GEOMETRICAL OLYMPIAD IN HONOR OF I. F. SHARYGIN

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Below is the list of problems for the first (correspondence) round of the VIII Sharygin Geometrical Olympiad and selected problems of the VII Olympiad.

The olympiad is intended for high-school students of 8–11 grades (these are four elder grades in Russian school). In the list below each problem is indicated by the numbers of school grades, for which it is intended. However, the participants are encouraged to solve problems for elder grades as well (solutions for younger grades will not be considered).

Your work containing solutions for the problems, written in Russian or in English, should be sent no later than by April 1, 2012, by e-mail to *geomolymp@mccme.ru* in `pdf`, `doc` or `jpg` files.

Winners of the correspondence round will be invited to take part in the final round to be held in Dubna town (near Moscow, Russia) in Summer 2012 .

More about Sharygin Olympiad see on www.geometry.ru.

Correspondence round of VIII Olympiad

1. *(8)* In triangle ABC a point M is the midpoint of the side AB, and a point D is the foot of altitude CD. Prove that $\angle A = 2\angle B$ if and only if $AC = 2MD$.

2. *(8)* A cyclic n-gon is divided by non-intersecting (inside the n-gon) diagonals to $n-2$ triangles. Each of these triangles is similar to at least one of the remaining ones.

For what n is this possible?

3. *(8)* A circle with center I touches sides AB, BC, CA of a triangle ABC at points C_1, A_1, B_1. Lines AI, CI, B_1I meet A_1C_1 in points X, Y, Z respectively. Prove that $\angle YB_1Z = \angle XB_1Z$

4. *(8)* Given triangle ABC. Point M is the midpoint of the side BC, and point P is the projection of B to the perpendicular bisector of segment AC. Line PM meets AB at a point Q. Prove that the triangle QPB is isosceles.

5. *(8)* Let D be an arbitrary point on the side AC of a triangle ABC. The tangent in D to the circumcircle of triangle BDC meets AB at point C_1; point A_1 is defined similarly. Prove that $A_1C_1||AC$.

6. *(8–9)* Point C_1 of hypotenuse AC of a right triangle ABC is such that $BC = CC_1$. Point C_2 on the cathetus AB is such that $AC_2 = AC_1$; point A_2 is defined similarly. Find angle AMC, where M is the midpoint of A_2C_2.

7. *(8–9)* In a non-isosceles triangle ABC the bisectors of angles A and B are inversely proportional to the respective side lengths. Find angle C.

8. *(8–9)* Let BM be the median of a right triangle ABC ($\angle B = 90°$). The incircle of the triangle ABM touches sides AB, AM in points A_1, A_2; points C_1, C_2 are defined similarly. Prove that the lines A_1A_2 and C_1C_2 meet on the bisector of angle ABC.

9. *(8–9)* In triangle ABC, given lines l_b and l_c containing the bisectors of angles B and C, and the foot L_1 of the bisector of angle A. Reconstruct triangle ABC.

10. In a convex quadrilateral all side lengths and all angles are pairwise different.

a) *(8–9)* Can the largest angle be adjacent to the largest side and at the same time the smallest angle be adjacent to the smallest side?

b) *(9–11)* Can the largest angle be non-adjacent to the smallest side and at the same time the smallest angle be non-adjacent to the largest side?

11. Given triangle ABC and point P. Points A', B', C' are the projections of P to BC, CA, AB respectively. A line passing through P and parallel to AB meets the circumcircle of triangle $PA'B'$ for the second time at point C_1. Points A_1, B_1 are defined similarly. Prove that

a) *(8-10)* lines AA_1, BB_1, CC_1 concur;

b) *(9–11)* triangles ABC and $A_1B_1C_1$ are similar.

12. *(9–10)* Let O be the circumcenter of an acute-angled triangle ABC. A line passing through O and parallel to BC meets AB and AC at points P and Q respectively. The sum of distances from O to AB and to AC is equal to OA. Prove that $PB + QC = PQ$.

13. *(9–10)* Points A, B are given. Find the locus of points C such that C, the midpoints of AC, BC and the centroid of triangle ABC are concyclic.

