GEOMETRY OF KIEPERT AND GRINBERG–MYAKISHEV HYPERBOLAS

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Abstract. A new synthetic proof of the following fact is given: if three points A', B', C' are the apices of isosceles directly-similar triangles BCA', CAB', ABC' erected on the sides BC, CA, AB of a triangle ABC, then the lines AA', BB', CC' concur. Also we prove some interesting properties of the Kiepert hyperbola which is the locus of concurrence points, and of the Grinberg–Myakishev hyperbola which is its generalization.

This paper is devoted to the following well-known construction.

A triangle ABC is given. Let A', B', C' be the apices of three isosceles directly-similar triangles BCA', CAB', ABC'. Then the lines AA', BB', CC' concur. When we change the angles of triangles BCA', CAB', ABC', the common point of these lines moves along an equilateral hyperbola isogonally conjugated to the line OL (where O and L are the circumcenter and the Lemoine point of $\triangle ABC$). It is called the Kiepert hyperbola.

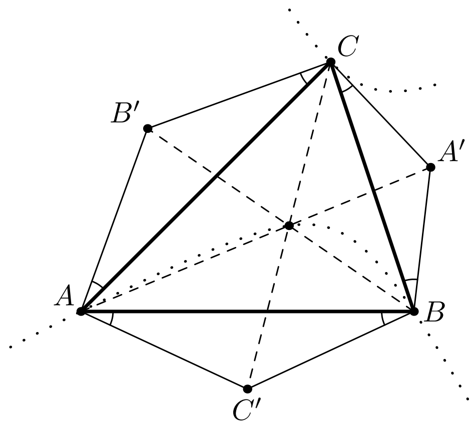

Fig. 1.

Usually this fact is proven using barycentric coordinates. A synthetic proof is based on the following simple but useful lemma.

Lemma 1. *Given two points A and B. Let f be a map sending lines passing through A to lines passing through B, and preserving the cross-ratio of the lines. Then the locus of points $l \cap f(l)$ (where l is a variable line through A) is a conic passing through A and B.*

Indeed, if X, Y, Z are three arbitrary points of this locus, then the lines l and $f(l)$ intersect the conic $ABXYZ$ in the same point (by the known fact that when

U, V, W, T are four fixed points on a conic and P is a variable point on the same conic, the cross-ratio of the lines PU, PV, PW, PT does not depend on P).

Consider now our initial construction. When we change the angles of the three isosceles triangles (keeping them directly similar), the points A' and B' move along the perpendicular bisectors of the segments BC and CA. Since the lines AB' and BA' rotate with equal velocities, the corresponding map between these perpendicular bisectors is projective. Thus, the map between the lines AA' and BB' also is projective. Using the Lemma, we obtain that the common point X of these lines moves along a conic passing through A and B. If the angle on the base of the three isosceles triangles is zero, X is the centroid M of $\triangle ABC$. If this angle is $\pi/2$, the point X becomes the orthocenter H. Also, if the angle on the base is equal to the angle $-C$, then X coincides with C. Therefore, the obtained conic is the hyperbola $ABCMH$. This hyperbola is equilateral because it passes through the vertices and the orthocenter of triangle ABC. The common point of the lines AA' and CC' moves on the same hyperbola (by a similar argument) and thus coincides with X. This proves that the lines AA', BB', CC' concur, and their point of concurrency lies on the equilateral hyperbola $ABCMH$. This hyperbola is clearly the isogonal conjugate of the line OL (since M and H are isogonally conjugate to L and O), and is known as the *Kiepert hyperbola* of triangle ABC.

Now consider some properties of the Kiepert hyperbola.

Denote by $X(\phi)$, $-\pi/2 \leq \phi \leq \pi/2$ the point on the hyperbola corresponding to isosceles triangles with base angle ϕ (where we say that $\phi > 0$ (respectively, $\phi < 0$) when the isosceles triangles are constructed on the external (respectively, internal) side of $\triangle ABC$). By the above, $X(0) = M$ and $X(\pm\pi/2) = H$. Here come some other examples.

