TWO APPLICATIONS OF A LEMMA ON INTERSECTING CIRCLES

VLADIMIR N. DUBROVSKY

Abstract. A useful property of the direct similitude that maps one of two intersecting circles on another and fixes their common point is applied to the configuration consisting of a triangle, its circumcircle, and a circle through its vertex and the feet of its two cevians.

This paper emerged from a discussion in *Hyacinthos* problem solving group at Yahoo started by Luis Lopes [3], which ended up with two theorems about the configuration consisting of a triangle, its circumcircle, and a circle through its vertex and the feet of its two cevians. The proofs of these theorems are given below preceded by a simple, but useful lemma on the direct similitude defined by two intersecting circles.

Lemma. *Let a and b be two intersecting circles and let P and Q be their common points. Then there is a unique direct similitude f with fixed point P that maps a to b, and for any X on a, the line XY, where $Y = f(X)$, passes through Q.*

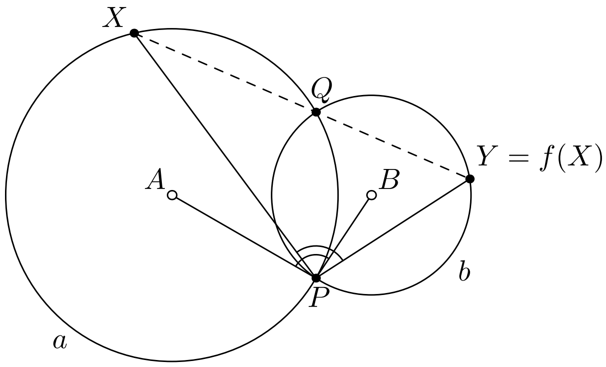

Fig. 1.

Proof. If A and B are the centers of the circles (Fig. 1), then, obviously, f can be represented as scaling by factor PB/PA with respect to P followed by the rotation through $\angle APB$ about P. Then all the triangles PXY are directly similar to triangle PAB, and therefore, the (signed) angle PXY is constant mod π; hence, by the Inscribed Angle Theorem, the second intersection point of XY and a is fixed. But for $X = f^{-1}(Q)$ this point is Q. □

Theorem 1. *Let AA_1, BB_1, and CC_1 be three cevians in a triangle ABC concurrent at P and let ω and α be the circumcircles of triangles ABC and AB_1C_1*

(Fig. 2). Denote by D the second intersection point of α and ω, and extend DA_1 to meet ω again at N. Then AN bisects B_1C_1.

Remark 1. Originally, this fact was reported by A. Zaslavsky [4] in the particular case where P is the triangle's orthocenter.

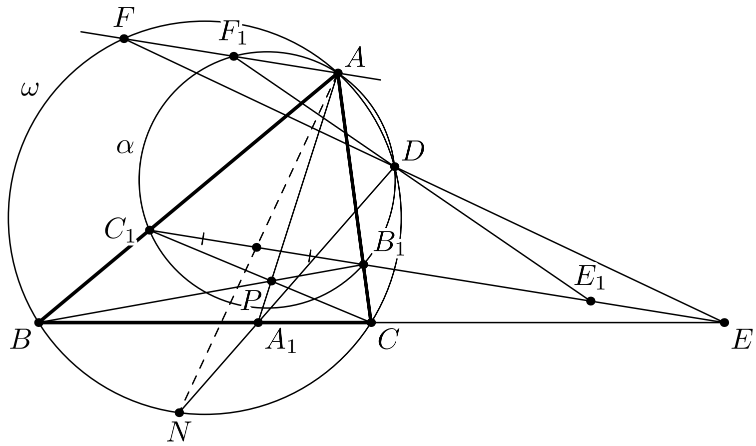

Fig. 2.

Proof. It will suffice to show that the line $n = AN$ is the harmonic conjugate, with respect to $b = AC$ and $c = AB$, of the line through A parallel to B_1C_1. Denote by E the meet of lines BC and B_1C_1 and by F, the second intersection of DE and ω. Since the lines DB, $DN = DA_1$, DC, and DE make a harmonic quadruple (because this is the fact for B, A_1, C, and E), the same is true for the lines AB, AN, AC, and AF. So it only remains to show that AF is parallel to B_1C_1. This can be done by means of the lemma given above.

Indeed, consider the direct similitude d that fixes D and takes ω to α. By the lemma, points B and C are taken by d to C_1 and B_1, respectively, F is taken to the second intersection point F_1 of AF and α, and E is taken to some point E_1. Since E is the meet of BC and FD, point E_1 is the meet of B_1C_1 and F_1D. By the definition of E_1 and F_1, triangles DFF_1 and DEE_1 are directly similar, hence the lines $FF_1(= AF)$ and $EE_1(= B_1C_1)$ are parallel. □

For P satisfying a special condition, we have an additional property of the same configuration.

Theorem 2. *In the setting of Theorem 1, assume that P is concyclic with A, B_1 and C_1. Then the line DP bisects the side BC.*

Proof. Let us draw a third circle γ, the circumcircle of triangle PBC (Fig. 3). We'll consider the composition m of the direct similitude d from the proof of Theorem 1 and another direct similitude q with center Q, the second intersection point of α and γ, which takes α to γ. By the Lemma, we have the following diagrams:

$$B \xrightarrow{d} C_1 \xrightarrow{q} C,\ \ C \xrightarrow{d} B_1 \xrightarrow{q} B,\ \ D \xrightarrow{d} D \xrightarrow{q} D_1,$$

where D_1 is the second intersection point of DP and γ. So m swaps B and C; hence, being a direct similitude, m is the reflection in the midpoint M of BC. It follows that the line $DP = DD_1$ passes through M. □

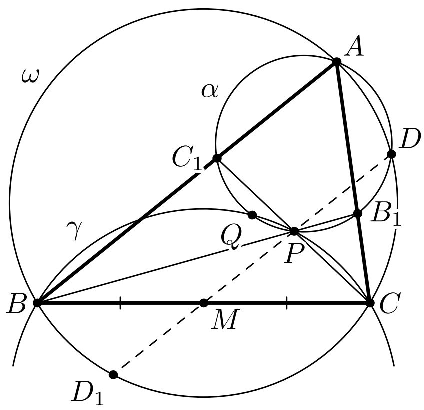

Fig. 3.

Remark 2. Obviously, the circle γ depends only on the triangle, not on point P, and is congruent to ω; it is known well that it passes through the orthocenter H. Thus, the point P in the theorem is, in fact, any point of the circumcircle of triangle BCH. In the original question posed by Lopes, P was just the orthocenter.

References

[1] V. N. Dubrovsky. Hyacinthos messages 20020, 20026, 20030, May 14, 16, 17 2011.
[2] F. Javier. Hyacinthos messages 20016, May 13 2011.
[3] L. Lopes. Hyacinthos messages 20014, May 14 2011.
[4] A. Zaslavsky. Hyacinthos messages 20025, May 16 2011.

E-mail address: vndubrovsky@gmail.com

The Kolmogorov School of Moscow State University