14. *(9–10)* For a convex quadrilateral $ABCD$, suppose $AC \cap BD = O$ and M is the midpoint of BC. Let $MO \cap AD = E$. Prove that $\frac{AE}{ED} = \frac{S_{\triangle ABO}}{S_{\triangle CDO}}$.

15. *(9–11)* Given triangle ABC. Consider lines l with the following property: the reflections of l in the sidelines of the triangle concur. Prove that all these lines have a common point.

16. *(9–11)* Given right triangle ABC with hypotenuse AB. Let M be the midpoint of AB and O be the center of the circumcircle ω of triangle CMB. Line AC meets ω for the second time in point K. Segment KO meets the circumcircle of triangle ABC in point L. Prove that segments AL and KM meet on the circumcircle of triangle ACM.

17. *(9–11)* A square $ABCD$ is inscribed into a circle. Point M lies on arc BC, AM meets BD at point P, DM meets AC at point Q. Prove that the area of quadrilateral $APQD$ is equal to the half of the area of the square.

18. *(9–11)* A triangle and two points inside it are marked. It is known that one of the triangle's angles is equal to 58°, one of the two remaining angles is equal to 59°, one of the two given points is the incenter of the triangle and the second one is its circumcenter. Using only a non-calibrated ruler determine the locations of the angles and of the centers.

19. *(10–11)* Two circles of radii 1 meet at points X, Y, and the distance between these points is also 1. Point C lies on the first circle, and lines CA, CB are tangents to the second one. These tangents meet the first circle for the second time at points B', A'. Lines AA' and BB' meet at point Z. Find angle XZY.

20. *(10–11)* Point D lies on the side AB of triangle ABC. Let ω_1 and Ω_1, ω_2 and Ω_2 be the incircles and the excircles (touching segment AB) of the triangles ACD and BCD respectively. Prove that the common external tangent to ω_1 and to ω_2, and the common external tangent to Ω_1 and to Ω_2 meet on AB.

21. *(10–11)* Two perpendicular lines pass through the orthocenter of an acute-angled triangle. The sidelines of the triangle cut two segments on each of these lines: one lying inside the triangle and another one lying outside it. Prove that the product of the two internal segments is equal to the product of the two external segments.

22. *(10–11)* A circle ω centered at I is inscribed into a segment of the disk, formed by an arc and a chord AB. Point M is the midpoint of that arc AB, and point N is the midpoint of the complementary arc. The tangents from N touch ω in points C and D. The opposite sidelines AC and BD of quadrilateral $ABCD$ meet at point X, and the diagonals of $ABCD$ meet at point Y. Prove that points X, Y, I and M are collinear.

23. *(10–11)* An arbitrary point is selected on each of twelve diagonals of the faces of a cube. The centroid of these twelve points is determined. Find the locus of all those centroids.

24. *(10–11)* Given are n $(n > 2)$ points on the plane. Assume that there are no three collinear points among them. In how many ways this set of n points can be divided into two non-empty subsets with non-intersecting convex envelopes?

Selected problems of VII Olympiad

1. *(B. Frenkin, correspondence round, 8)* Given triangle ABC. The perpendicular bisector of side AB meets one of the remaining sides at a point C'. Points A' and B' are defined similarly. For which triangles ABC the triangle $A'B'C'$ is regular?

Answer. For regular triangles and for triangles with angles equal to 30, 30 and 120 grades.

Solution. Consider a non-regular triangle ABC. Let AB be its largest side. Then points A', B' lie on segment AB. From the assumption we conclude that $C'C_0$, where C_0 is the midpoint of AB, is the bisector of the segment $A'B'$. Thus, $CA' = A'B = AB' = CB'$, i.e. C' coincides with C, and the triangle ABC is isosceles. Also we have $2\angle A = \angle A + \angle CAB' = \angle CB'B = 60°$, so $\angle A = \angle B = 30°$.

2. *(A. Akopyan, correspondence round, 8)* Two unit circles ω_1 and ω_2 intersect at points A and B. M is an arbitrary point of ω_1, N is an arbitrary point of ω_2. Two unit circles ω_3 and ω_4 pass through both points M and N. Let C be the second common point of ω_1 and ω_3, and D be the second common point of ω_2 and ω_4. Prove that $ACBD$ is a parallelogram.