$X(\pm\pi/3) = T_{1,2}$ are the Fermat-Torricelli points. Note that their isogonally conjugated Appollonius points (also known as the isodynamic points) are inverse to each other wrt the circumcircle of $\triangle ABC$. Thus the Torricelli points are two opposite points of the Kiepert hyperbola, i.e. the midpoint of the segment T_1T_2 is the center of the hyperbola and therefore lies on the Euler circle of $\triangle ABC$. (The Euler circle is also known as the nine-point circle.)

$X(\pm\pi/6) = N_{1,2}$. These points are called the *Napoleon points*.

The following properties of Torricelli and Napoleon points are well-known:

– the lines T_1T_2 and N_1N_2 pass through L;
– the lines T_1N_1 and T_2N_2 pass through O;
– the lines T_1N_2 and N_1T_2 pass through the center O_9 of the Euler circle.

Using these properties we obtain the following theorem.

Theorem 2. *For an arbitrary $\phi \in [-\pi/2, \pi/2]$,*
– the line $X(\phi)X(-\phi)$ passes through L;
– the line $X(\phi)X(\pi/2 - \phi)$ passes through O;
– the line $X(\phi)X(\pi/2 + \phi)$ passes through O_9.

Proof. We will prove only the first assertion of the theorem – the other two can be proven similarly. By projecting from C, we obtain a projective transformation from the Kiepert hyperbola to the perpendicular bisector of AB. This transformation transforms the points $X(\phi)$ and $X(-\phi)$ to two points symmetric with respect to the line AB. Thus the map from the hyperbola to itself which sends every $X(\phi)$ to $X(-\phi)$ is projective. Consider now the map transforming a point X to the second common point of the hyperbola with the line XL. Since these two projective maps coincide in the points T_1, T_2, N_1, and N_2 they coincide in all points of the Kiepert hyperbola. □

Remark. The properties of the Torricelli and Napoleon points we used above can be avoided if so desired. All we needed for the proof of Theorem 2 were three distinct angles ϕ for which the statement of Theorem 2 is known to hold. Using the properties of the Torricelli and Napoleon points, we found four such angles ($\pi/3$, $\pi/6$, $-\pi/3$, and $-\pi/6$), but there are three angles which can be easily seen to satisfy Theorem 2: namely, $-A$, $-B$, $-C$. (These are distinct only if $\triangle ABC$ is scalene – otherwise, a limiting argument will do the trick.) Why do these angles satisfy Theorem 2? Notice first that $X(-A) = A$. To prove that the line $X(-A)X(A)$ passes through L, show that for $\phi = A$ the apex A' is a vertex of the triangle formed by the tangents to the circumcircle of $\triangle ABC$ at A, B and C, and conclude that AX is a symmedian of $\triangle ABC$. To prove that the line $X(-A)X(\pi/2 + A)$ passes through O, notice that for $\phi = \pi/2 + A$ the apex A' is O. To prove that the line $X(-A)X(\pi/2 - A)$ passes through O_9, check that for $\phi = \pi/2 - A$ the apex A' is the reflection of A in O_9.

As a special case of Theorem 2 we obtain that the tangents to the Kiepert hyperbola in the points M and H meet in the point L. The point L thus is the pole of the line HM with respect to the Kiepert hyperbola. Therefore L is the center of the inscribed conic touching the sidelines of the triangle in the feet of its altitudes. (Here we are applying the known fact (see Theorem 4.8 in [1]) that if a conic touches the sidelines of $\triangle ABC$ at the feet of the cevians from a point P, then the center of the conic is the pole of the line PM with respect to the conic.)

Two first assertions of Theorem 2 can be generalized.

Theorem 3. *Consider the pairs of points $X(\phi_1)$, $X(\phi_2)$, where ϕ_1 and ϕ_2 are variable angles satisfying $\phi_1 + \phi_2 = \text{const}$. All such lines $X(\phi_1)X(\phi_2)$ meet the line OL at the same point.*

Proof. Denote the sum $\phi_1 + \phi_2$ by $2\phi_0$. The quadrilateral

$$X(\phi_1)X(\phi_0)X(\phi_2)X(\pi/2 + \phi_0)$$

is harmonic (this can be seen, for instance, by using the projective transformation from the Kiepert hyperbola to the perpendicular bisector of AB – in fact, the quadrilateral $X(\phi_1)X(\phi_0)X(\phi_2)X(\pi/2 + \phi_0)$ is, under this transformation, a preimage of a "quadrilateral" which is more easily seen to be harmonic, and projective transformations preserve harmonicity). Therefore, the line $X(\phi_1)X(\phi_2)$ passes through the pole of the line $X(\phi_0)X(\pi/2 + \phi_0)$ wrt the Kiepert hyperbola.

