ON CERTAIN TRANSFORMATIONS PRESERVING PERSPECTIVITY OF TRIANGLES

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Abstract. To any pair of perspective triangles we assign a family F of projective transformations such that image of the second triangle under any transformation from F is perspective to the first triangle. This helps us to solve some interesting problems.

All constructions under consideration are on the complex projective plane. We abbreviate "projective transformation" as PT and "affine transformation" as AT. We denote by H_O^c the homothety with center O and factor c.

Let ABC and $A'B'C'$ be perspective triangles with the perspector O and the perspectrix ℓ. We try to find some good PT taking ABC to $A'B'C'$. For $\ell = \infty$, this is the homothety H_O^c for some complex number c. Now let us generalize this to arbitrary ℓ. A PT is called a *phomothety* if it takes each point P to the point P' such that O, P and P' are collinear and $(P', P, O, K) = c$, where $K = \ell \cap OP$. It is easy to see that, for $\ell = \infty$, the phomothety is the homothety H_O^c. It is clear that $f^{-1} \circ H^c_{f(O),f(\ell)} \circ f = H^c_{O,\ell}$ for any PT f. For a PT f preserving O and taking ℓ to ∞, we have $f^{-1} \circ H_O^c \circ f = H^c_{O,\ell}$.

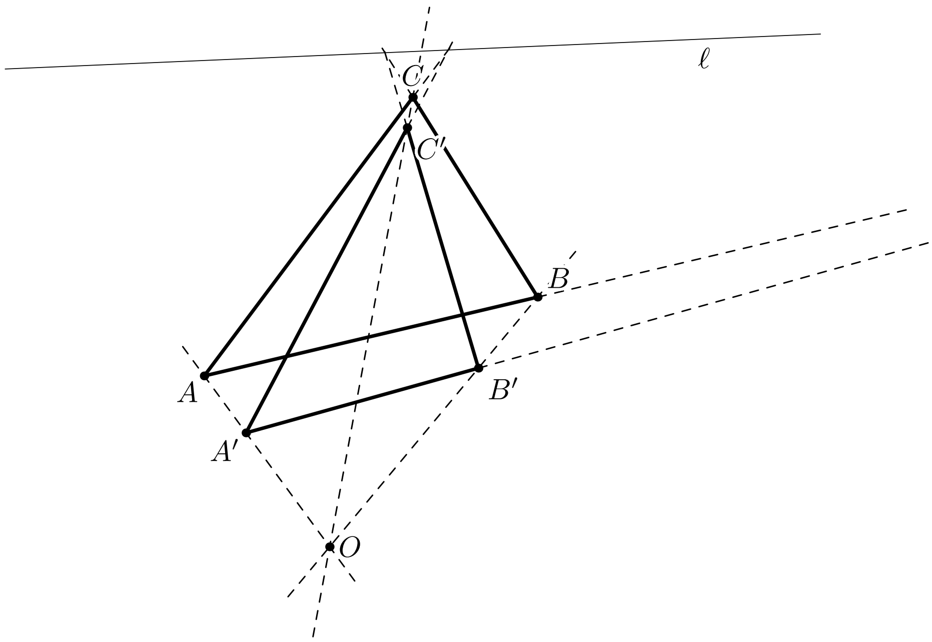

Fig. 1.

We can prove the following properties of phomothety:
1) If $A'B'C' = H^c_{O,\ell}(ABC)$ then $A'B'C'$ and ABC are perspective with perspector O and perspectrix ℓ.

We have $A'B'C' = f^{-1}(H^c_O(f(ABC)))$ and $f(A'B'C') = H^c_O(f(ABC))$, hence $f(ABC)$ and $f(A'B'C')$ are perspective with perspector O and perspectrix ∞, so ABC and $A'B'C'$ are perspective with perspector $f^{-1}(O) = O$ and perspectrix $f^{-1}(\infty) = \ell$.
2) If $A'B'C'$ and ABC are perspective with perspector O and perspectrix ℓ then $A'B'C' = H^c_{O,\ell}(ABC)$ for some complex number c.

The triangles $f(ABC)$ and $f(A'B'C')$ are perspective with perspector $f(O) = O$ and perspectrix $f(\ell) = \infty$, hence for some complex c we have $f(A'B'C') = H^c_O(f(ABC))$ and $A'B'C' = H^c_{O,\ell}(ABC)$.
3) $H^c_{O,\ell} \circ H^d_{O,\ell} = H^{cd}_{O,\ell}$

$$H^{cd}_{O,\ell} = f^{-1} \circ H^{cd}_O \circ f = f^{-1} \circ H^c_O \circ H^d_O \circ f = f^{-1} \circ H^c_O \circ f \circ f^{-1} \circ H^d_O \circ f = H^c_{O,\ell} \circ H^d_{O,\ell}.$$

Suppose that ω is a conic, A is a point, and $H^c_{O,\ell}$ is a phomothety with ℓ being the polar of O in ω. Let m be the polar of A in ω and m' be the polar of A in $\psi = H^c_{O,\ell}(\omega)$ (ψ is obviously a conic, since $H^c_{O,\ell}$ is a PT). How are m and m' related?