Solution. Let O_i be the center of circle ω_i. From the assumption it follows that O_1AO_2B, O_1CO_3M, O_3MO_4N, O_4NO_2D are rhombuses with sides equal to 1. Then $\overrightarrow{O_1C} = \overrightarrow{MO_3} = \overrightarrow{O_4N} = \overrightarrow{DO_2}$ and $\overrightarrow{O_1A} = \overrightarrow{BO_2}$. Thus $\overrightarrow{AC} = \overrightarrow{DB}$, q.e.d.

3. *(D. Shvetsov, correspondence round, 8–10)* The excircle of an equilateral triangle ABC ($\angle B = 90^\circ$) touches the side BC at point A_1 and touches the line AC at point A_2. Line A_1A_2 meets the incircle of ABC for the first time at point A'; point C' is defined similarly. Prove that $AC||A'C'$.

Solution. Let I be the incenter and PQ be the diameter of the incircle parallel to AC (Fig. 1). Since $\angle PIC = \angle ACI = \angle BCI$ and $CA_1 = (AB+BC-AC)/2 = r = IP$, the quadrilateral IPA_1C is an isosceles trapezoid. Then the line A_1P is parallel to IC, i.e it coincides with A_1A_2. So P coincides with A', and similarly Q coincides with C'.

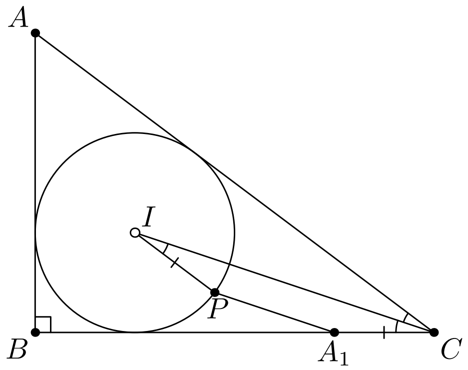

Fig. 1.

4. *(P. Dolgirev, correspondence round, 9–10)* Given are triangle ABC and a line ℓ. The reflections of ℓ with respect to AB and and with respect to AC meet at a point A_1. Points B_1, C_1 are defined similarly. Prove that

a) lines AA_1, BB_1, CC_1 concur;

b) their common point lies on the circumcircle of ABC;

c) two points constructed in this way for two perpendicular lines are opposite.

Solution. At first, note that if ℓ moves parallel with constant velocity, then the reflections of ℓ with respect to AC and BC also move with constant velocities. Therefore, C_1 moves along the line passing through C. This means that the common point of CC_1 with the circumcircle depends only on the direction of l. Now let A', B' be the common points of ℓ with BC and AC respectively (Fig. 2). Then $\angle C_1B'C = \angle CB'A'$, $\angle C'AC = \angle BA'C_1$. Thus, C is the incenter or the excenter of the triangle $A'B'C_1$, i.e. C_1C bisects the angle $A'C_1B'$ or the adjacent angle. However, the angle between lines $A'C_1$ and $B'C_1$ does not depend on l, hence the angle between CC_1 and C_1A' also does not depend on ℓ. So, when ℓ rotates, the lines AA_1, BB_1, CC_1 rotate with the same velocity. This proves all the assertions.

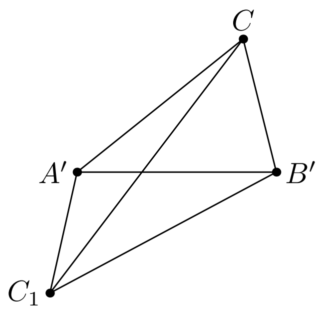

Fig. 2.

5.*(B. Frenkin, correspondence round, 9–11)* a) Does there exist a triangle, whose shortest median is longer that the longest bisector?

b) Does there exist a triangle in which the shortest bisector is longer that the longest altitude?

Solution. a) No, it does not. Let the lengths of sides BC, AC, AB be equal to a, b, c respectively and $a \le b \le c$. Let also CM be the median, AL be the bisector. If the angle C is not acute, then $AL > AC$. Since $BC \le AC$ and $\angle CMA$ is not acute, we see rhat $CM \le AC$ and $CM < AL$.