But this line, for every ϕ_0, passes through O_9 (by Theorem 2). Thus all lines $X(\phi_1)X(\phi_2)$ pass through some fixed point of the polar of O_9 which coincides with OL by Theorem 2. $\square$

Since the Kiepert hyperbola is isogonally conjugated to the line OL, the point obtained in Theorem 3 is the point $X'(\phi_3)$, isogonally conjugated to some point $X(\phi_3)$ of the hyperbola. The relation between ϕ_3 and ϕ_1, ϕ_2 can be obtained using two isogonal pairs theorem.

Let $X(\phi_1)$, $X(\phi_2)$ be two points of the Kiepert hyperbola and $X'(\phi_1)$, $X'(\phi_2)$ be their respective isogonal conjugates on the line OL. By the two isogonal pairs theorem (Corollary from Theorem 3.16 in [1]), the lines $X(\phi_1)X'(\phi_2)$ and $X'(\phi_1)X(\phi_2)$ meet on the hyperbola. By Theorem 3 the corresponding angle is equal to $f(\phi_1)-\phi_2$. On the other hand it is equal to $f(\phi_2)-\phi_1$, where f is some unknown function[1]. Therefore $f(\phi_1)+\phi_1 = f(\phi_2)+\phi_2 = \text{const}$. Taking $\phi_2 = 0$ we obtain $f(\phi) = -\phi$. Thus we can formulate the following Theorem:

Theorem 4. *The three points $X(\alpha)$, $X(\beta)$, $X'(\gamma)$ are collinear iff $\alpha+\beta+\gamma$ is an integer multiple of π.*

An interesting partial case is obtained when $X(\alpha)$ and $X(\beta)$ are the two infinite points of the hyperbola. Then $X(\gamma)$ is isogonally conjugate to the infinite point of the line OL, i.e. it coincides with the fourth common point of the Kiepert hyperbola and the circumcircle. Thus the sum of the angles corresponding to this point and the two infinite points is an integer multiple of π.

Finally note that triangle $A'B'C'$ is not only perspective to ABC, but orthologic to it with orthology center O. The locus of second orthology centers is also the Kiepert hyperbola and this fact can be formulated elementarily.

Theorem 5. *Let $AB'C$ and $CA'B$ be similar isosceles triangles with angles on the bases CA and BC equal to ϕ. Let the perpendicular from C to $A'B'$ meet the perpendicular bisector of AB in point C_1. Then $\angle AC_1B = 2\phi$ and $\angle C_1BA = \angle BAC_1 = \pi/2-\phi$.*

Using this Theorem 5, we immediately see that the point $X(\pi/2-\phi)$ is the orthology center.

Proof. Let A'', B'' be the images of A', B' under the homothety with center C and coefficient 2. Then $\angle CAB'' = \angle CBA'' = \pi/2$ and $A''B'' \parallel A'B'$. Let C_2 be the common point of CC_1 and $A''B''$. Since the quadrilateral CC_2BA'' is inscribed into the circle with diameter CA'', we have $\angle BC_2A'' = \angle BCA'' = \phi$. Similarly $\angle AC_2B'' = \phi$. Thus $A''B''$ is the exterior bisector of angle AC_2B. By orthogonality, C_2C_1 must thus be the bisector of angle AC_2B, and its common point with the perpendicular bisector of AB must lie on the circumcircle of ABC_2

[1] Let $X'(\phi)$ be an arbitrary point of the line OL and $X(\phi_1)$, $X(\phi_2)$ be two common points of the Kiepert hyperbola and some line l passing through $X'(\phi)$. Then by Theorem 3 $\phi_1+\phi_2$ depends only on ϕ and not on l. We denote the corresponding function by $f(\phi)$. By Theorem 2 $f(0)=0$.

