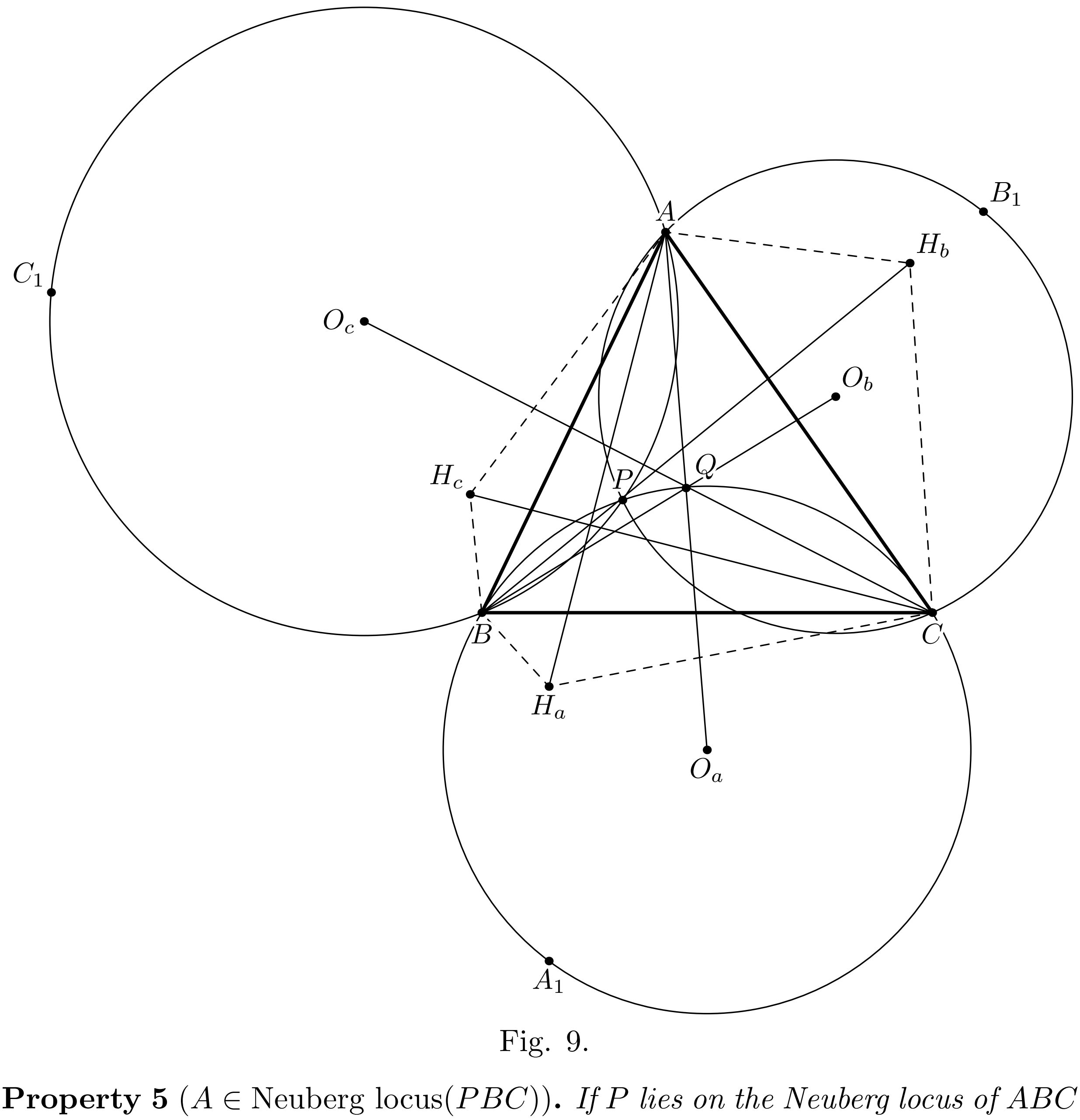

Fig. 9.

Property 5 ($A \in$ Neuberg locus(PBC))**.** *If P lies on the Neuberg locus of ABC, then A lies on the Neuberg locus of BPC and similarly.*

Proof. Note that, we have proved if the Euler lines of PBC, PCA, PAB are concurrent, then they concur on the Euler line of ABC. So using Property 3, we get that A lies on the Neuberg locus of PBC. □

Property 6 (Orthocenters of A_iBC)**.** *Given a triangle ABC and a point P, suppose that A_1, B_1, C_1 are the intersection points of AP, BP, CP with $\odot PBC$, $\odot PCA$, $\odot PAB$. H_a, H_b, H_c are the orthocenters of $\triangle A_1BC$, $\triangle B_1CA$, $\triangle C_1AB$. Prove that AH_a, BH_b, CH_c are concurrent iff P lies on the Neuberg locus of $\triangle ABC$ or the circles with diameter BC, CA, AB or the line at infinity.*

Proof. Suppose, O_a, O_b, O_c are the circumcenters of PBC, PCA, PAB. Note that, CO_a, CH_b are isogonal conjugates wrt $\angle ACB$ and similar for others. Suppose, $A' = BH_c \cap CH_b$. Similarly, define B', C'. Consider the case when P lies on the Neuberg locus of ABC. Clearly, AO_a, BO_b, CO_c concur at some point Q. Let Q^* be the isogonal conjugate of Q wrt ABC. Note that, AA', BB', CC' concur at Q^*.

So applying converse of Brianchon's theorem on $BA'CB'AC'$ we get that, there exists a conic touching BH_a, H_aC, CH_b, H_bA, AH_c, H_cB. So applying Brianchon's theorem on $AH_cBH_aCH_b$ we get that AH_a, BH_b, CH_c are concurrent.

The other two cases are just special cases which can be verified easily.

□

Property 7 (Concurrency of Brocard axes)**.** *Given a triangle ABC and point P, prove that Brocard axes of PBC, PCA, PAB are concurrent iff P lies on the Neuberg locus of ABC or the circumcircle of ABC or the line at infinity.*

Proof. Suppose, O_a, H_a, K_a are the circumcenter, orthocenter and symmedian point of PBC. Let O be the circumcenter of ABC. Note that, its enough to prove that,

$$\prod_{cyc}(O_aO_b, O_aO_c; O_aP, O_aK_a) = 1.$$

Suppose, $P'B'C'$ is the Lemoine axis of PBC. Then note that,

$$(O_aO_b, O_aO_c; O_aP, O_aK_a) = (PC, PB; PP', B'C') = \frac{C'P'}{B'P'}.$$

Let $P_1B_1C_1$ be the isotomic line of $P'B'C'$. A line through P parallel to BC intersects B_1C_1 at U. Using Lemma 5 we get that, $\dfrac{B_1U}{C_1U} = \dfrac{B'P'}{P'C'}$. Hence,

$$(PB, PC; PU, B_1C_1) = (PB, PC; PP', B'C').$$

However,

$$(PB, PC; PU, B_1C_1) = (O_aO_c, O_aO_b; O_aO, O_aH_a).$$

So,

$$(O_aO_c, O_aO_b; O_aO, O_aH_a) = (O_aO_c, O_aO_b; O_aP, O_aK_a).$$

But since the lines O_aH_a are concurrent, so

$$\prod_{cyc}(O_aO_c, O_aO_b; O_aO, O_aH_a) = 1,$$

and we are done. □

Property 8 (Quadrangles Involutifs [3])**.** *For a triangle ABC, a point P lies on its Neuberg locus if and only if*

$$\begin{vmatrix} 1 & BC^2+AP^2 & BC^2\cdot AP^2 \\ 1 & CA^2+BP^2 & CA^2\cdot BP^2 \\ 1 & AB^2+CP^2 & AB^2\cdot CP^2 \end{vmatrix} = 0;$$

Which is equivalent with

$$(AP^2-AB^2)(BP^2-BC^2)(CP^2-CA^2) = (AP^2-AC^2)(BP^2-BA^2)(CP^2-CB^2).$$

Proof. From the proof of Property 6(we will use the same notations in this proof too), we get that

$$\prod_{cyc} \frac{C'P'}{P'B'} = 1.$$

However,

$$\frac{C'P'}{P'B'} = \frac{C'B}{BA}\cdot\frac{AC}{CB'} = \frac{BC^2}{BC^2-PB^2}\cdot\frac{BC^2-PC^2}{BC^2} = \frac{BC^2-PC^2}{BC^2-PB^2};$$

And our result follows. □

4. Proof of The Neuberg Problem.

Given a triangle ABC and a point P, suppose, P^* is the isogonal conjugate of P wrt ABC; P_a, P_b, P_c are the circumcenters of $\triangle BPC$, $\triangle CPA$, $\triangle APB$; P_a^*, P_b^*, P_c^* be the circumcenters of $\triangle BP^*C$, $\triangle CP^*A$, $\triangle AP^*B$, respectively.

Suppose, P lies on the Neuberg locus of $\triangle P_aP_bP_c$, so that AP_a, BP_b, CP_c are concurrent. Clearly, it implies AP_a^*, BP_b^*, CP_c^* are concurrent.

Suppose, O is the circumcenter of $\triangle ABC$. Note that, the triangle formed by the circumcenters of $\triangle OP_bP_c$, $\triangle OP_cP_a$, $\triangle OP_aP_b$ is homothetic to $\triangle ABC$. So if we draw parallels to the Euler lines of $\triangle OP_bP_c$, $\triangle OP_cP_a$, $\triangle OP_aP_b$ through A, B, C they will meet at some point Q. Similarly define Q^* for P^*.

Now note that, P_bP_c is anti-parallel to P_b^*, P_c^* wrt $\angle P_bOP_c$. So the Euler line of $\triangle OP_bP_c$ is anti-parallel to the Euler line of $\triangle OP_b^*P_c^*$ wrt $\angle P_bOP_c$, and similar for others. Therefore, Q^* is the isogonal conjugate of Q wrt ABC. Note that, the triangle formed by the centroids of $\triangle OP_bP_c$, $\triangle OP_cP_a$, $\triangle OP_aP_b$ is homothetic to $\triangle P_aP_bP_c$. So if we draw parallels to AQ, BQ, CQ through P_a, P_b, P_c, then they will concur. So using Lemma 1 we get that, Q lies on $\Gamma(ABC)$ wrt $\triangle P_aP_bP_c$ which is the rectangular hyperbola passing through A, B, C, H, P where H is the orthocenter of $\triangle ABC$.

So Q^* lies on OP^*. Similarly, Q lies on OP. Thence, using Lemma 3, we get that, $Q = OP \cap HP^*$ and $Q^* = OP^* \cap HP$.

Now note that, if R is the concurrency point of AP_a, BP_b, CP_c and R^* is the concurrency point of AP_a^*, BP_b^*, CP_c^*, then from Corollary 1 of Property 2, we have $R = OP \cap HP^*$ and $R^* = OP^* \cap HP$.

All of this leads to $R \equiv Q$ and $R^* \equiv Q^*$.

From Corollary 2 of Property 2, we get that the PQ is parallel to the Euler line of $\triangle P_aP_bP_c$. But now we have, Q lies on OP, and O is the isogonal conjugate of P wrt $P_aP_bP_c$.

So the line joining P and isogonal conjugate of P wrt $P_aP_bP_c$ is parallel to the Euler line of $P_aP_bP_c$. Hence proved.

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GENERALIZATION OF A PROBLEM WITH ISOGONAL CONJUGATE POINTS

TRAN QUANG HUNG AND PHAM HUY HOANG

ABSTRACT. In this note we give a generalization of the problem that was used in the All-Russian Mathematical Olympiad and a purely sythetic proofs.

The following problem was proposed by Andrey Badzyan on All-Russian Mathematical Olympiad (2004–2005, District round, Grade 9, Problem 4).

Problem 1. *Let ABC be a triangle with circumcircle (O) and incircle (I). M is the midpoint of AC, N is the midpoint of the arc AC which contains B. Prove that $\angle IMA = \angle INB$.*

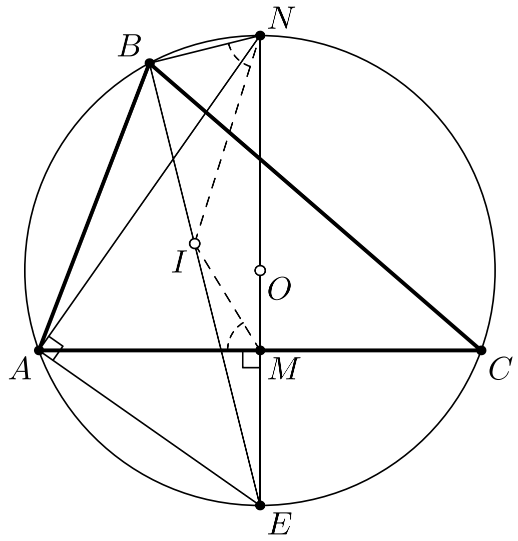

Fig. 1.

Official solution by the Committee. Denote by E be the midpoint of the arc AC which does not contain B. It is clear that B, I, E are collinear, since the line formed by these points is the angle bisector of $\angle ABC$.

Additionally, N, O, M, E are also collinear, since these points all belong to the perpendicular bisector of AC and it is well-known that $AE = EC = IE$.

Since $\angle NAE = \angle AME = 90°$ it is easy to see that $\triangle AME \sim \triangle NAE$ which implies that $|ME| \cdot |NE| = |AE|^2 = |EI|^2$. Hence, we have $\triangle EIM \sim \triangle ENI$ from which we get $\angle IME = \angle EIN$.

Note the following

$$90° + \angle IMA = \angle AME + \angle IMA = \angle IME = \angle EIN =$$
$$= \angle INB + \angle IBN = \angle INB + 90°.$$

We get the required equality

$$\angle IMA = \angle INB.$$

□

Darij Grinberg in [1] gave a solution using the idea of excircle construction while another member named *mecrazywong* on the same forum suggested a different solution by making use of similarity and angle chasing. Now we give a generalized problem.

Problem 2. *Let ABC be a triangle with circumcircle (O). Suppose P, Q are two points lying in the triangle such that P is the isogonal conjugate of Q with respect to $\triangle ABC$. Denote by D the point of intersection of AP and (O) in which $D \neq A$. OD consecutively cuts BC at M and again cuts (O) at N. Prove that $\angle PMB = \angle QNA$.*

If points P and Q are coincide with the incenter I, Problem 2 is coincide with problem 1.