Now let $\angle C$ be acute. Since AB is the largest side, $\angle C \ge 60^\circ$ and the angles A, B are acute. Then the base H of altitude AH lies on the segment BC. Thus, AH (and AL) is at least $AC \cos\ 60^\circ = b\sqrt{3}/2$. But the square of CM is equal to $\frac{2a^2+2b^2-c^2}{4} \le \frac{2a^2+b^2}{4} \le \frac{3b^2}{4}$. So CM is not greater than $b\sqrt{3}/2$ and can not exceed the bisector of angle A.

b) No, it does not. Let $a \le b \le c$ and l be the bisector of angle C. Then $(al + bl)\sin\frac{C}{2} = 2S_{ABC} = ab\sin C$, i.e. $l = \frac{2ab\cos\frac{C}{2}}{a+b}$. On the other hand, the altitude from A is equal to $h = b\sin C$. Since $a + b \ge 2a$ and $C \ge 60^\circ$, we have $h/l = (a+b)\sin\frac{C}{2}/a \ge 1$.

Remark. It is easy to construct a triangle such that its shortest median is longer than its longest altitude.

6.*(A. Zaslavsky, correspondence round, 9–11)* Given n straight lines in general position on the plane (every two of them are not parallel and every three of them do not concur). These lines divide the plane into several parts. What is

a) the minimal;

b) the maximal

number of these parts that can be angles?

Solution. a) **Answer.** 3. Consider the convex envelope of all common points of those lines. Two lines passing through some vertex of this envelope divide the plane into four angles, and one of them contains all the remaining points. Thus the remaining lines do not intersect the vertical angle and the number of angles can not be less than three. An example with three angles can be constructed by induction: the next line must intersect all previous lines inside the triangle which is the convex envelope of common points.

b) **Answer.** n, in case n is odd; $n-1$ in case $n > 2$ is even. Let us construct a circle containing all common points. Our lines divide it into $2n$ arcs. Let AB, BC

be two adjacent arcs, X, Y be the common points of the line passing through B, with the lines passing through A and C respectively. If X lies on the segment BY, then the part containing the arc BC can not be an angle. Thus, only one of the two parts containing the adjacent arcs can be an angle. Thus, the number of angles is not greater than n, and an equality is possible only when the part containing each second arc is an angle. However, if n is even, this yields that there exist two angles containing the opposite arcs. Since these two angles are formed by the same lines, this is not possible for $n > 2$. If n is odd then n parts formed by the sidelines of a regular n-gon are angles. Obviously, we can add one line without reduction of the number of angles.

Second solution of a). *(A. Goncharuk, Kharkov)* Let a polygon T be the union of all bounded parts. Then all angles are vertical to the angles of T, which are less than $180°$. From the formula for the sum of angles we obtain that there exist three such angles. The polygon with three angles can be constructed in the following way. Take a point D inside the triangle ABC, inscribe a sufficiently small circle in the angle ADB and take $n-4$ points on the smaller arc formed by the touching points. The tangents in these points and the lines AC, BC, AD, BD form the desired polygon.

7. *(A. Zaslavsky, correspondence round, 9–11)* Does there exist a non-isosceles triangle such that the altitude from one vertex, the bisector from the second one and the median from the third one are equal?

Solution. Yes, it does. Fix some vertices A, B, construct a point D which is the reflection of A wityh respect to B, and consider an arbitrary point C such that $\angle BCD = 150°$. The altitude of triangle ABC from A is equal to the distance DH from D to BC, i.e. $CD/2$. The median BM from B as the medial line in triangle ACD also is equal to $CD/2$ (Fig. 3). Now we move the point C along the arc BD containing angle $150°$. As C tends to B, the bisector from C tends to zero and the median from B tends to $AB/2$. As C tends to D, the median from B tends to zero and the bisector stays not smaller than BC. Thus, there exists a point C for which the bisector is equal to two remaining segments.

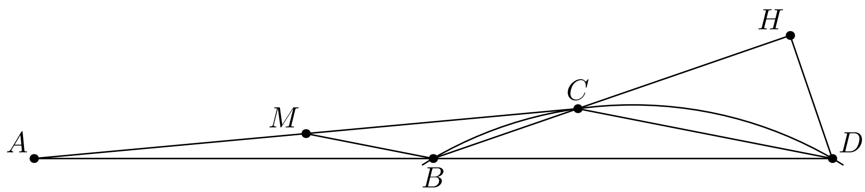

Fig. 3.