(Fig. 2). Therefore $\angle AC_1B = \angle AC_2B = 2\phi$ and $\angle C_1BA = \angle BAC_1 = \pi/2 - \phi$. □

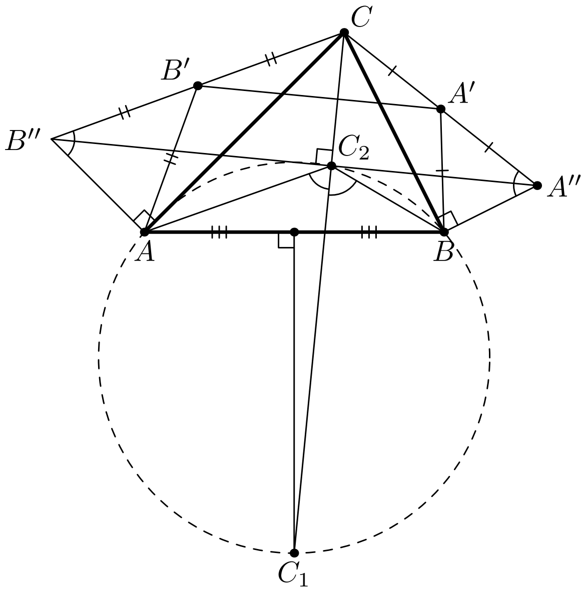

Fig. 2.

D. Grinberg and A. Myakishev [2] found the following generalization of the Kiepert hyperbola.

Theorem 6. *Let $A_1B_1C_1$ be the cevian triangle of a point P with respect to $\triangle ABC$. Let AC_aC_1, C_1C_bB, BA_bA_1, A_1A_cC, CB_cB_1, B_1B_aA be isosceles directly-similar triangles with bases AC_1, C_1B, BA_1, A_1C, CB_1, B_1A. Then, the triangle formed by the lines A_bA_c, B_cB_a, C_aC_b is perspective to $\triangle ABC$.*

If the point P is fixed and the angle ϕ on the bases of isosceles triangles changes, then the perspectivity center moves along some circumconic of $\triangle ABC$.

The authors proposed only an analytic proof of this assertion. The following synthetic proof was found recently.

Proof of Theorem 6. First note that the perspectivity of triangles follows from the Desargues theorem. In fact the ratio of distances from the points C_a and C_b to the line AB is equal to AC_1/BC_1 and does not depend on ϕ. Thus the common point C_2 of the lines AB and C_aC_b also does not depend on ϕ. Defining similarly points A_2 and B_2, we obtain from Ceva and Menelaos theorems that A_2, B_2, C_2 are collinear if the lines AA_1, BB_1, and CC_1 concur (in fact, homothetic triangles yield $\frac{BA_1}{A_1C} = \frac{BA_b}{A_1A_c} = \frac{BA_2}{A_1A_2}$ and similarly $\frac{CA_1}{A_1B} = \frac{CA_2}{A_1A_2}$; dividing these two equalities by each other results in $\left(\frac{BA_1}{A_1C}\right)^2 = \frac{BA_2}{CA_2}$, and similar equalities can be found by cyclic shifting). Therefore it is sufficient to prove that the common point of the lines AA' and BB' moves along some conic, where A', B', C' are the vertices of the triangle formed by A_bA_c, B_cB_a, C_aC_b.