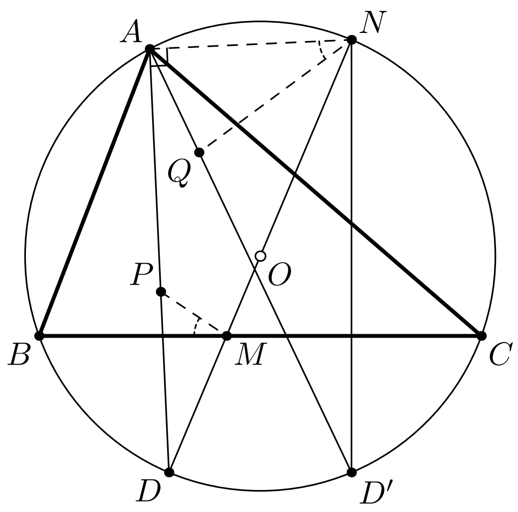

Fig. 2.

Proof. Denote the intersection of AQ and (O) by D'. Since $\angle DAB = \angle D'AC$ we have that $BCD'D$ is an isosceles trapezoid.

We have,

- $\angle PDB = \angle BD'Q$
- $\angle BPD = \angle BAP + \angle PBA = \angle CBD' + \angle QBC = \angle QBD'$.

So $\triangle PBD \sim \triangle BQD'$ and it is easy to conclude that

$$\frac{|PD|}{|BD'|} = \frac{|BD|}{|QD'|} \Rightarrow |PD| \cdot |QD'| = |BD| \cdot |BD'|. \tag{1}$$

On the other hand,

- $\angle MBD = \angle BND'$ (since $\angle MBD = \frac{1}{2} m\, \overset{\frown}{CD} = \frac{1}{2} m\, \overset{\frown}{BD'} = \angle BND'$)
- $\angle BDM = \angle BD'N$

so $\triangle BMD \sim \triangle NBD'$. Hence

$$|BD| \cdot |BD'| = |MD| \cdot |ND'|. \tag{2}$$

From (1) and (2) it is follows that $|PQ| \cdot |QD|' = |MD| \cdot |ND'|$, or $\dfrac{|PD|}{|MD|} = \dfrac{|QD'|}{|ND'|}$. Since $\angle PDM = \angle QD'N$ we get $\triangle PDM \sim \triangle ND'Q$, thus $\angle PMD = \angle NQD'$.

Also, from $\triangle BMD \sim \triangle NBD'$ we get $\angle BMD = \angle NBD' = \angle NAD'$.

Hence

$$\angle PMD - \angle BMD = \angle NQD' - \angle NAD' \Rightarrow \angle PMB = \angle QNA.$$

The proof is completed. □

From the above general problem, we get some corollaries

Corollary 1. *Let ABC be a triangle with bisector AD. Let M be the midpoint of BC. Suppose P and Q are two points on the segment AD such that $\angle ABP = \angle CBQ$, then the circumcenter of the triangle PQM lies on a fixed line when P, Q vary.*

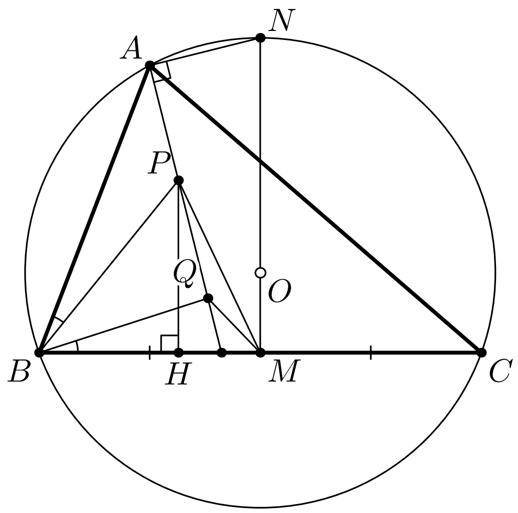

Fig. 3.

Proof. Let H the a point on BC such that $PH \perp BC$. Denote by N the midpoint of the arc BC which contains A. It is easy to see that P, Q are two isogonal conjugate points with repsect to triangle ABC. From our generalized problem, we have $\angle QNA = \angle PMB$ which yields $\angle AQN = \angle HPM = \angle PMN$ (note that $\angle NAD = 90°$), thus $QPMN$ is a concyclic quadrilateral. Therefore the circumcenter of triangle PQM lies on the perpendicular bisector of MN, which is a fixed line. We are done. □

Corollary 2. *From the generalized problem it is follows that $\angle PMN = \angle AQN$, thus if we denote the intersection of PM and AQ by T, then Q, M, N, T are concyclic. Moreover, $PM \parallel AQ$ if and only if $Q \in OM$.*

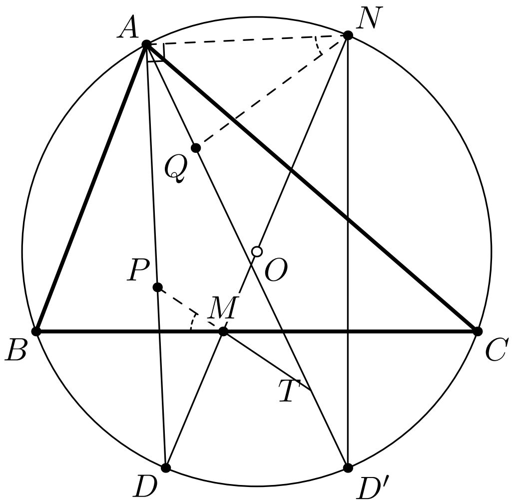

Fig. 4.

Proof. We have $\angle NMT = \angle NMC + \angle CMT = \angle MNB + \angle NBM + \angle PMB = \angle D'AC + \angle NAC + \angle QNA = \angle QAN + \angle QNA = \angle D'QN$. Hence Q, M, N, T are concyclic.

Therefore

$$PM \parallel AQ \iff (PM, AQ) = 0 \iff (NQ, ND) = 0 \iff NQ \equiv ND.$$

We are done.

□

Hence from the above corollary, we can make a new problem.

Problem 3. *Let ABC be a triangle with circumcircle (O). Let d be a line which passes through O and intersects BC at M. Suppose Q is a point on d and P is the isogonal conjugate of Q. Prove that AP and d intersect at a point lying on (O) if and only if $PM \parallel AQ$.*

The proof directly follows from Corollary 2.

References

[1] Incenter, circumcircle and equal angles, All-Russian MO Round 4, 2005. *http://www.artofproblemsolving.com/Forum/viewtopic.php?t=32163.*

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ON CIRCLES TOUCHING THE INCIRCLE

ILYA I. BOGDANOV, FEDOR A. IVLEV, AND PAVEL A. KOZHEVNIKOV

Abstract. For a given triangle, we deal with the circles tangent to the incircle and passing through two its vertices. We present some known and recent properties of the points of tangency and some related objects. Further we outline some generalizations for polygons and polytopes.

1. Case of triangle

Throughout this section, we use the following notation[1].

Let ABC be a triangle. The circles γ and Γ are its incircle and circumcircle with centers I and O and radii r and R, respectively. The sides BC, CA, and AB touch γ at points A_1, B_1, and C_1, respectively.

Now construct the circle ω_A passing through B and C and tangent to γ at some point X_A. Define the circles ω_B, ω_C and the points of tangency X_B, X_C in a similar way. This paper is devoted to the investigation of the objects related to the constructed circles.

Let M_A be the second meeting point of ω_A and X_AA_1; define the points M_B and M_C analogously.

We start with the following description of the points M_A, M_B, and M_C.

Theorem 1. *Line OM_A is the perpendicular bisector of BC. Further, point M_A is the radical center of γ, B, and C (here we regard B and C as degenerate circles).*

Proof. Consider the homothety with center X_A taking γ to ω_A. This homothety takes BC to the line t_A touching ω_A at M_A, hence $t_A \parallel BC$. Thus M_A is the midpoint of arc BC, and OM_A is the perpendicular bisector of BC.

Consequently, there exists an inversion ι with center M_A which takes BC to ω_A. We have $\iota(A_1) = X_A$, $\iota(B) = B$, $\iota(C) = C$, hence $M_AA_1 \cdot M_AX_A = M_AB^2 = M_AC^2$. This means that M_A has equal powers with respect to the circles γ, B, and C. □

[1]All the results from this section allow various generalizations which are shown in the next section. So, throughout this section we put into footnotes the statements and approached that work only in this particular case, and fail up to further generalizations.

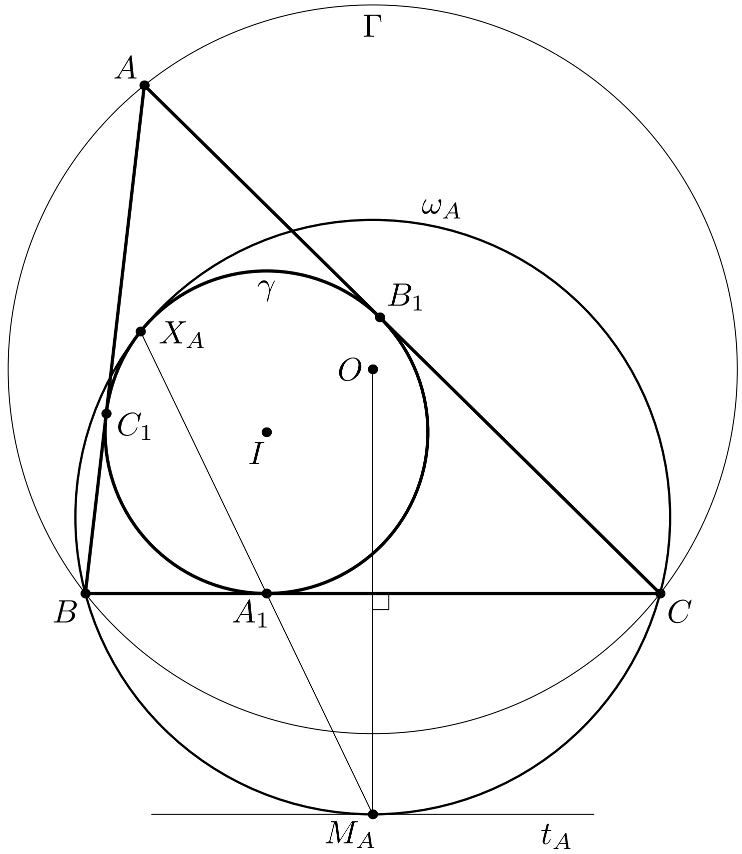

Fig. 1.

Remark 1. One could finish the proof without using inversion in the following way. In triangles M_ABA_1 and M_AX_AB we have $\angle A_1BM_A = \frac{1}{2}\overparen{M_AC} = \frac{1}{2}\overparen{M_AB} = \angle BX_AM_A$. Hence $\triangle M_ABA_1 \sim \triangle M_AX_AB$, and $M_AA_1 \cdot M_AX_A = M_AB^2$.

Let ℓ_A be the radical axis of circles γ and A; hence ℓ_A passes through the midpoints of AB_1 and AC_1. Define the lines ℓ_B and ℓ_C in a similar way. From Theorem 1 and analogous statements for M_B and M_C, we obtain the following

Corollary 2. $\ell_A = M_BM_C$, $\ell_B = M_CM_A$, $\ell_C = M_AM_B$.

Theorem 3. *The circumcenter of triangle $M_AM_BM_C$ is O. Next, triangles $M_AM_BM_C$ and $A_1B_1C_1$ are homothetical with some center K.*

Proof. By Corollary 2, we have $M_BM_C \perp AI$. Next, by Theorem 1 we get $M_BO \perp AC$ and $M_CO \perp AB$. Consequently, $\angle OM_BM_C = \angle CAI = \angle IAB = \angle M_BM_CO$; therefore, $M_BO = M_CO$. Analogously, we get $M_AO = M_BO$, and the points M_A, M_B, and M_C lie on some circle Ω with center O.

Consider now the homothety h with positive coefficient which takes γ to Ω; thus $h(I) = O$. Notice that the vectors $\overrightarrow{IA_1}$ and $\overrightarrow{OM_A}$ are codirectional, so $h(A_1) = M_A$; analogously, $h(B_1) = M_B$ and $h(C_1) = M_C$, as desired[2]. □

[2]Let us present an alternative proof. For the second claim of Theorem, it suffices to show that the respective sides of triangles $M_AM_BM_C$ and $A_1B_1C_1$ are parallel. From Corollary 2 we get $\ell_B = M_AM_C$. Since $\ell_B \perp BI$, we have $\ell_B \parallel A_1C_1$. Similarly, $\ell_A \parallel B_1C_1$ and $\ell_C \parallel A_1B_1$.

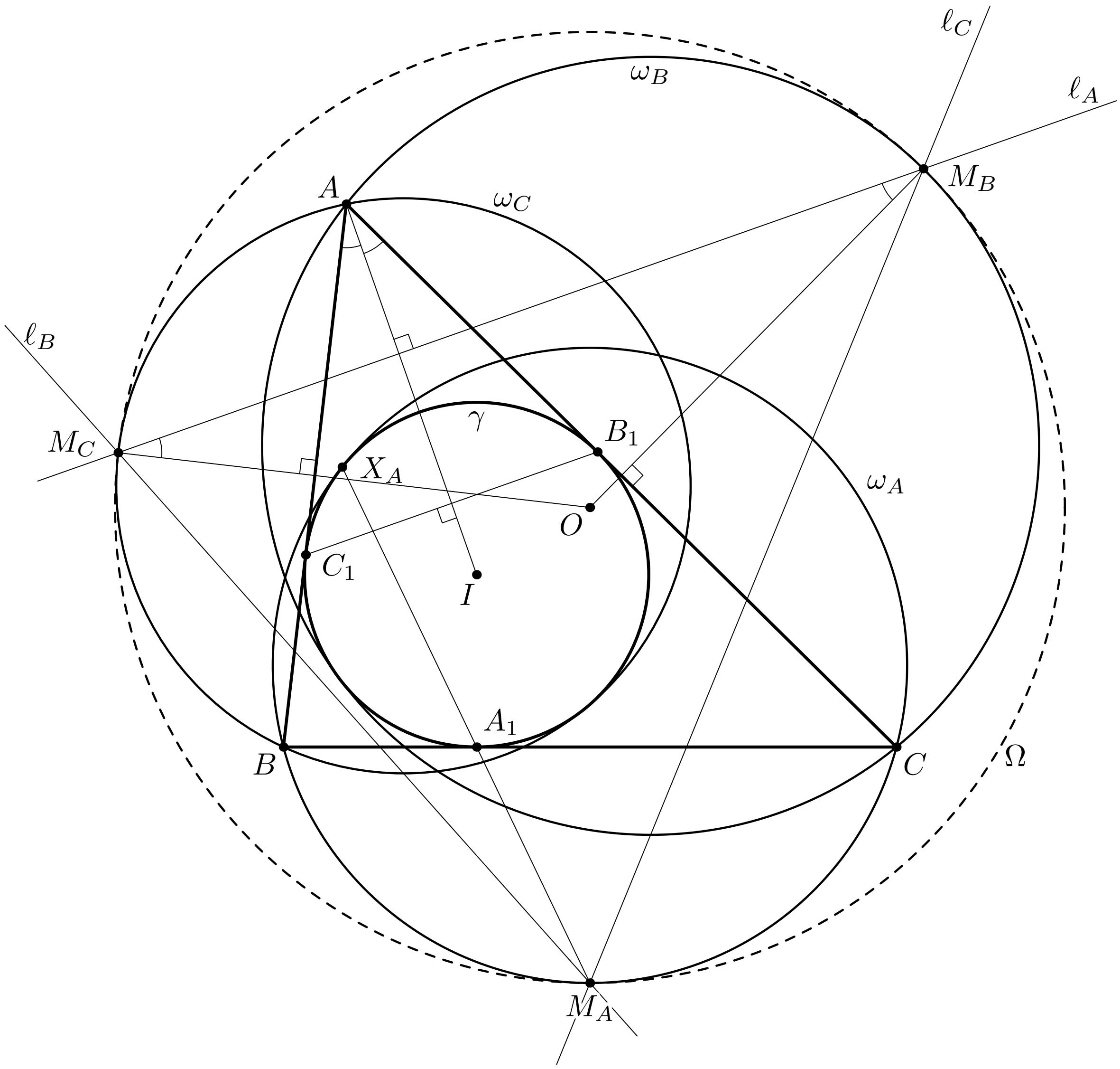

Fig. 2.