Remark. When C moves from B to D, the bisector increases and the altitude decreases. Thus the angles of the desired triangle are determined in a unique way.

8. *(G. Feldman, correspondence round, 10–11)* Let CX and CY be tangents from vertex C of a triangle ABC to the circle passing through the midpoints of its sides. Prove that the lines XY, AB and the tangent to the circumcircle of ABC at point C concur.

Solution. The homothety with center C and factor 1/2 transforms the line XY to the radical axis of point C and the circle passing through the midpoints A', B', C' of BC, CA, AB respectively. On the other hand, the tangent at C to the circumcircle touches also the circle $A'B'C'$, i.e. it is the radical axis of this circle and the point C. The common point of these radical axes lies on $A'B'$. Using the inverse homothety we obtain the assertion.

9. *(N. Beluhov, correspondence round, 10–11)* Three congruent regular tetrahedrons have a common center. Is it possible that all faces of the polyhedron formed by their intersection are congruent?

Solution. Yes, it is possible. Let the first tetrahedron touch their common inscribed sphere at points A, B, C, D. Rotate these points around the common perpendicular (and bisector) of the segments AB and CD by 120° to obtain A', B', C', D' and by 240° to obtain A'', B'', C'', D'' (the twelve points form two regular hexagons). The tangential planes to the sphere in these twelve points form the three tetrahedrons needed. Indeed, for any two of these points there exists an isometry that maps this set of twelve points onto itself and maps one of these two points to another one. These isometries enable us to map any facet of the obtained polygon onto any other one.

10. *(T. Golenishcheva-Kutuzova, final round, 8)* Peter made a paper rectangle, put it on an identical rectangle and pasted both rectangles along their perimeters. Then he cut the upper rectangle along one of its diagonals and along the perpendiculars to this diagonal from two remaining vertices. After this he turned back the obtained triangles in such a way that they, along with the lower rectangle, form a new rectangle.

Let this new rectangle be given. Reconstruct the original rectangle using compass and ruler.

Solution. Let $ABCD$ be the obtained rectangle; O be its center; K, M be the midpoints of its shortest sides AB and CD; L, N be the meets of BC and AD respectively with the circle with diameter KM (Fig. 4). Then $KLMN$ is the desired rectangle. In fact, let P be the projection of M to LN. Since $\angle CLM = \angle OML = \angle MLO$, the triangles MCL and MPL are equal. Thus the bend along ML matches these triangles. Similarly the bend along MN matches triangles MDN and MPN. Finally, since the construction is symmetric with respect to the point O, the bend along KL and KN matches triangles BKL and AKN with triangle NKL.

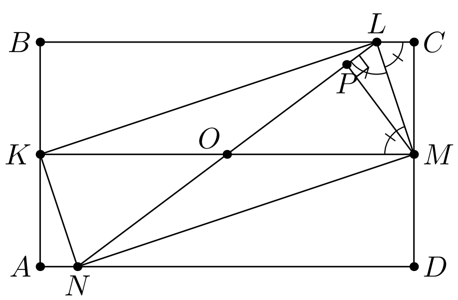

Fig. 4.

11.(*A. Shapovalov, final round, 8*) Given the circle of radius 1 and several its chords with the sum of lengths 1. Prove that one can inscribe a regular hexagon into that circle so that its sides do not intersect those chords.

Solution. Paint the smallest arcs corresponding to given chords. If we rotate the painted arcs in such a way that the corresponding chords form a polygonal line, then the distance between the ends of it is less than 1, and since a chord with length 1 corresponds to an arc equal to 1/6 of the circle, the total length of painted arcs is less than 1/6 of the circle.

Now inscribe a regular hexagon into the circle and mark one of its vertices. Rotate the hexagon, and when the marked vertex coincides with a painted point, paint the points corresponding to all remaining vertices. The total length of painted arcs increases at most 6 times, therefore there exists an inscribed regular hexagon with non-painted vertices. Obviously its sides do not intersect the chords.