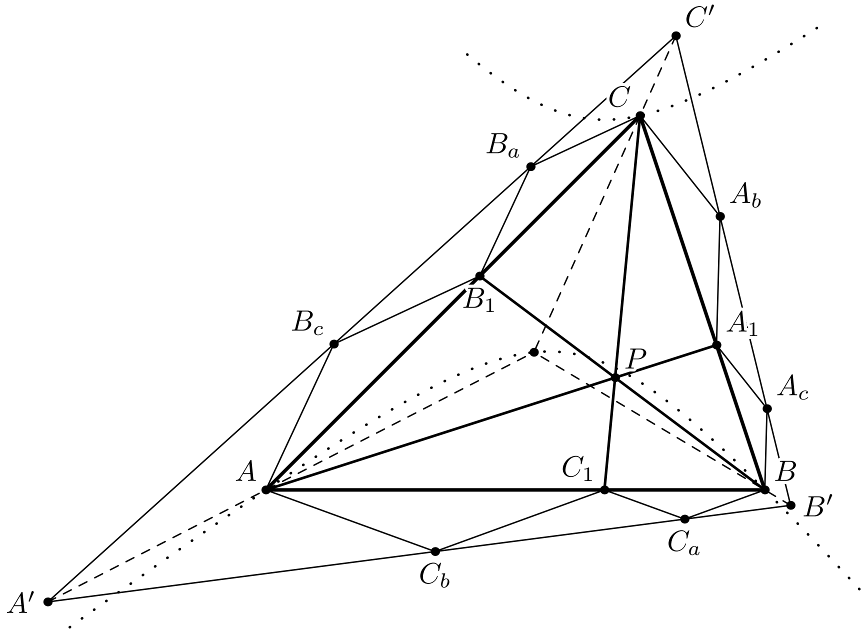

Fig. 3.

Let us find the trajectory of the point A'. This point lies on the lines B_cB_a and C_aC_b passing through the fixed points B_2 and C_2, respectively. These lines also pass through the points B_a and C_a moving on the perpendicular bisectors of the segments AB_1 and AC_1. It is evident that the obtained map between these bisectors is projective, thus by Lemma 1 the point A' moves along some conic α passing through B_2 and C_2. This conic also passes through A, because when $\phi = 0$ the points A and A' coincide. Similarly the point B' moves along a conic β passing through A_2, C_2 and B.

Now, note that the map between A' and B_a is the projection of α from C_2 to the perpendicular bisector of AB_1. Thus this map is projective. Similarly the map between B' and A_b is projective. Therefore the map between A' and B' is a projective map between the conics α and β. Since A lies on α and B lies on β, the map between the lines AA' and BB' is also projective. Using Lemma 1 we obtain that the common point of these lines moves on some conic passing through A and B. Similarly we obtain that this conic passes through C.

$\square$

Grinberg and Myakishev proved by calculations that this conic always is a hyperbola. They also obtained some results concerning the dependence of this hyperbola on P. The main part of their results is very complicated. Therefore we consider only one interesting partial case.

Statement 7. *If P is the orthocenter of $\triangle ABC$, then the Kiepert and Grinberg–Myakishev hyperbolas coincide.*

Proof. It is sufficient to find two common points of these hyperbolas distinct from A, B, C. We are going to prove that $Y\left(\frac{\pi}{4}\right) = X\left(-\frac{\pi}{4}\right)$ and $Y\left(-\frac{\pi}{4}\right) = X\left(\frac{\pi}{4}\right)$, where $Y(\phi)$ is the point of the Grinberg–Myakishev hyperbola corresponding to the angle ϕ.

Let AA_1, BB_1 be the altitudes of $\triangle ABC$, and let AB_aB_1, B_1B_cC, CA_cA_1, A_1A_bB be isosceles right-angled triangles with apices lying on the external side of $\triangle ABC$; let AC_0B be an isosceles right-angled triangle with apex lying on the internal side of $\triangle ABC$. We have to prove that the lines B_cB_a, A_bA_c and CC_0 concur.

Since the points A_1 and C_0 lie on the circle with diameter AB, $\angle CA_1C_0 = \angle BAC_0 = \frac{\pi}{4}$. Thus, the points A_b, A_1, C_0 are collinear and $C_0A_b \parallel CA_c$. Similarly $C_0B_a \parallel CB_c$ (Fig. 4). Also it is clear that

$$\frac{C_0A_b}{C_0B_a} = \frac{\sin\angle C_0BA_b}{\sin\angle C_0AB_a} = \frac{\sin B}{\sin A} = \frac{CA_1}{CB_1} = \frac{CA_c}{CB_c}.$$

Therefore triangles CA_cB_c and $C_0A_bB_a$ are homothetic, which yields $Y\left(\frac{\pi}{4}\right) = X\left(-\frac{\pi}{4}\right)$. Similarly we obtain the second equality. □

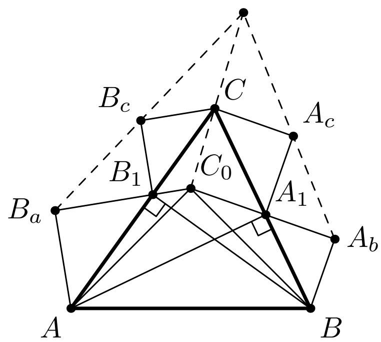

Fig. 4.

Acknowledgements

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References

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