Till the end of this section, we fix the notation of Ω as the circumcircle of $M_A M_B M_C$, h as the homothety taking $A_1 B_1 C_1$ to $M_A M_B M_C$, and K as its center. Denote by $R(\Omega)$ the radius of Ω.

From the homothety h, we immediately obtain the following.

Corollary 4. *Lines* $A_1 X_A = X_A M_A$, $B_1 X_B = X_B M_B$, *and* $C_1 X_C = X_C M_C$ *are concurrent at* K.

Corollary 5. *Points* K, I, *and* O *are collinear, and* $\frac{KO}{KI} = \frac{R(\Omega)}{r}$.

Now we describe point K in some other terms.

Theorem 6. K *is the radical center of* ω_A, ω_B, *and* ω_C.

For the first part, consider the homothety h taking $\triangle A_1 B_1 C_1$ to $\triangle M_A M_B M_C$. Let Ω be the circumcircle of $M_A M_B M_C$; then $h(\gamma) = \Omega$. Hence $h(I)$ is the center of Ω. Next, h takes IA_1 to the line passing through M_A and perpendicular to BC, which is the perpendicular bisector of BC. Similarly, $h(IB_1)$ is the perpendicular bisector of AC. Since the perpendicular bisectors of BC and AC pass through O, we get $h(I) = O$.

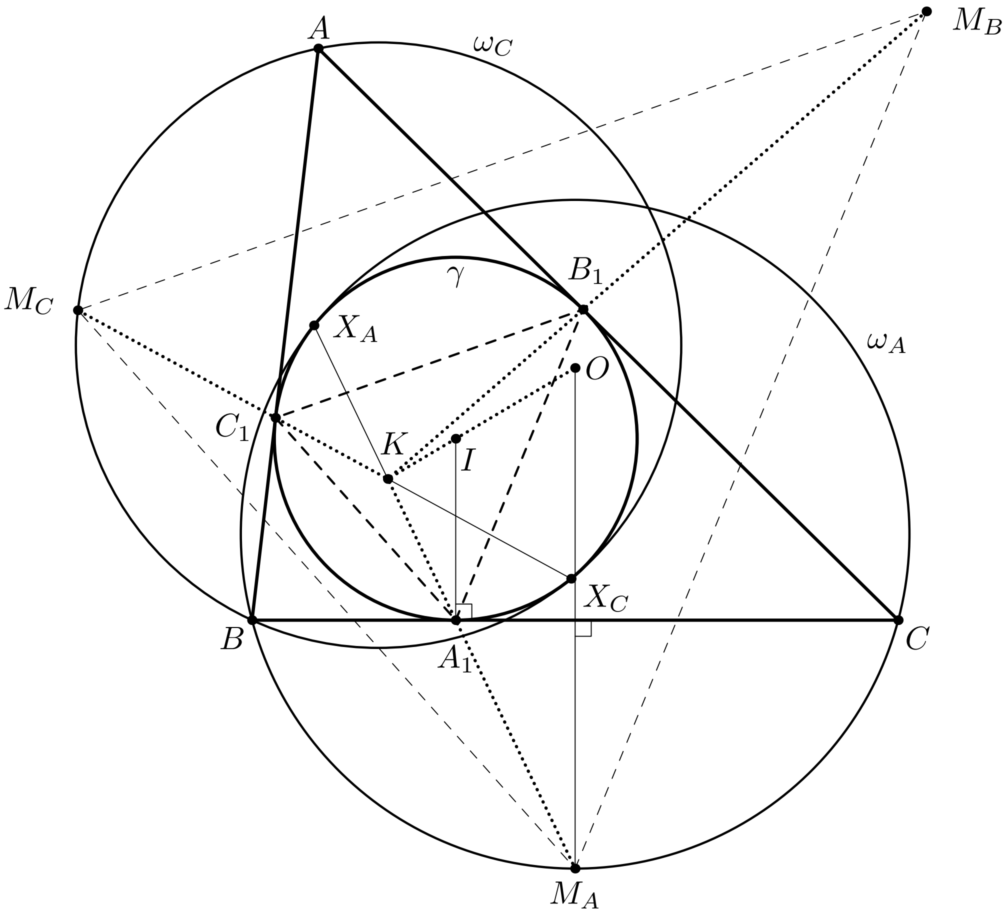

Fig. 3.

Proof. From the homothety h we get $\frac{KA_1}{KC_1} = \frac{KM_A}{KM_C}$. Hence the relation $KX_A \cdot KA_1 = KX_C \cdot KC_1$ implies $KX_A \cdot KM_A = KX_C \cdot KM_C$. Thus we obtain the equality of powers of K with respect to ω_A and ω_C. The same is true for ω_A and ω_B. □

Remark 2. In fact, this theorem is an instance of a more general fact. Consider two fixed circles γ and Ω, and let ω be a variable circle tangent to γ at X and to Ω at M (with a fixed type of tangencies — internal or external). Then, by Monge's theorem, all the lines KM pass through a fixed point K which is a center of homothety taking γ to Ω. Moreover, K is the radical center of all such circles ω: if Y is the second common point of XM and γ, then $KX \cdot KM$ is proportional to $KX \cdot KY$ which is fixed.

Remark 3. Theorem 6 combined with the first statement of Corollary 5 forms the statement of a problem on the Romanian Masters of Mathematics 2012 olympiad proposed by F. Ivlev [6, Problem 3].

Next, we introduce some more objects related to our construction. Let m_A be the line passing through A and perpendicular to IA. Define lines m_B and m_C

similarly. Finally, let us define the points[3] $I_A = m_B \cap m_C$, $I_B = m_C \cap m_A$, and $I_C = m_A \cap m_B$.

Theorem 7. *M_A is the midpoint of the segment A_1I_A.*

Proof. By its definition, line ℓ_B is the midline between parallel lines A_1C_1 and m_B; hence m_B intersects A_1M_A at point I'_A such that $M_AI'_A = M_AA_1$. Similarly, m_C also intersects A_1M_A at the same point, hence $I'_A = I_A$, and M_A is the midpoint of A_1I_A. □

Let S_A be the circumcenter of triangle BIC. Define S_B and S_C similarly[4]. Denote by Γ_S the circumcircle of triangle $S_AS_BS_C$; let R_S be the radius of this circle[5].

Theorem 8. *The circle Γ_S has O as its center, and the equality $R_S = R(\Omega) - r/2$ holds[6]. Moreover, triangles $S_AS_BS_C$ and $M_AM_BM_C$ are homothetical with center O.*

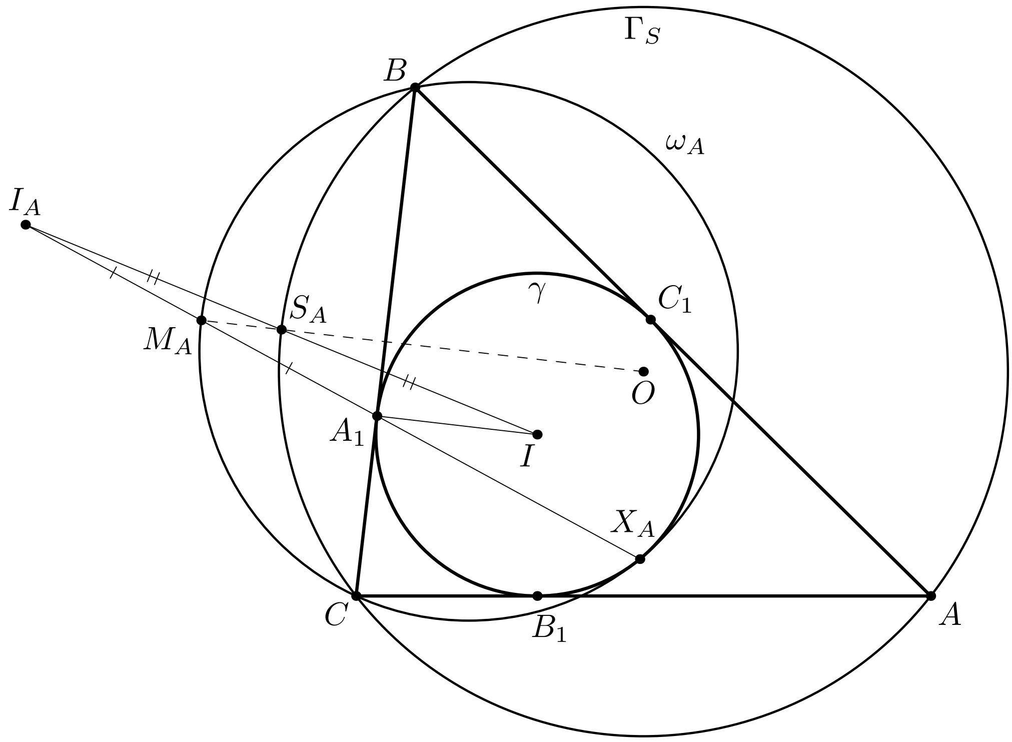

Fig. 4.

Proof. Notice that all the points M_A, S_A, and O lie on the perpendicular bisector to BC, hence they are collinear. Next, we have $\overrightarrow{OM_A} = \overrightarrow{OS_A} + \overrightarrow{S_AM_A}$. By the definition, S_A belongs to perpendicular bisectors of BI and CI. Hence S_A is the image of I_A under the homothety with center I and ratio $\frac{1}{2}$; thus S_A is the

[3]Notice that m_A, m_B, and m_C are external bisectors of $\angle A$, $\angle B$, and $\angle C$, respectively, while I_A, I_B, and I_C are the excenters of $\triangle ABC$.

[4]S_A, S_B, and S_C are the midpoints of arcs BC, CA, and AB of circle Γ, respectively.

[5]In our case of triangle, $\Gamma_S = \Gamma$.

[6]Thus $R(\Omega) = R + r/2$.

midpoint of II_A. Using Theorem 7, we conclude that S_AM_A is the midline in triangle I_AIA_1, hence $\overrightarrow{S_AM_A} = \overrightarrow{IA_1}/2$. Since the vectors $\overrightarrow{OS_A}$ and $\overrightarrow{IA_1}$ are codirectional, we get $OS_A = |\overrightarrow{OS_A}| = |\overrightarrow{OM_A}| - |\overrightarrow{IA_1}|/2 = R(\Omega) - r/2$. Similarly, we obtain that the segments OS_B and OS_C have the same length, so O is the center of Γ_S. Finally, the homothety with center O which takes Ω to Γ_S takes $M_AM_BM_C$ to $S_AS_BS_C$. □

Remark 4. Using the relation between ratios of similarity of triangles $A_1B_1C_1$, $I_AI_BI_C$, $M_AM_BM_C$, and $S_AS_BS_C$, one may obtain the relation between radii in alternative way: we have $R(\Omega) = R(M_AM_BM_C) = \big(R(I_AI_BI_C) + R(A_1B_1C_1)\big)/2$ and $R(I_AI_BI_C) = 2R(S_AS_BS_C) = 2R_S$; hence $R(\Omega) = R_S + r/2$.

Remark 5. The radius R_S could be calculated in terms of R, r, and OI: $R_S = \frac{R^2 - OI^2}{2r}$ (see, for example, the Problem from Sharygin olympiad, 2005 [7, Problem 20]). Combining with Corollary 5 we get $\frac{KO}{KI} = \frac{R^2 + r^2 - OI^2}{2r^2}$ thus obtaining the ratio of our homothety[7].

At the end of this section, let us make some remarks on the results obtained above.