12.(*A. Zaslavsky, final round, 8*) Using only the ruler, divide the side of a square table into n equal parts. All lines drawn must lie on the surface of the table.

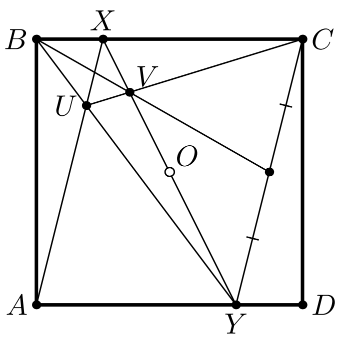

Fig. 5.

Solution. First, we bisect the side. Find center O of square $ABCD$ as a common point of its diagonals. Now let a point X lie on the side BC, Y be a common point of XO and AD, U be a common point of AX and BY, V be a common point of UC and XY (Fig. 5). Then the line BV bisects the bases of the trapezoid $CYUX$. The line passing through O and the midpoint of CY bisects sides AB and CD.

Now suppose that two opposite sides are divided into k equal parts. Let us demonstrate how to divide it into $k+1$ equal parts. Let $AX_1 = X_1X_2 = \cdots = X_{k-1}B$, $DY_1 = Y_1Y_2 = \cdots Y_{k-1}C$. Then by the Thales theorem, the lines $AY_1, X_1Y_2, \ldots, X_{k-1}C$ divide the diagonal BD into $k+1$ equal parts (Fig. 5). Dividing similarly the second diagonal and joining the corresponding points by the lines parallel to BC we divide side AB into $k+1$ equal parts.

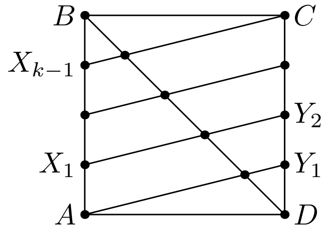

Fig. 6.

13. *(D.Cheian, final round, 9)* In triangle ABC, $\angle B = 2\angle C$. Points P and Q on the perpendicular bisector to CB are such that $\angle CAP = \angle PAQ = \angle QAB = \frac{\angle A}{3}$. Prove that Q is the circumcenter of triangle CPB.

Solution. Let D be the reflection of A in the perpendicular bisector to BC. Then $ABCD$ is the isosceles trapezoid and its diagonal BD is the bisector of angle B. Thus $CD = DA = AB$. Now $\angle DAP = \angle C + \angle A/3 = (\angle A + \angle B + \angle C)/3 = 60°$. Thus triangle ADP is equilateral and $AP = AB$. Since AQ is the bisector of angle PAB, $QP = QB = QC$ (Fig. 7).

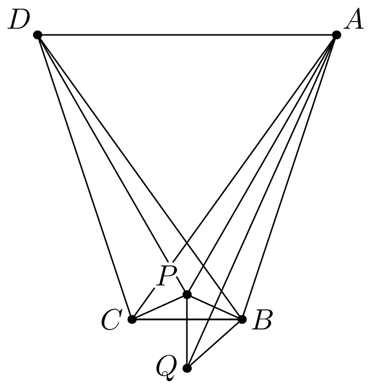

Fig. 7.

14. *(A. Zaslavsky, final round, 9)* A quadrilateral $ABCD$ is inscribed into a circle with center O. The bisectors of its angles form a cyclic quadrilateral with circumcenter I, and its external bisectors form a cyclic quadrilateral with circumcenter J. Prove that O is the midpoint of IJ.

Solution. Let the bisectors of angles A and B, B and C, C and D, D and A meet at points K, L, M, N respectively (Fig. 8). Then line KM bisects the angle formed by lines AD and BC. If this angle is equal to ϕ, then by external angle theorem we obtain that $\angle LKM = \angle B/2 - \phi/2 = (\pi - \angle A)/2 = \angle C/2$ and thus $\angle LIM = \angle C$. On the other hand, the perpendiculars from L and M to BC and CD respectively form the angles with ML equal to $(\pi - \angle C)/2$, i.e. the triangle formed by these perpendiculars and ML is isosceles and the angle at its vertex is equal to C. Thus the vertex of this triangle coincides with I. So the perpendiculars from the vertices of $KLMN$ to the corresponding sidelines of $ABCD$ pass through I. Similarly the perpendiculars from the vertices of triangle formed by external bisectors pass through J.

Now let K' be the common point of external bisectors of angles A and B. Since quadrilateral $AKBK'$ is inscribed into the circle with diameter KK', the projections of K and K' to AB are symmetric with respect to the midpoint of AB. From this and the above assertion, the projections of I and J to each side of $ABCD$ are symmetric with respect to the midpoint of this side, whci this is equivalent to the assertion.

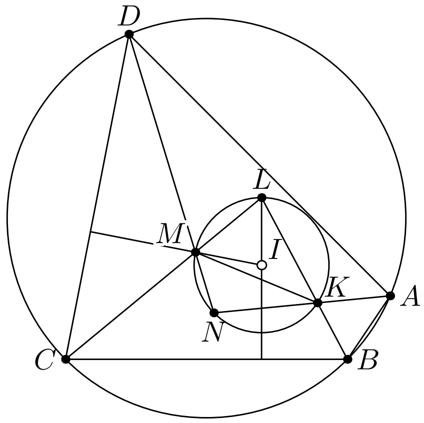

Fig. 8.

Remark. A similar property of a triangle is well-known: the circumcenter is the midpoint of the segment between the incenter and the circumcenter of the triangle formed by its external bisectors.

15. *(I. Bogdanov, final round, 9)* Circles ω and Ω are inscribed into one angle. Line ℓ meets the sides of angles, ω and Ω at points A and F, B and C, D and E respectively (the order of points on the line is A, B, C, D, E, F). It is known that $BC = DE$. Prove that $AB = EF$.

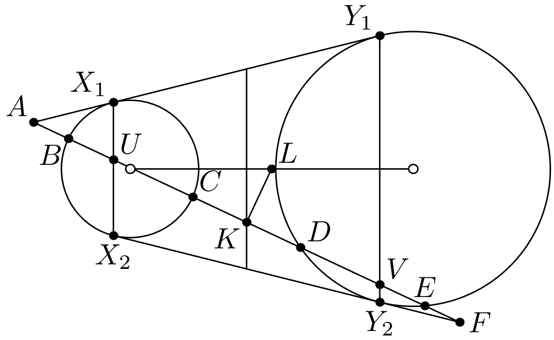

Fig. 9.

First solution. Let one side of the angle touch ω and Ω at points X_1, Y_1, and the second side touch them at points X_2, Y_2; U, V are the common points of X_1X_2 and Y_1Y_2 with AF. The midpoint of CD lies on the radical axis of the circles, i.e. the medial line of trapezoid $X_1Y_1Y_2X_2$, thus $BU = EV$ and $CU = DV$ (Fig. 9). This yields that $X_1U \cdot X_2U = Y_1V \cdot Y_2V$. Hence $FY_2/FX_2 = Y_2V/X_2U = X_1U/Y_1V = AX_1/AY_1$, i.e. $AX_1 = FY_2$. Now from $AB \cdot AC = AX_1^2 = FY_2^2 = FE \cdot FD$ we obtain the assertion of the problem.

Second solution. Prove that there exists exactly one line passing through a fixed point A on a side of the angle which satisfies the condition of the problem. In fact, the distances from the midpoint K of segment CD to the projections of the centers of the circles to the sought line are equal, thus K coincides with the projection of the midpoint L of the segment between the centers. Hence K is the common point of the circle with diameter AL and the radical axis, distinct from the midpoint of segment X_1Y_1. On the other hand, if F is a point such that $AX_1 = Y_2F$ then $AB \cdot AC = FE \cdot FD$ and $AD \cdot AE = FC \cdot FB$, thus AF is the sought line.

16. *(L. Emelyanov, final round, 10)* Quadrilateral $ABCD$ is circumscribed. Its incircle touches sides AB, BC, CD, DA in points K, L, M, N respectively. Points A', B', C', D' are the midpoints of segments LM, MN, NK, KL. Prove that the quadrilateral formed by lines AA', BB', CC', DD' is cyclic.

Solution. Let us begin with the assertion which follows by a calculation of angles.

Lemma. Points A, B, C, D lie on the same circle if and only is the bisectors of angles formed by lines AB and CD are parallel to the bisectors of angles formed by lines AD and BC.