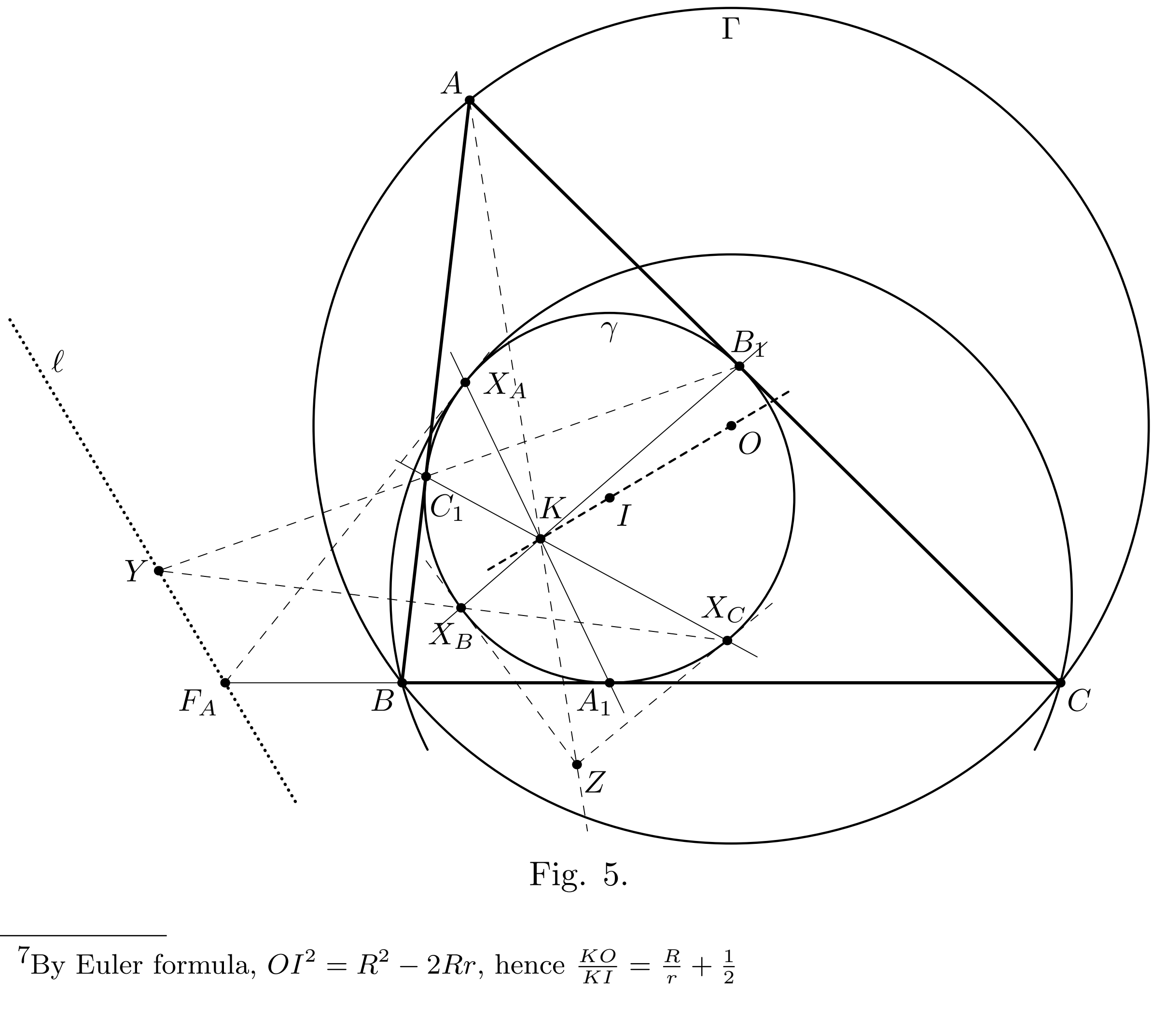

Fig. 5.

[7]By Euler formula, $OI^2 = R^2 - 2Rr$, hence $\frac{KO}{KI} = \frac{R}{r} + \frac{1}{2}$

Remark 6. Here, we present an alternative approach to considered construction (without using points M_A, M_B, and M_C) related to a result by S. Ilyasov and A. Akopyan [5, Problem M2244]. We present here an equivalent reformulation of this statement; for the completeness, we also provide its proof.

Proposition 1. *Let Γ and γ be two fixed circles, and let A, B be variable points on Γ. Suppose two circles ω_1 and ω_2 (each may degenerate to a line) passing through A and B are tangent to γ at X and Y, respectively. Then the line XY passes through a fixed point.*

Proof. Let ℓ be the radical axis of γ and Γ, and let K be its pole with respect to γ. We claim that XY passes through K.

Denote by F the point of intersection of AB with the common tangent to γ and ω_1 at X. Then F is the radical center of γ, Γ, and ω_1, hence it belongs to ℓ. Since the powers of F with respect to ω_1 and ω_2 are equal (in fact, they are equal to $FA \cdot FB$), this point lies also on the common tangent to γ and ω_2 at Y. Hence XY is the polar line of F with respect to γ, so it passes through K. □

In our case, one may apply this fact to the pairs of points (A, B), (B, C), and (C, A) obtaining that the lines X_AA_1, X_BB_1, and X_CC_1 are concurrent at K. This proves Corollary 4 with one more description of point K. Next, since $\ell \perp OI$ and $IK \perp \ell$, we have $K \in OI$, thus proving Corollary 5. Note that from this new description of K one could easily derive that $\frac{KO}{KI} = \frac{R^2+r^2-OI^2}{2r^2}$.

Further, let the tangents to γ at X_B and X_C intersect at point Z; this point is the radical center of ω_B, ω_C, and γ. Hence AZ is the radical axis of ω_B and ω_C. To finish an alternative proof of Theorem 6, it suffices to show that $K \in AZ$. For that, notice that the point Y of intersection of lines B_1C_1 and X_BX_C lies on the polar line ℓ of K with respect to γ. Then the polar line of Y contains the poles of lines B_1C_1, ℓ, and X_BX_C, which are A, K, and Z, respectively.

One may also notice that this approach allows to generalize some of the facts to the case of curved triangle ABC (when its sides are the circular arcs). Namely, in this case the lines A_1X_A, B_1X_B, and C_1X_C are also concurrent at some point collinear with O and I.

Remark 7. Let us mention two facts related to the considered construction.

a) The line containing points A_1, I_A, M_A, and X_A passes also through the midpoint of the altitude from A (see a problem on the Moscow mathematical olympiad 2001 [4, Problem 10.3], and also a problem proposed by Bulgaria for the IMO in 2002 [3, Problem 2002-G7, p. 319]). To prove this one can use the homothety with center A taking the incircle to the excircle.

b) Lines AX_A, BX_B, and CX_C are concurrent (see a problem by A. Badzyan in [1, Problem M2268]; this fact was independently noticed by D. Shvetsov).

Remark 8. All the results from this section could be reformulated if the incircle is replaced by one of the excircles.

2. A general case: polygons and polytopes

All Theorems and Corollaries as well as remarks from the previous section admit generalizations to the case when the base figure is a polygon which is simultaneously circumscribed and inscribed, and also to the case when the base figure is a polytope in space (or in n-dimensional space) which is simultaneously circumscribed and inscribed.

Let us start from the general set up. Further we use the terminology for the case of 3-dimensional space though everything is appropriate for n-dimensional case for all $n \geq 2$. For $n = 2$ one could replace *faces* by *sides*, *planes* by *lines*, and *spheres* by *circles.* For $n > 3$ one could replace *faces* by $(n-1)$-dimensional *hyperfaces*, *planes* by *hyperplanes*, *spheres* by $(n-1)$-*dimensional spheres*, and *circles* by $(n-2)$-*dimensional spheres.*

Let $\mathcal{P}$ be a convex polytope with vertices $P_1, \ldots, P_n$; let $F_1, \ldots, F_k$ be the planes determined by the faces of $\mathcal{P}$. Suppose that $\mathcal{P}$ has both an inscribed sphere γ and a circumscribed sphere Γ. Let I, O and r, R be the centers and the radii of these spheres, respectively.

For every $i \in \{1, 2, \ldots, k\}$, define $\Gamma_i = F_i \cap \Gamma$ (thus, Γ_i is the circumcircle of the face lying in F_i). Let γ touch F_i at point Q_i.

Now let us construct the sphere ω_i passing through Γ_i and tangent to γ at point X_i. Let M_i be the second meeting point of ω_i and X_iQ_i. Let $\mathcal{Q}$ and $\mathcal{M}$ be the convex polytopes with vertices $Q_1, \ldots, Q_k$ and $M_1, \ldots, M_k$, respectively. For every $j \in \{1, \ldots, n\}$, let ℓ_j be the radical plane of spheres γ and P_j (hence ℓ_j passes through the midpoints of P_jQ_i for all i such that $P_j \in F_i$).

Let m_j be the plane passing through P_j perpendicular to IP_j. Let S_i be the circumcenter of sphere passing through Γ_i and I.

Now we proceed with the analogues of results from the previous section.

In the proof of Theorem 11 below we clarify how to generalize the proof of Theorem 3 to the case of a polytope. All the other proofs of Theorems below are completely analogous to those in the case of triangle; thus we omit these proofs.

Theorem 9 (cf. Theorem 1)**.** *Line OM_i is the perpendicular to F_i. Further, point M_i has equal powers with respect to γ and any point $X \in \Gamma_i$ (here X is regarded as a degenerate sphere).*

Corollary 10 (cf. Corollary 2)**.** *If $P_j \in F_i$, then ℓ_j passes through M_i.*

Theorem 11 (cf. Theorem 3)**.** *Points $M_1, \ldots, M_k$ lie on some sphere Ω with center O. Next, polytopes $\mathcal{M}$ and $\mathcal{Q}$ are homothetical with some center K.*

Proof. To prove the first statement, it suffices to prove the equality $OM_i = OM_s$ for every pair of indices i, s such that F_i and F_s correspond to adjacent faces[8].

Let $F_{i,s}$ be the plane containing I and $F_i \cap F_s$; then $F_{i,s}$ is the bisector plane of F_i and F_s. By Theorem 1, each of M_i and M_s has equal powers with respect to γ and all the points in $\Gamma_i \cap \Gamma_s$, hence M_iM_s is perpendicular to the plane spanned

[8]In n-dimensional case, these two hyperfaces should have a common $(n-2)$-dimensional face.

by $\Gamma_i \cap \Gamma_s$ and I which is $F_{i,s}$. Finally, by the same Theorem we have $OM_i \perp F_i$ and $OM_s \perp F_s$, hence $\angle OM_iM_s = \angle OM_sM_i$, as required.

The proof of the second statement of Theorem is completely analogous to that in the case of triangle. □

Now, as in the case of triangle, we denote by h the obtained homothety taking $\mathcal{Q}$ to $\mathcal{M}$, by K its center, and by Ω the circumsphere of $\mathcal{M}$ (then $\Omega = h(\gamma)$). Denote also by $R(\Omega)$ the radius of Ω.

Remark 9. Notice that the first statement of Theorem 11 in the case of tetrahedron appeared as a problem proposed by F. Bakharev in All-Russian mathematical olympiad in 2003 [2, Problem 744].

Corollary 12 (cf. Corollary 4)**.** *Lines $Q_iX_i = X_iM_i$ ($i = 1, \dots, k$) are concurrent at point K.*

Corollary 13 (cf. Corollary 5)**.** *Points K, I, and O are collinear, and $\frac{KO}{KI} = \frac{R(\Omega)}{r}$.*

Theorem 14 (cf. Theorem 6)**.** *K has equal powers with respect to spheres ω_i ($i = 1, \dots, k$).*

Theorem 15 (cf. Theorem 7)**.** *Let $i \in \{1, \dots, k\}$. Consider a point I_i such that M_i is the midpoint of Q_iI_i. Then for all $j \in \{1, \dots, n\}$ with $P_j \in F_i$, plane m_j passes through I_i.*

Remark 10. From Theorem 15 we see that the convex polytope $\mathcal{I}$ with vertices $I_1, \dots, I_k$ is also homothetical to the polytopes $\mathcal{Q}$ and $\mathcal{M}$. A particular case of this fact (for an inscribed and circumscribed quadrilateral in the plane) appeared as a problem by S. Berlov, L. Emelyanov, and A. Smirnov in All-Russian mathematical olympiad in 2004 [2, Problem 755].

Theorem 16 (cf. Theorem 8)**.** *All the points S_i lie on the sphere Γ_S with center O and radius $R_S = R(\Omega) - r/2$.*

Remark 11. The alternative approach mentioned in the previous section also works in the general case. It uses the following generalized statement.

Proposition 2. *Let Γ and γ be two fixed spheres, and let $\Gamma' \subset \Gamma$ be a circle. Suppose two spheres ω_1 and ω_2 (each may degenerate to a plane) passing through Γ' are tangent to γ at X and Y, respectively. Then the line XY passes through a fixed point K that is the pole of radical plane of Γ and γ with respect to γ.*

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ONE PROPERTY OF THE JERABEK HYPERBOLA AND ITS COROLLARIES

ALEXEY A. ZASLAVSKY

Abstract. We study the locus of the points P having the following property: if $A_1B_1C_1$ is the circumcevian triangle of P with respect to the given triangle ABC, and A_2, B_2, C_2 are the reflections of A_1, B_1, C_1 in BC, CA, AB, respectively, then the triangles ABC and $A_2B_2C_2$ are perspective. We show that this locus consists of the infinite line and the Jerabek hyperbola of ABC. This fact yields some interesting corollaries.

We start with the following well-known fact [1, p. 4.4.5].

Statement 1. *Let the tangents to the circumcircle of ABC at A and B meet in C_0. The line CC_0 meets the circumcircle of ABC for the second time in C_1, and C_2 is the reflection of C_1 in AB. Then CC_2 is a median in ABC.*

Proof. Let C' be the common point of CC_1 and AB, and A'', B'' be the common points of AC_2, BC_2 with BC, AC, respectively. Since $\angle C'CB = \angle C_1AB = \angle BAA''$, the triangles BCC' and BAA'' are similar; therefore, $BA'' = \frac{AB}{BC} \cdot BC'$. But CC' is a symedian in ABC, so $BC' = \frac{BC^2}{BC^2+AC^2} \cdot AB$. Therefore, $\frac{BA''}{BC} = \frac{AB^2}{AC^2+BC^2}$. Analogously, the ratio $\frac{AB''}{AC}$ has the same value, yielding the claim. □

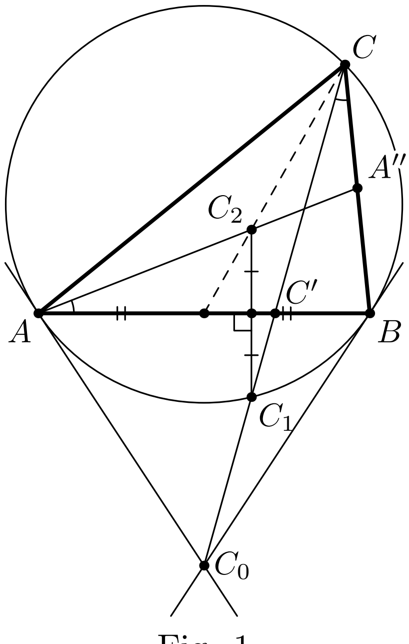

Fig. 1.

This fact yields the following corollary: let $A_1B_1C_1$ be the circumcevian triangle of the Lemoine point, L, and let A_2, B_2, C_2 be the reflections of A_1, B_1,

C_1 in BC, CA, AB, respectively. Then the triangles ABC and $A_2B_2C_2$ are perspective.

Our goal is to find the locus of the points sharing this property. To this end, first we formulate the following assertion.