In fact, consider the case when $ABCD$ is a convex quadrilateral, rays BA and DC meet at point E, rays DA and BC meet at point F. Then the angles between the bisectors of angles BED and BFD are equal to half-sums of opposite angles of the quadrilateral. This proves the lemma. Other cases are considered in the same way.

Now let us turn to the solution of the problem. Let I be the incenter of $ABCD$, r be the radius of its incircle. Then $IC' \cdot IA = r^2 = IA' \cdot IC$, i.e. points A, C, A', C' lie on the circle. By the lemma, the bisectors of angles between AA' and CC' are parallel to the bisectors of angles between IA and IC, and hence to the bisectors of the angles between perpendicular lines KN and LM. Similarly the bisectors of the angles between BB' and DD' are parallel to the bisectors of the angles between KL and MN. Using again the lemma we obtain the assertion of the problem.

17. *(V. Mokin, final round, 10)* Point D lies on the side AB of triangle ABC. The circle inscribed in angle ADC touches internally the circumcircle of triangle ACD. Another circle inscribed in angle BDC touches internally the circumcircle of triangle BCD. These two circles touch segment CD in the same point X. Prove that the perpendicular from X to AB passes through the incenter of triangle ABC.

Solution. Let us first prove the following auxiliary fact.

Lemma. Let a circle touch the sides AC, BC of a triangle ABC in points U, V and touch internally its circumcircle in point T. Then the line UV passes through the incenter I of triangle ABC.

Proof of Lemma. Let the lines TU, TV intersect the circumcircle for the second time at points X, Y. Since the circles ABC and TUV are homothetic with center T, points X, Y are the midpoints of arcs AC, BC, i.e. lines AY and BX meet at point I (Fig. 10). Thus the assertion of the lemma follows from Pascal theorem applied to the hexagon $AYTXBC$.

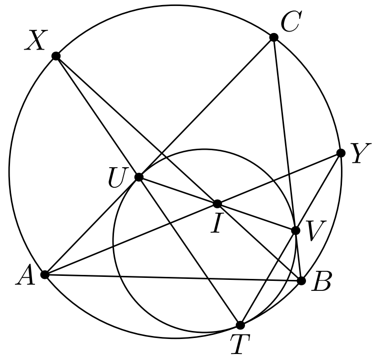

Fig. 10.

From the lemma and from the assumption it follows that DI_1XI_2 is a rectangle, where I_1, I_2 are incenters of triangles ACD and BCD respectively (Fig. 11). Let Y, C_1, C_2 be the projections of points X, I_1, I_2 to AB. Then $BY - AY = BC_2 + C_2Y - AC_1 - C_1Y = (BC_2 - DC_2) - (AC_1 - DC_1) = (BC - CD) - (AC - CD) = BC - AC$. Thus, Y is the touching point of AB with the incircle.

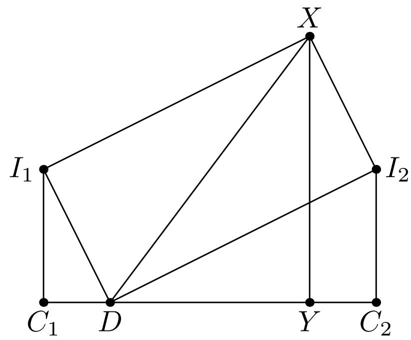

Fig. 11.

18. *(S. Tokarev, final round, 10)* Given a sheet of tin 6×6. It is allowed to bend it and to cut it so that it does not fall to pieces. How to make a cube with edge 2, divided by partitions into unit cubes?

Solution. The desired development is presented on Fig. 12. Bold lines describe the cuts, thin and dotted lines describe the bends up and down. The central 2×2 square corresponds to the horizontal partition of the cube.

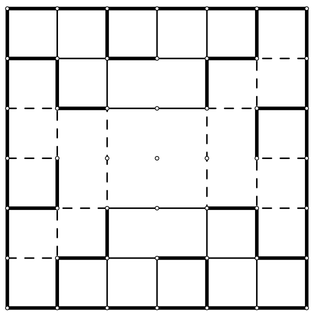
Fig. 12.