Lemma 1. *Let CC_1 divide AB in ratio $x : y$. Then CC_2 divides AB in ratio*

$$x(b^2(x+y) - c^2x) : y(a^2(x+y) - c^2y).$$

In order to prove this, it suffices to repeat the argument by which we demonstrated the previous assertion and then to apply Ceva's theorem.

Now let P be the point with barycentric coordinates $(x : y : z)$. Using Lemma 1 and Ceva's theorem, we see that a point P has the property in question iff it lies on some cubic c. From the following assertion, we infer that c is degenerated.

Statement 2. *Let three parallel lines passing through the vertices of ABC meet its circumcircle in A_1, B_1, C_1. The points A_2, B_2, C_2 are the reflections of A_1, B_1, C_1 in BC, CA, AB, respectively. Then the lines AA_2, BB_2, CC_2 are concurrent.*

Proof. Consider the three lines which pass through A_1, B_1, C_1 and are parallel to BC, CA, AB, respectively. It is easy to see that they meet at a point on the circumcircle of ABC. The points A_2, B_2, C_2 are the reflections of this point in the midpoints of the sides of ABC. Therefore, the triangles ABC and $A_2B_2C_2$ are centrosymmetric. □

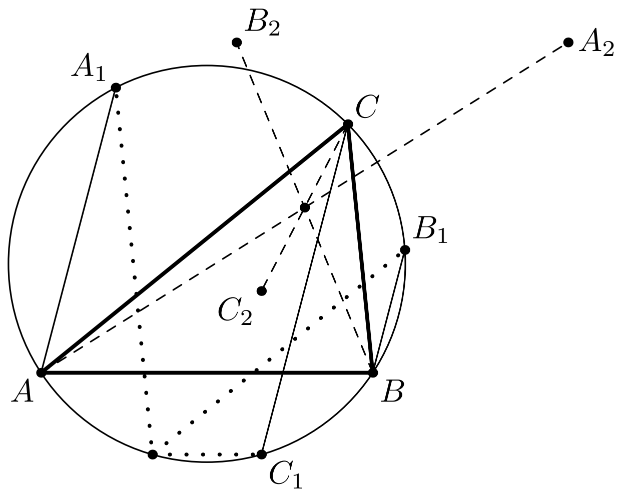

Fig. 2.

Note also that the center of perspective in this claim lies on the Euler circle.

And so, c consists of the infinite line and some conic k. In order to determine k completely, it suffices to indicate five point lying on it. We already know that k contains the Lemoine point, L. Furthermore, k contains the vertices of ABC as well as its orthocenter H (in this case, all of A_2, B_2, C_2 coincide with H). Therefore, k is the Jerabek hyperbola.

Here follow some corollaries of this fact.

Statement 3. *Let P be a point on the Euler line of ABC, $A_1B_1C_1$ be the circumcevian triangle of P, and A_2, B_2, C_2 be the reflections of A_1, B_1, C_1 in the midpoints of BC, CA, AB, respectively. Then the lines AA_2, BB_2, CC_2 are concurrent.*

Proof. The isogonal conjugate, Q, of P lies on the Jerabek hyperbola. Let CQ meet the circumcircle of ABC for the second time in C_3. Then C_2 and C_3 are symmetric with respect to AB. □

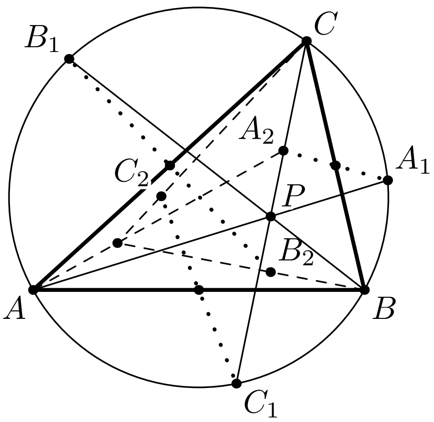

Fig. 3.

Statement 4. *Let P be a point on the Euler line of ABC, A_0, B_0, C_0 be the midpoints of BC, CA, AB, respectively, and A_1, B_1, C_1 be the projections of the circumcenter O of ABC onto AP, BP, CP, respectively. Then the lines A_0A_1, B_0B_1, C_0C_1 are concurrent.*

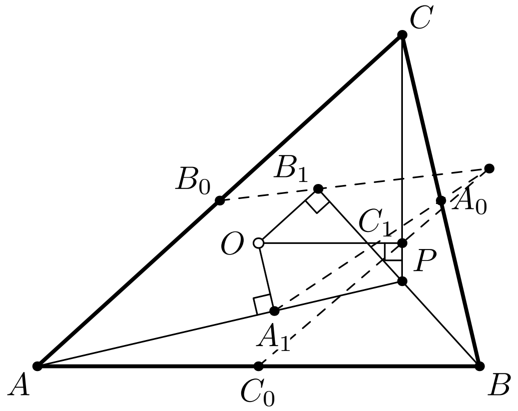

Fig. 4.

Proof. It suffices to apply homothety of center the centroid of ABC and coefficient $-\frac{1}{2}$ to the configuration of the previous claim. □

Statement 5. *Let O, I be the circumcenter and incenter of ABC. An arbitrary line perpendicular to OI meets BC, CA, AB in A_1, B_1, C_1, respectively. Then the circumcenters of the triangles IAA_1, IBB_1, ICC_1 are collinear.*

Proof. Applying an inversion of center I and the previous assertion we see that the circumcircles of the three triangles in question have a common point other than I. □

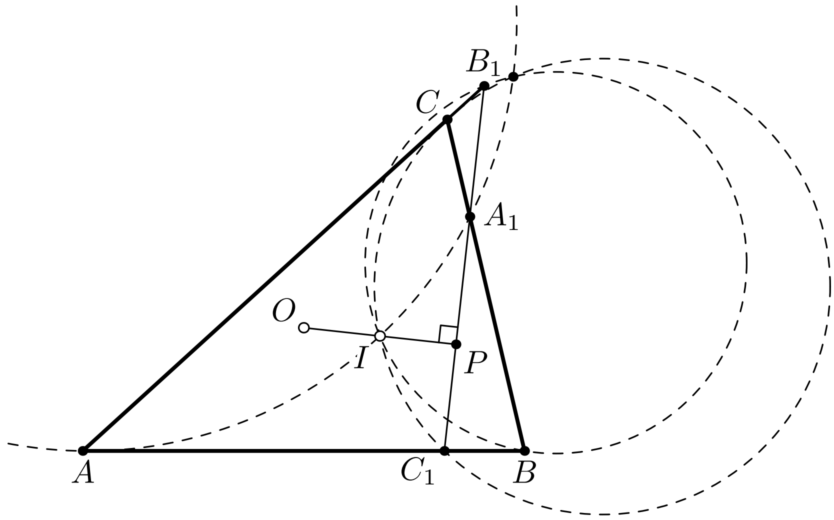

Fig. 5.

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A GENERALIZATION OF THE DANDELIN THEOREM

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Abstract. We prove three apparently new theorems related to the doubly tangent circles of conics including a generalization of the Dandelin theorem on spheres inscribed in a cone. Also we discuss the focal properties of doubly tangent circles of conics.

1. Introduction

The main result of this paper is the following generalization of the celebrated Dandelin theorem on spheres inscribed in a cone. We say that a sphere is *inscribed* in a quadric surface of revolution if the sphere touches the quadric along a circle. Evidently, the center of such sphere lies on the axis of revolution.

Theorem 1.1. *Let an inclined plane intersect a quadric surface of revolution in a conic. Let a sphere be inscribed in the quadric and tangent to the plane. Then the tangency point of the sphere and the plane is a focus of the conic. The plane containing the contact points of the sphere and the quadric intersects the inclined plane in a directrix of the conic; see Figure 1.*

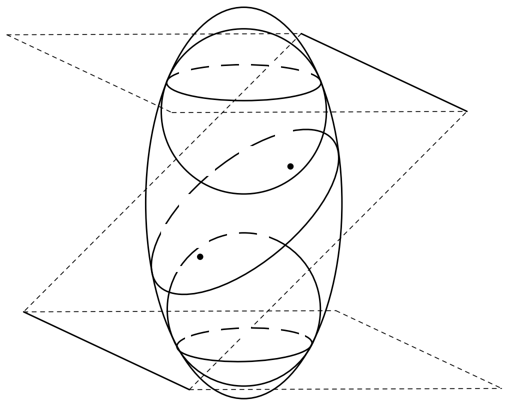

Fig. 1. A generalization of the Dandelin theorem.

Note that the particular case in which the quadric is a one-sheeted hyperboloid of revolution was already considered by Dandelin; see [3, §11]. We use the generalized "focus-directrix" property of conics stated in Section 2 to prove Theorem 1.1.

Using similar approach we prove the following theorem on ellipses doubly tangent to two given nested circles.

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Theorem 1.2. *Let a family of ellipses be such that each ellipse in the family is doubly tangent to two given nested circles. Then*
(a) Similarity property. *All the ellipses in the family are similar;*
(b) Tangency property. *For each given circle the tangency points of the circle and each ellipse in the family are collinear with a limiting point of the pencil of circles generated by the two given circles; see Figure 2.*

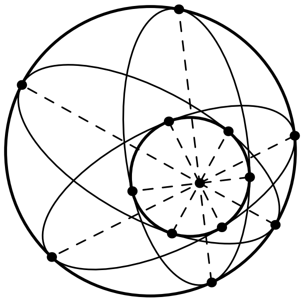

Fig. 2. Tangency property.

(c) Foci property. *The foci of all the ellipses in the family lie on a fixed circle concentric with the larger given circle; see Figure 3.*

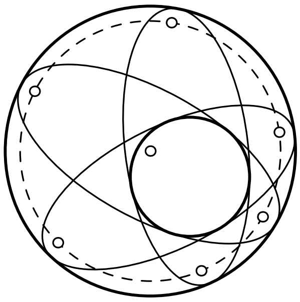

Fig. 3. Foci property.

(d) Orthogonality property. *The four common points of any two ellipses in the family lie on two perpendicular lines intersecting in a limiting point of the pencil of circles generated by the two given circles; see Figure 4.*

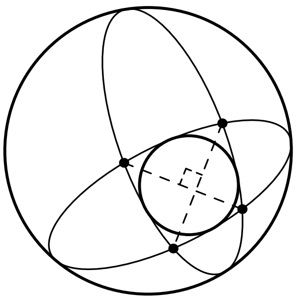

Fig. 4. Orthogonality property.

The following theorem was firstly proved by E.H. Neville in 1936; see [5] and [4].

The Neville Theorem. *If three ellipses are such that each pair selected from them has one common focus and two intersection points, then their three common chords are concurent; see Figure 5.*

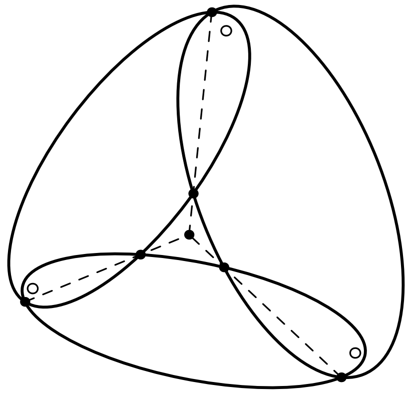

Fig. 5. The Neville theorem.

We prove the following generalization of the Neville theorem.

Theorem 1.3. *Let three ellipses be such that each pair selected from them has a common doubly tangent circle with the center lying on the major axes of the ellipses. Suppose that each pair of the ellipses has four common points. Then the intersection points of the three ellipses lie on* 6 *lines forming a complete quadrangle; see Figure 6.*

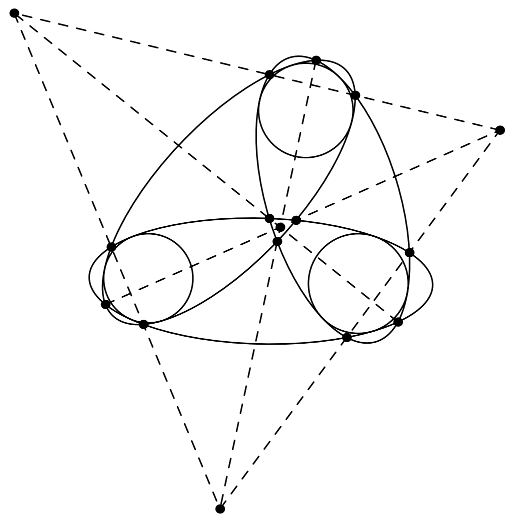

Fig. 6. A generalization of the Neville theorem.

The paper is organized as follows. In Section 2 we discuss generalized focal properties of conics. In Section 3 we prove Theorem 1.1, Theorem 1.2, and Theorem 1.3.

2. Generalized focal properties of conics

In this section we discuss generalized focal properties involving *doubly tangent circles* of conics. We start with the following definition.

Definition. Consider a noncircular conic. A circle having two tangency points with the conic is called a *doubly tangent circle* of this conic.

The following proposition is obvious.

Proposition. *See Figure 7. Each conic distinct from a parabola or a circle has two families of doubly tangent circles. The centers of the circles in the first family lie on the major axis. The centers of the circles in the second family lie on the minor axis. Any parabola has exactly one family of doubly tangent circles.* □

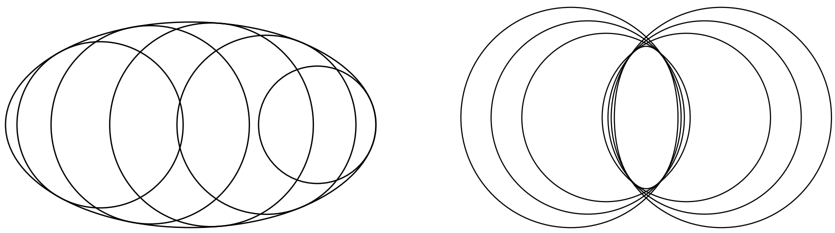

Fig. 7. Two families of doubly tangent circles of an ellipse.

Let P be a point and ω be a circle. Denote by r the radius of ω. Denote by d the distance between P and the center of ω. In what follows we use the following notations. We define the *tangent distance* $t(P, \omega)$ from P to ω to be $\sqrt{|d^2 - r^2|}$. If P lies in the exterior of ω, then $t(P, \omega)$ is the length of a tangent segment from P to ω. Let λ be a line. Denote by $d(P, \lambda)$ the distance from P to λ.

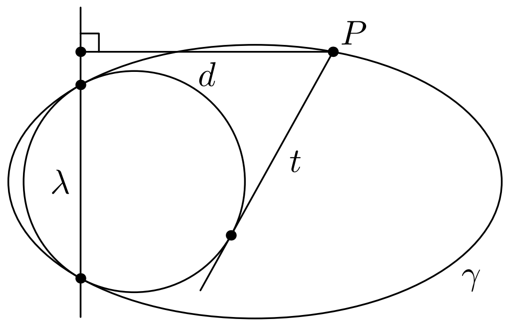

Fig. 8. *Generalized "focus-directrix" property of conics.* For an arbitrary point $P \in \gamma$ we have $t/d = \text{const}$.

Theorem 2.1. Generalized "focus-directrix" property. (Cf. [3, Theorem 1]) See Figure 8. *Let γ be a noncircular conic. Let ω be an arbitrary doubly tangent circle of γ. Let λ be the line passing through the tangency points of ω and γ. If the center of ω lies on the major axis of γ, then for an arbitrary point $P \in \gamma$ we have $\frac{t(P,\omega)}{d(P,\lambda)} = \varepsilon$, where ε is the eccentricity of γ. If the center of ω lies on the minor axis of γ, then for an arbitrary point $P \in \gamma$ we have $\frac{t(P,\omega)}{d(P,\lambda)} = \varepsilon'$, where $\varepsilon' = \frac{\varepsilon}{\sqrt{|1-\varepsilon^2|}}$.*

Proof. It suffices to consider the following 2 cases.

Case 1. *The center of ω lies on the major axis of the conic γ* or *γ is a hyperbola and the center of ω lies on the minor axis.* The proof is found in [3, Theorem 1].

Case 2. *The conic γ is an ellipse and the center of ω lies on the minor axis.* Denote by Π the plane containing the circle ω. Consider the sphere Σ such that ω is a great circle of Σ; see Figure 9. Consider a circle $\theta \subset \Sigma$ such that γ is the orthogonal projection of θ onto the plane Π. It is easy to see that the eccentricity ε of γ is equal to $\sin \varphi$, where φ is the angle between Π and the plane containing θ. Denote by O the center of ω. Consider the point N on θ such that $PN \perp \Pi$. Note that $|PN| = \sqrt{|ON|^2 - |OP|^2} = t(P, \omega)$. Therefore,

$$\frac{t(P,\omega)}{d(P,\lambda)} = \frac{|PN|}{d(P,\lambda)} = \tan\varphi = \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} = \varepsilon'.$$

Generalized "focus-directrix" property is proved. □

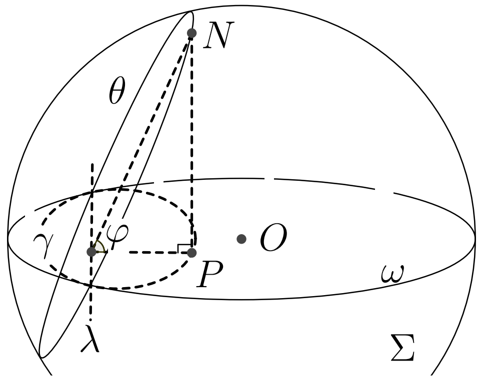

Fig. 9.

Theorem 2.2. Generalized "bifocal" property. (Cf. [3, Theorem 3]) See Figure 10. *Let γ be a noncircular conic. Let ω_1 and ω_2 be two arbitrary doubly tangent circles of γ such that the centers of the circles lie on the same axis of symmetry of γ. Let λ_i be the line passing through the tangency points of γ and ω_i. If a point $P \in \gamma$ moves between the lines λ_1 and λ_2, then $t(P, \omega_1) + t(P, \omega_2) = \text{const}$. Otherwise $|t(P, \omega_1) - t(P, \omega_2)| = \text{const}$.*

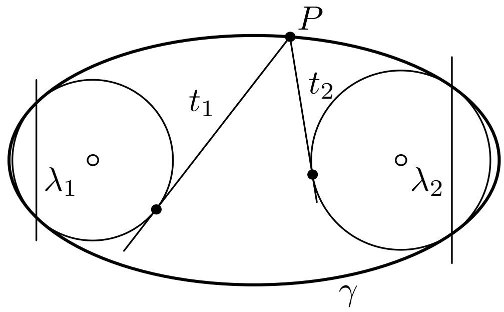

Fig. 10. *Generalized "bifocal" property of conics.* If P moves between the lines λ_1 and λ_2, then $t_1 + t_2 = \text{const}$. Otherwise $|t_1 - t_2| = \text{const}$.

Proof. Assume that the centers of the circles ω_1 and ω_2 lie on the major axis of the conic γ. Let ε be the eccentricity of γ. Denote by d the distance between the parallel lines λ_1 and λ_2. If the point P lies between the lines λ_1 and λ_2, then $t(P, \omega_1) + t(P, \omega_2) = \varepsilon \cdot (d(P, \lambda_1) + d(P, \lambda_2)) = \varepsilon \cdot d = \text{const}$, where the first equality follows from the generalized "focus-directrix" property. If the point P does not lie between the lines λ_1 and λ_2, we have $|t(P, \omega_1) - t(P, \omega_2)| = \varepsilon \cdot |d(P, \lambda_1) - d(P, \lambda_2)| = \varepsilon \cdot d = \text{const}$. If the centers of the circles ω_1 and ω_2 lie on the minor axis of the conic γ, the proof is analogous. □

Analogous focal properties of conics on the sphere and in the hyperbolic plane are discussed in [6].

3. Proofs

Proof of Theorem 1.1. Assume that the center of the given sphere lies on the major axis of axial sections of the given quadric. Denote by F the tangency point of the given sphere and the inclined plane; see Figure 11. Denote by λ the intersection line of the inclined plane and the plane containing the contact points of the sphere and the quadric. Denote by φ the angle between these planes. Let P be an arbitrary point on the given conic. Consider the plane containing the axis of the quadric and passing through P. Let this plane intersect the quadric, the given sphere, and the plane containing the contact points in the conic γ_P, in the circle ω_P, and in the line λ_P, respectively. Evidently, the circle ω_P is doubly tangent to γ_P, and λ_P is the line passing through the tangency points. We have $\frac{d(P,\lambda_P)}{d(P,\lambda)} = \sin\varphi$. Note that $|PF| = t(P,\omega_P)$, because $|PF|$ and $t(P,\omega_P)$ are the lengths of tangent segments from the point P to the given sphere. By the generalized "focus-directrix" property, the ratio $\frac{t(P,\omega_P)}{d(P,\lambda_P)}$ equals the eccentricity ε of γ_P and does not depend on the point P lying on the given conic. Therefore, we have

$$\frac{|PF|}{d(P,\lambda)} = \frac{t(P,\omega_P)}{d(P,\lambda_P)} \cdot \frac{d(P,\lambda_P)}{d(P,\lambda)} = \varepsilon \cdot \sin\varphi = \text{const}.$$

Thus the point F and the line λ are a focus and a directrix of the given conic. If the center of the given sphere lies on the minor axis of axial sections of the given quadric, then the proof is analogous. □

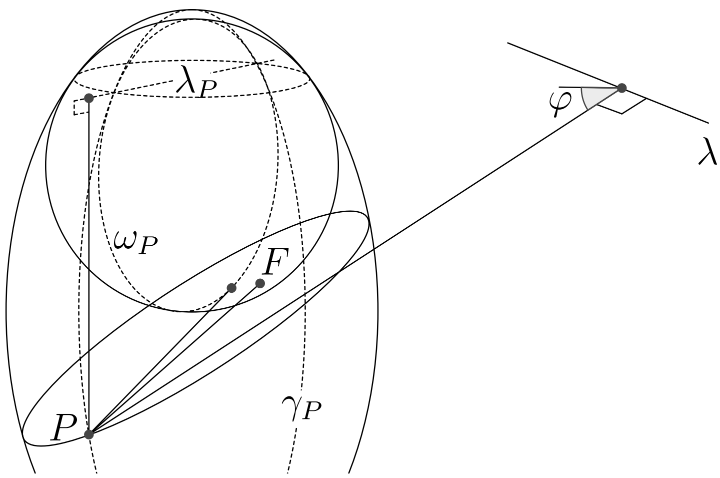

Fig. 11.

In the proof of Theorem 1.2 we use the following notations. Denote by α and β the given circles, with β lies inside α. Denote by L the limiting point of the pencil of circles generated by α and β.

Proof of Theorem 1.2(a). Consider an arbitrary ellipse from the given family; see Figure 12. Denote by P a tangency point of the ellipse and α. Denote by λ the line passing through the tangency points of the ellipse and β.

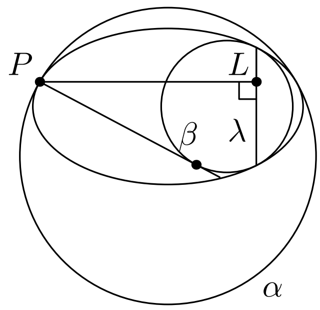

Fig. 12.

From Theorem 1.2(b) it follows that the lines PL and λ are perpendicular. By the generalized "focus-directrix" property, the ratio $\frac{t(P,\beta)}{|PL|}$ is equal to the eccentricity of the ellipse. By the well-known *geometric characterization of a hyperbolic pencil of circles* (see [2, Theorem 2.12]), this ratio does not depend on the point P lying on the circle α. Thus all the ellipses doubly tangent to α and β are similar. □

Proof of Theorem 1.2(b). Apply a projective transformation that preserves the circle α and takes L to the center O of α. Let this transformation take the circle β to an ellipse β'. Then O is a limiting point of the pencil of conics generated by α and β'. Evidently, O is the center of β'. So this transformation takes each ellipse γ from the given family to an ellipse γ' with the center O. The line passing through the tangency points of γ' and β' (or α) is passing through O. Thus the line passing through the tangency points of γ and β (or α) is passing through L. □

Proof of Theorem 1.2(c). Consider an arbitrary ellipse γ doubly tangent to α and β. Let F_1 and F_2 be the foci of γ; see Figure 13. Denote by ε the eccentricity of γ. Denote by I the center of β. Denote by r the radius of β. Let T be a tangency point of γ and β.

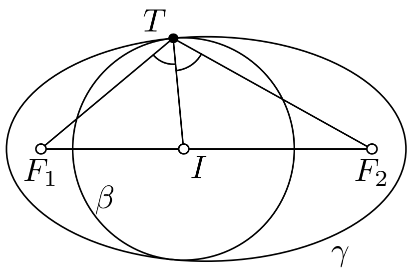

Fig. 13.

From the *optical property of an ellipse* (see [2, Theorem 1.1]) it follows that TI is the bisector of the angle F_1TF_2. Therefore,

$$\frac{|F_1I|}{|F_1T|} = \frac{|F_2I|}{|F_2T|} = \frac{|F_1I| + |F_2I|}{|F_1T| + |F_2T|} = \varepsilon.$$

Therefore, we have

$$|F_1I| \cdot |F_2I| = \frac{\varepsilon^2}{1-\varepsilon^2} \cdot (|F_1T| \cdot |F_2T| - |F_1I| \cdot |F_2I|) =$$
$$= \frac{\varepsilon^2}{1-\varepsilon^2} \cdot |TI|^2 = \frac{\varepsilon^2}{1-\varepsilon^2} \cdot r^2 = \text{const.}$$

Here the first equality follows from the previous equation; the second equality follows from the known formula for the length of the angle bisector; the fourth equality follows from Theorem 1.2(a). Since the perpendicular bisector of the segment F_1F_2 passes through the center of the circle α and the product $|F_1I| \cdot |F_2I|$ does not depend on an ellipse γ in the given family, it follows that the foci of all the ellipses in the family lie on a fixed circle concentric with α. □

Proof of Theorem 1.2(d). The assertion is a corollary of Theorem 1.2(a) and the following lemma.

Lemma on two similar conics. *Let two similar conics have four common points. Suppose there exists a circle doubly tangent to the conics such that the center of this circle is the intersection point of either the major axes or the minor axes. Then the four common points of these conics lie on the bisectors of the angle between the lines passing through the tangency points of each conic and the circle; see Figure 14.*

Proof. Denote by γ_1 and γ_2 the given conics. Since γ_1 and γ_2 are similar, we see that the eccentricity of γ_1 is equal to the eccentricity of γ_2, i. e. $\varepsilon_1 = \varepsilon_2 = \varepsilon$. Denote by ω the given circle.

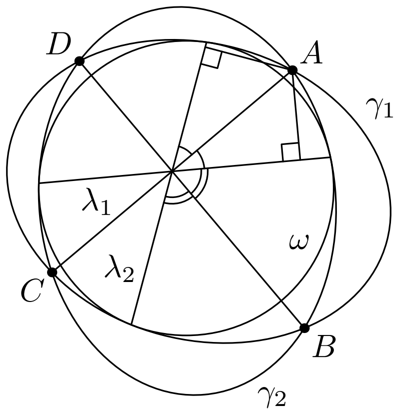

Fig. 14.

Assume that the center of ω lies on the major axes of γ_1 and γ_2. Denote by λ_i the line passing through the tangency points of γ_i and ω. Denote by A, B, C, and D the intersection points of γ_1 and γ_2. By the generalized "focus-directrix" property, we get

$$\frac{t(A,\omega)}{d(A,\lambda_1)} = \varepsilon = \frac{t(A,\omega)}{d(A,\lambda_2)}.$$

Therefore, $d(A,\lambda_1) = d(A,\lambda_2)$. Analogously, we have $d(B,\lambda_1) = d(B,\lambda_2)$, $d(C,\lambda_1) = d(C,\lambda_2)$, and $d(D,\lambda_1) = d(D,\lambda_2)$. Thus the points A, B, C, and D

lie on the bisectors of the angle between λ_1 and λ_2. If the center of ω lies on the minor axes of γ_1 and γ_2, the proof is analogous. □

Proof of Theorem 1.3. Denote by Π the plane containing the given ellipses γ_1, γ_2, and γ_3. Denote by ω_{ij} the common doubly tangent circle of the ellipses γ_i and γ_j. Consider the spheres Σ_{12}, Σ_{23}, and Σ_{31} such that ω_{12}, ω_{23}, and ω_{31} are great circles of these spheres, respectively. Consider three prolate spheroids Γ_1, Γ_2, and Γ_3 having γ_1, γ_2, and γ_3 as axial sections, respectively. The sphere Σ_{ij} is inscribed in Γ_i and Γ_j. By Monge's theorem, we have that the intersection $\Gamma_i \bigcap \Gamma_j$ consists of two ellipses $\tau_{ij}^p, p = 1, 2$. Obviously, the plane containing each ellipse τ_{ij}^p is perpendicular to Π. Thus the orthogonal projection of τ_{ij}^p onto Π is a segment of the straight line λ_{ij}^p passing through the intersection points of γ_i and γ_j. Since the ellipses τ_{ij}^p and τ_{jk}^q lie on Γ_j, it follows that they intersect in two real or imaginary points. Evidently, the real line passing through these points is perpendicular to Π and intersects Π in the common point of λ_{ij}^p and λ_{jk}^q. Note that the common points of τ_{ij}^p and τ_{jk}^q lie on τ_{ki}^r for some r. Thus the lines λ_{ij}^p, λ_{jk}^q, and λ_{ki}^r have the common point. By the same argument, the other corresponding lines are concurrent. □

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ON ROTATION OF A ISOGONAL POINT

ALEXANDER SKUTIN

Abstract. In this short note we give a synthetic proof of the problem posed by A. V Akopyan in [1]. We prove that if Poncelet rotation of triangle T between circle and ellipse is given then the locus of the isogonal conjugate point of any fixed point P with respect to T is a circle.

We will prove more general problem:

Problem. *Let T be a Poncelet triangle rotated between external circle ω and internal ellipse with foci Q and Q' and P be any point. Then the locus of points P' isogonal conjugates to P with respect to T is a circle.*

Proof. First, prove the following lemma:

Lemma. *Suppose that ABC is a triangle and P, P' and Q, Q' are two pairs of isogonal conjugates with respect to ABC. Let H be a Miquel point of lines PQ, PQ', $P'Q$ and $P'Q'$. Then H lies on (ABC).*

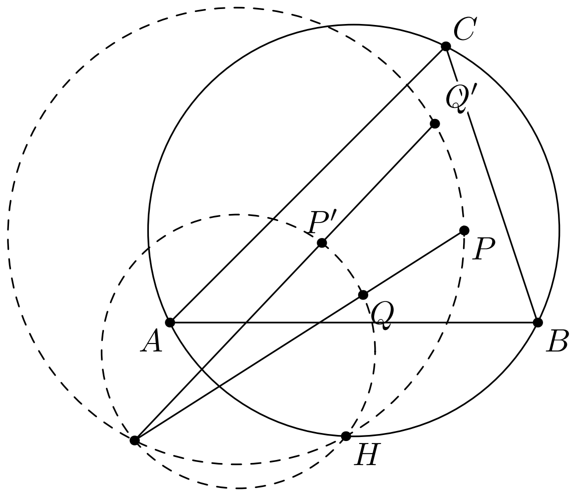

Fig. 1.

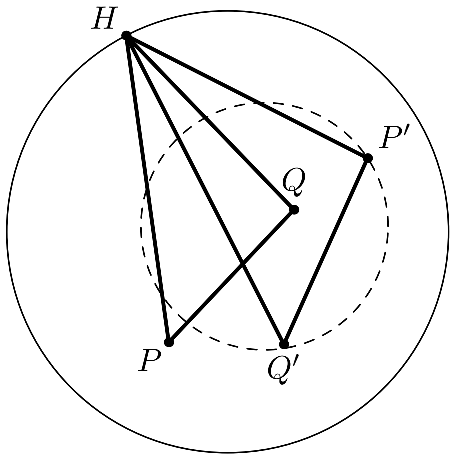

Fig. 2.

Proof. From here, the circumcircle of a triangle XYZ is denoted by (XYZ) and the oriented angle between lines ℓ and m is denoted by $\angle(\ell, m)$. Let A^* and B^* be such points that $A^*AH \sim B^*BH \sim PQH$. It is clear that $HPQ \sim HQ'P'$. From construction it immediately follows that there exists a similarity with center H which maps the triangle QBP' to the triangle PB^*Q'. So $HPB^*Q' \sim HQBP'$, and similarly $A^*PQ'H \sim AQP'H$. From the properties of isogonal conjugation it can be easily seen that $\angle(Q'A^*, A^*P) = \angle(P'A, AQ) = \angle(Q'A, AP)$, hence points A^*, A, P, and Q' are cocyclic. Similarly the quadrilateral PB^*BQ' is inscribed in a circle. Let lines AA^* and BB^* intersect in a point F. Indeed $ABQH \sim A^*B^*PH$, so $\angle(BQ, QA) = \angle(B^*P, PA^*)$. Obviously $\angle(B^*P, PA^*) = \angle(B^*B, BQ') + \angle(Q'A, AF)$. Thus

$$\angle(B^*P, PA^*) + \angle(BQ', Q'A) = \\ = \angle(FB, BQ') + \angle(BQ', Q'A) + \angle(Q'A, AF) = \angle(BF, FA),$$

but we have proved that

$$\angle(B^*P, PA^*) + \angle(BQ', Q'A) = \angle(BQ, QA) + \angle(BQ', Q'A) = \angle(AC, CB),$$

so F is on (ABC). We know that $A^*AH \sim B^*BH$, so $\angle(A^*A, AH) = \angle(B^*B, BH)$, hence $AFHB$ is inscribed in a circle. From that it is clear that H is on (ABC).

□

Now the problem can be reformulated in the following way. Suppose that ω is a circle, P, Q and Q' are fixed points, H is a variable point on ω. Let P' be such a point that $PQH \sim Q'P'H$. We need to prove that locus of points P' is a circle.

It is clear that the transformation which maps H to P' is a composition of an inversion, a parallel transform and rotations. Indeed, denote by z_x the coordinate of a point X in the complex plane. Than this transformation have the following equation:

$$z_h \to z_{q'} + (z_h - z_{q'})\frac{z_q - z_p}{z_h - z_p}.$$

Therefore, the image of the circle ω under this transformation is a circle.

□

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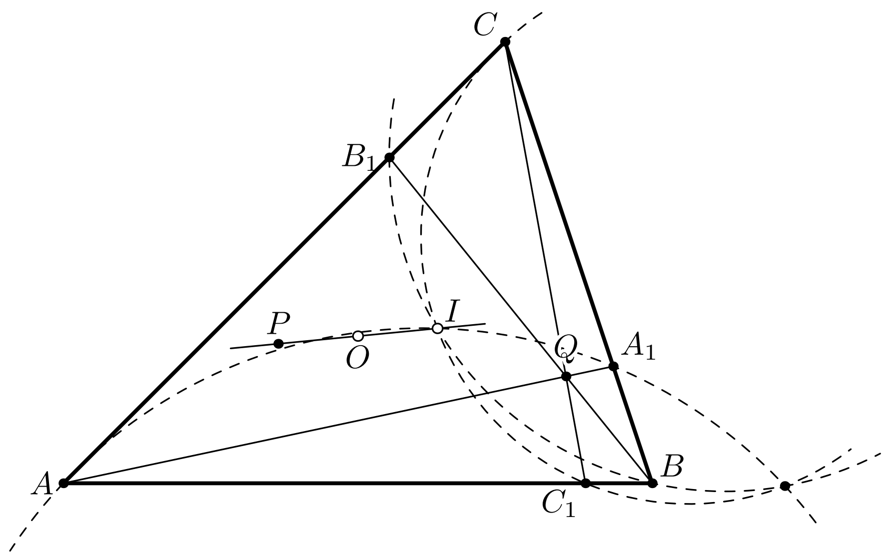
PROBLEM SECTION

In this section we suggest to solve and discuss problems provided by readers of the journal. The authors of the problems do not have purely geometric proofs. We hope that interesting proofs will be found by readers and will be published. Please send us solutions by email: `editor@jcgeometry.org`, as well as interesting "unsolved" problems for publishing in this Problems Section. All problems in this issue belong to Tran Quang Hung (`analgeomatica@gmail.com`).

Tran Quang Hung, **Coaxal circles generated by point on the Feuerbach hyperbola.**

Let ABC be a triangle with circumcenter O and incenter I. Let P be a point on the line OI and let Q be the isogonal conjugate point of P with respect to triangle ABC. It is known that the locus of points Q is the Feuerbach hyperbola. Let $A_1B_1C_1$ be the cevian triangle of the point Q with respect to the triangle ABC. Prove that circumcircles of triangles AIA_1, BIB_1, CIC_1 are coaxal.



Tran Quang Hung, **Two pairs of perspective triangles with the common circumcircle**

Here is the series of two problems. Let ABC and $A'B'C'$ be two triangles inscribed in the same circle ω and perspective from point P.

1. Let ω_a be the circumcircle of the triangle formed by the lines AB, AC and $B'C'$. Let A_1 be the second point of intersection of ω and ω_a. The points B_1, C_1, A_1', B_1' and C_1' we define analogously. Prove that the triangles $A_1B_1C_1$ and $A_1'B_1'C_1'$ are perspective.

2. Let ω_a be the circle which touches ω and segments AB and $B'A'$ as is shown on Fig. 2. Denote the touching point of ω_a and ω by A_1. The points B_1, C_1, A_1', B_1' and C_1' we define analogously. Prove that the triangles $A_1B_1C_1$ and $A_1'B_1'C_1'$ are perspective.

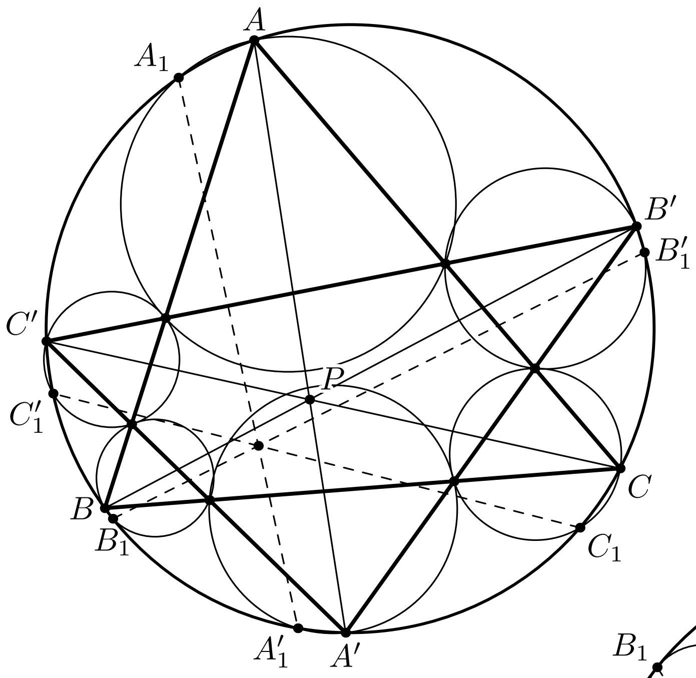


Fig. 1.

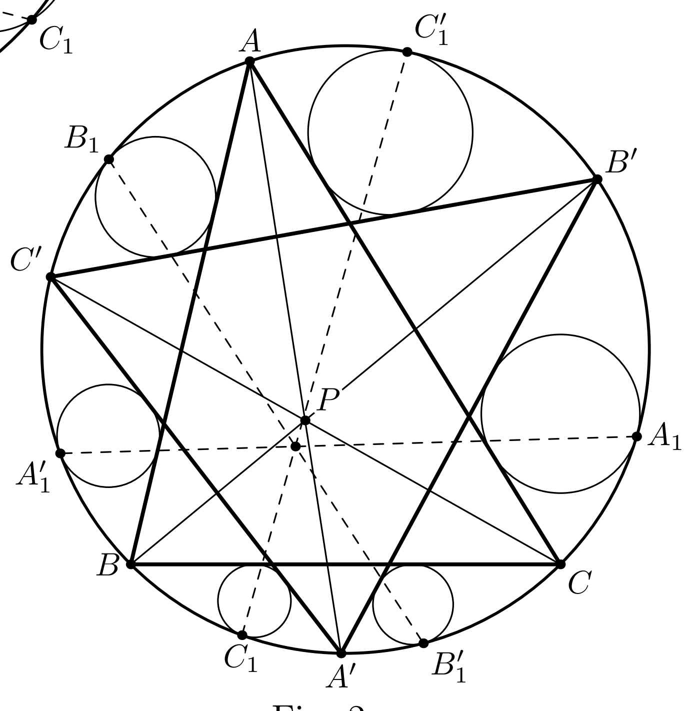


Fig. 2.

IX GEOMETRICAL OLYMPIAD IN HONOUR OF I. F. SHARYGIN

The Correspondence Round

Below is the list of problems for the first (correspondence) round of the IX Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of 8–11 grades (these are four elder grades in Russian school). In the list below, each problem is indicated by the numbers of school grades, for which it is intended. However, the participants may solve problems for elder grades as well (solutions of problems for younger grades will not be considered).

Your work containing solutions for the problems, written in Russian or in English, should be sent not later than by April 1, 2013 to *geomolymp@mccme.ru* in `pdf`, `doc` or `jpg` files.

Winners of the correspondence round will be invited to take part in the final round to be held in Dubna town (near Moscow, Russia) in Summer 2013 . More about Sharygin Olympiad see on `www.geometry.ru/olimpgeom.htm`.

(1) (8) Let ABC be an isosceles triangle with $AB = BC$. Point E lies on side AB, and ED is the perpendicular from E to BC. It is known that $AE = DE$. Find $\angle DAC$.

(2) (8) Let ABC be an isosceles triangle ($AC = BC$) with $\angle C = 20°$. The bisectors of angles A and B meet the opposite sides in points A_1 and B_1 respectively. Prove that triangle A_1OB_1 (where O is the circumcenter of ABC) is regular.

(3) (8) Let ABC be a right-angled triangle ($\angle B = 90°$). The excircle inscribed into angle A touches the extensions of sides AB, AC in points A_1, A_2 respectively; points C_1, C_2 are defined similarly. Prove that the perpendiculars from A, B, C to C_1C_2, A_1C_1, A_1A_2 respectively concur.

(4) (8) Let ABC be a nonisosceles triangle. Point O is its circumcenter, and point K is the center of the circumcircle w of triangle BCO. The altitude of ABC from A meets w in point P. Line PK intersects the circumcircle of ABC in points E and F. Prove that one of segments EP and FP is equal to segment PA.

(5) (8) Four segments join some point inside a convex quadrilateral with its vertices. Four obtained triangles are equal. Can we assert that this quadrilateral is a rhombus?

(6) (8–9) Diagonals AC, BD of trapezoid $ABCD$ meet in point P. The circumcircles of triangles ABP, CDP intersect line AD for the second time in points X, Y. Point M is the midpoint of segment XY. Prove that $BM = CM$.

(7) (8–9) Let BD be a bisector of triangle ABC. Points I_a, I_c are the incenters of triangles ABD, CBD. Line I_aI_c meets AC in point Q. Prove that $\angle DBQ = 90°$.

(8) (8–9) Let X be an arbitrary point inside the circumcircle of triangle ABC. Lines BX and CX meet the circumcircle in points K and L respectively. Line LK intersects BA and AC in points E and F respectively. Find the locus of points X such that the circumcircles of triangles AFK and AEL touch.

(9) (8–9) Let T_1 and T_2 be the touching points of the excircles of triangle ABC with sides BC and AC respectively. It is known that the reflection of the incenter of ABC in the midpoint of AB lies on the circumcircle of triangle CT_1T_2. Find $\angle BCA$.

(10) (8–9) The incircle of triangle ABC touches the side AB in point C'; the incircle of triangle ACC' touches sides AB and AC in points C_1, B_1; the incircle of triangle BCC' touches the sides AB and BC in points C_2, A_2. Prove that lines B_1C_1, A_2C_2 and CC' concur.

(11) (8–9) a) Let $ABCD$ be a convex quadrilateral. Let $r_1 \le r_2 \le r_3 \le r_4$ be the radii of the incircles of triagles ABC, BCD, CDA, DAB. Can the inequality $r_4 > 2r_3$ hold?

b)The diagonals of a convex quadrilateral $ABCD$ meet in point E. Let $r_1 \le r_2 \le r_3 \le r_4$ be the radii of the incircles of triangles ABE, BCE, CDE, DAE. Can the inequality $r_2 > 2r_1$ be correct?

(12) (8–11) On each side of triangle ABC, two distinct points are marked. It is known that these points are the feet of the altitudes and the bisectors.

a) Using only a ruler determine which points are the feet of the altitudes and which points are the feet of the bisectors.

b) Solve p.a) drawing only three lines.

(13) (9–10) Let A_1 and C_1 be the touching points of the incircle of triangle ABC with BC and AB respectively, A' and C' be the touching points of the excircle inscribed into angle B with the extensions of BC and AB respectively. Prove that the orthocenter H of triangle ABC lies on A_1C_1 iff lines $A'C_1$ and BA are perpendicular.

(14) (9–11) Let M, N be the midpoints of diagonals AC, BD of right-angled trapezoid $ABCD$ ($\angle A = \angle D = 90°$). The circumcircles of triangles ABN, CDM meet line BC in points Q, R. Prove that the distances from Q, R to the midpoint of MN are equal.

(15) (9–11) a) Triangles $A_1B_1C_1$ and $A_2B_2C_2$ are inscribed into triangle ABC so that $C_1A_1 \perp BC$, $A_1B_1 \perp CA$, $B_1C_1 \perp AB$, $B_2A_2 \perp BC$, $C_2B_2 \perp CA$, $A_2C_2 \perp AB$. Prove that these triangles are equal.

b) Points $A_1, B_1, C_1, A_2, B_2, C_2$ lie inside triangle ABC so that A_1 is on segment AB_1, B_1 is on segment BC_1, C_1 is on segment CA_1, A_2 is on segment AC_2, B_2 is on segment BA_2, C_2 is on segment CB_2 and angles $BAA_1, CBB_1, ACC_1, CAA_2, ABB_2, BCC_2$ are equal. Prove that triangles $A_1B_1C_1$ and $A_2B_2C_2$ are equal.

(16) (9–11) The incircle of triangle ABC touches BC, CA, AB in points A', B', C' respectively. The perpendicular from incenter I to the median from vertex C meets line $A'B'$ in point K. Prove that $CK \parallel AB$.

(17) (9–11) An acute angle between the diagonals of a cyclic quadrilateral is equal to ϕ. Prove that an acute angle between the diagonals of any another quadrilateral having the same sidelengths is less than ϕ.

(18) (9–11) Let AD be a bisector of triangle ABC. Points M and N are the projections of B and C to AD. The circle with diameter MN intersects BC in points X and Y. Prove that $\angle BAX = \angle CAY$.

(19) (10–11) a) The incircle of triangle ABC touches AC and AB in points B_0 and C_0 respectively. The bisectors of angles B and C meet the medial perpendicular to the bisector AL in points Q and P respectively. Prove that lines PC_0, QB_0 and BC concur.

b) Let AL be the bisector of triangle ABC. Points O_1 and O_2 are the circumcenters of triangles ABL and ACL respectively. Points B_1 and C_1 are the projections of C and B to the bisectors of angles B and C respectively. Prove that lines O_1C_1, O_1B_1 and BC concur.

c) Prove that two points obtained in pp. a) and b) coincide.

(20) (10–11) Let C_1 be an arbitrary point on side AB of triangle ABC. Points A_1 and B_1 of rays BC and AC are such that $\angle AC_1B_1 = \angle BC_1A_1 = \angle ACB$. Lines AA_1 and BB_1 meet in point C_2. Prove that all lines C_1C_2 have a common point.

(21) (10–11) Given are a circle ω and a point A outside it. One of two lines drawn through A intersects ω in points B and C, the second one intersect it in points D and E (D lies between A and E). The line passing through D and parallel to BC, meets ω for the second time in point F, and line AF meets ω in point T. Let M be the common point of lines ET and BC, and N be the reflection of A in M. Prove that the circumcircle of triangle DEN passes through the midpoint of segment BC.

(22) (10–11) The common perpendiculars to the opposite sidelines of a non-planar quadrilateral are mutually perpendicular. Prove that they are complanar.

(23) (10–11) Two convex polygons A and B don't intersect. Polygon A have exactly 2012 planes of symmetry. What is the maximal number of symmetry planes of the union of A and B when B has a) 2012, b) 2013 symmetry planes?

c) What is the answer to the question of p.b) when the symmetry planes are replaced by the symmetry axes?

IX GEOMETRICAL OLYMPIAD IN HONOUR OF I. F. SHARYGIN

Final round. Ratmino, 2013, August 1-2

8 grade. First day

8.1. Let $ABCDE$ be a pentagon with right angles at vertices B and E and such that $AB = AE$ and $BC = CD = DE$. The diagonals BD and CE meet at point F. Prove that $FA = AB$.

8.2. Two circles with centers O_1 and O_2 meet at points A and B. The bisector of angle O_1AO_2 meets the circles for the second time at points C and D. Prove that the distances from the circumcenter of triangle CBD to O_1 and to O_2 are equal.

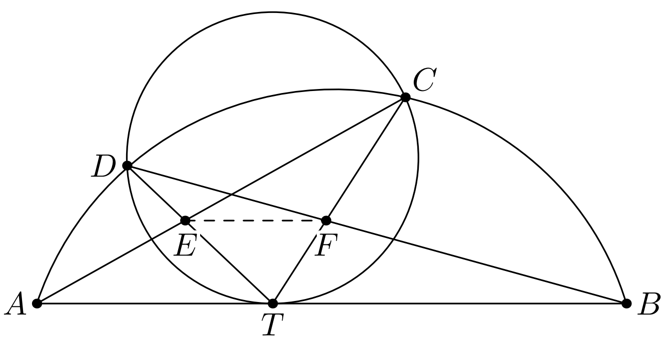
8.3. Each vertex of a convex polygon is projected to all nonadjacent sidelines. Can it happen that each of these projections lies outside the corresponding side?

8.4. The diagonals of a convex quadrilateral $ABCD$ meet at point L. The orthocenter H of the triangle LAB and the circumcenters O_1, O_2, and O_3 of the triangles LBC, LCD, and LDA were marked. Then the whole configuration except for points H, O_1, O_2, and O_3 was erased. Restore it using a compass and a ruler.

8 grade. Second day

8.5. The altitude AA', the median BB', and the angle bisector CC' of a triangle ABC are concurrent at point K. Given that $A'K = B'K$, prove that $C'K = A'K$.

8.6. Let α be an arc with endpoints A and B (see fig.). A circle ω is tangent to segment AB at point T and meets α at points C and D. The rays AC and TD meet at point E, while the rays BD and TC meet at point F. Prove that EF and AB are parallel.



8.7. In the plane, four points are marked. It is known that these points are the centers of four circles, three of which are pairwise externally tangent, and all these three are internally tangent to the fourth one. It turns out, however, that it is impossible to determine which of the marked points is the center of the fourth (the largest) circle. Prove that these four points are the vertices of a rectangle.

8.8. Let P be an arbitrary point on the arc AC of the circumcircle of a fixed triangle ABC, not containing B. The bisector of angle APB meets the bisector of angle BAC at point P_a; the bisector of angle CPB meets the bisector of angle BCA at point P_c. Prove that for all points P, the circumcenters of triangles PP_aP_c are collinear.

9 grade. First day

9.1. All angles of a cyclic pentagon $ABCDE$ are obtuse. The sidelines AB and CD meet at point E_1; the sidelines BC and DE meet at point A_1. The tangent at B to the circumcircle of the triangle BE_1C meets the circumcircle ω of the pentagon for the second time at point B_1. The tangent at D to the circumcircle of the triangle DA_1C meets ω for the second time at point D_1. Prove that $B_1D_1 \parallel AE$.

9.2. Two circles ω_1 and ω_2 with centers O_1 and O_2 meet at points A and B. Points C and D on ω_1 and ω_2, respectively, lie on the opposite sides of the line AB and are equidistant from this line. Prove that C and D are equidistant from the midpoint of O_1O_2.

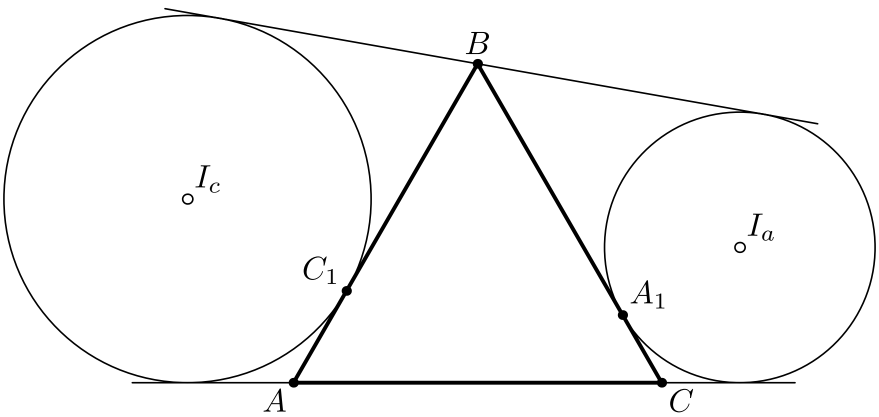
9.3. Each sidelength of a convex quadrilateral $ABCD$ is not less than 1 and not greater than 2. The diagonals of this quadrilateral meet at point O. Prove that $S_{AOB} + S_{COD} \leqslant 2(S_{AOD} + S_{BOC})$.

9.4. A point F inside a triangle ABC is chosen so that $\angle AFB = \angle BFC = \angle CFA$. The line passing through F and perpendicular to BC meets the median from A at point A_1. Points B_1 and C_1 are defined similarly. Prove that the points A_1, B_1, and C_1 are three vertices of some regular hexagon, and that the three remaining vertices of that hexagon lie on the sidelines of ABC.

9 grade. Second day

9.5. Points E and F lie on the sides AB and AC of a triangle ABC. Lines EF and BC meet at point S. Let M and N be the midpoints of BC and EF, respectively. The line passing through A and parallel to MN meets BC at point K. Prove that $\frac{BK}{CK} = \frac{FS}{ES}$.

9.6. A line ℓ passes through the vertex B of a regular triangle ABC. A circle ω_a centered at I_a is tangent to BC at point A_1, and is also tangent to the lines ℓ and AC. A circle ω_c centered at I_c is tangent to BA at point C_1, and is also tangent to the lines ℓ and AC.



9.7. Two fixed circles ω_1 and ω_2 pass through point O. A circle of an arbitrary radius R centered at O meets ω_1 at points A and B, and meets ω_2 at points C and D. Let X be the common point of lines AC and BD. Prove that all the points X are collinear as R changes.

9.8. Three cyclists ride along a circular road with radius 1 km counterclockwise. Their velocities are constant and different. Does there necessarily exist (in a sufficiently long time) a moment when all the three distances between cyclists are greater than 1 km?

10 grade. First day

10.1. A circle k passes through the vertices B and C of a triangle ABC with $AB > AC$. This circle meets the extensions of sides AB and AC beyond B and C at points P and Q, respectively. Let AA_1 be the altitude of ABC. Given that $A_1P = A_1Q$, prove that $\angle PA_1Q = 2\angle BAC$.

10.2. Let $ABCD$ be a circumscribed quadrilateral with $AB = CD \neq BC$. The diagonals of the quadrilateral meet at point L. Prove that the angle ALB is acute.

10.3. Let X be a point inside a triangle ABC such that $XA \cdot BC = XB \cdot AC = XC \cdot AB$. Let I_1, I_2, and I_3 be the incenters of the triangles XBC, XCA, and XAB, respectively. Prove that the lines AI_1, BI_2, and CI_3 are concurrent.

10.4. We are given a cardboard square of area $1/4$ and a paper triangle of area $1/2$ such that all the squares of the side lengths of the triangle are integers. Prove that the square can be completely wrapped with the triangle. (In other words, prove that the triangle can be folded along several straight lines and the square can be placed inside the folded figure so that both faces of the square are completely covered with paper.)

10 grade. Second day

10.5. Let O be the circumcenter of a cyclic quadrilateral $ABCD$. Points E and F are the midpoints of arcs AB and CD not containing the other vertices of the quadrilateral. The lines passing through E and F and parallel to the diagonals of $ABCD$ meet at points E, F, K, and L. Prove that line KL passes through O.

10.6. The altitudes AA_1, BB_1, and CC_1 of an acute-angled triangle ABC meet at point H. The perpendiculars from H to B_1C_1 and A_1C_1 meet the rays CA and CB at points P and Q, respectively. Prove that the perpendicular from C to A_1B_1 passes through the midpoint of PQ.

10.7. In the space, five points are marked. It is known that these points are the centers of five spheres, four of which are pairwise externally tangent, and all these four are internally tangent to the fifth one. It turns out, however, that it is impossible to determine which of the marked points is the center of the fifth (the largest) sphere. Find the ratio of the greatest and the smallest radii of the spheres.

10.8. In the plane, two fixed circles are given, one of them lies inside the other one. For an arbitrary point C of the external circle, let CA and CB be two chords of this circle which are tangent to the internal one. Find the locus of the incenters of triangles ABC.