Lemma 1. $m' = H^{c^2}_{O,\ell}(m)$.

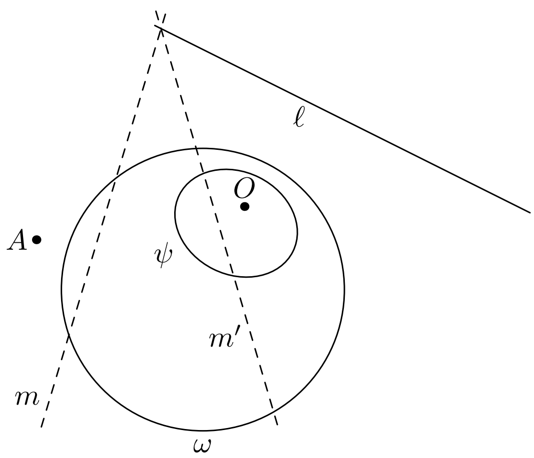

Fig. 2.

Proof. Indeed, find a PT f_1 taking ℓ to ∞, and an AT f_2 making $f_1(\omega)$ a circle, both of them preserving O and A. Denote the composition $f_2 \circ f_1$ by f. We have $f(\ell) = \infty$, $f(O) = O$, $f(O)$ is polar of $f(\ell)$ in $f(\omega)$, therefore O is the center of $f(\omega)$. Since $f(\ell) = \infty$, we have $H^c_{O,\ell} = f^{-1} \circ H^c_O \circ f$, $\psi = H^c_{O,\ell}(\omega)$, hence $f(\psi) = H^c_O(f(\omega))$. Thus $f(\psi)$ and $f(\omega)$ are concentric circles, $f(m)$ is the polar of A in $f(\omega)$, $f(m')$ is the polar of A in $f(\psi)$.

It is clear that $f(m') = H^{c^2}_O(f(m))$, so $m' = H^{c^2}_{O,\ell}(m)$. □

This means that we can easily perform some phomotheties and watch how the polars of some points vary.

The following well-known lemma gives us a pair of perspective triangles in a different way:

Lemma 2. *Let ω be a conic and ABC be a triangle. Denote by A', B', C' the poles of BC, CA, AB in ω. Then ABC and $A'B'C'$ are perspective, and their perspectrix is the polar of their perspector in ω.*

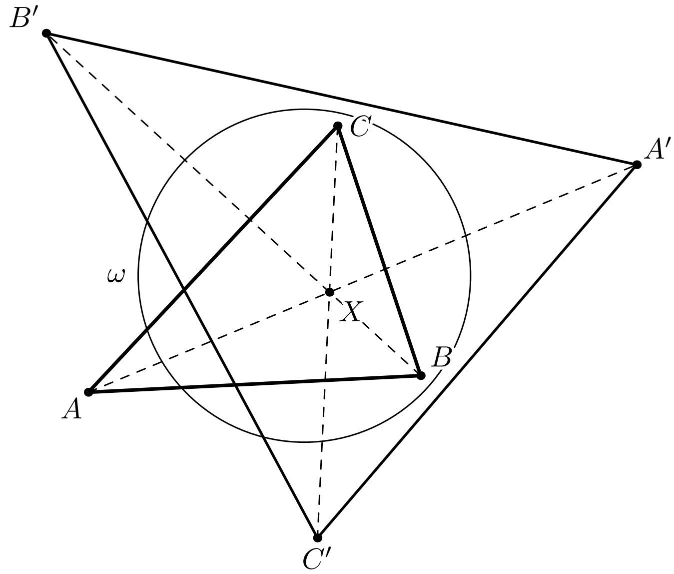

Fig. 3.

Proof. Let $X = AA' \cap BB'$. Applying a PT make ω a circle; after this do another PT preserving ω and taking X to the center of ω. We have $XA' \perp BC$, and X, A, A' are collinear. Similarly, we have $AX \perp BC$, $BX \perp CA$, therefore X is the orthocenter of ABC, hence $CX \perp AB$. Since $XC' \perp AB$, the points C, C', X are collinear. This means that ABC and $A'B'C'$ are perspective, and their perspectrix ∞ is the polar of their perspector X in ω. □

Now we are ready to prove the following theorem:

Theorem 3. *Let ω be a conic and ABC a triangle. Let A', B', C' be the poles of BC, CA, AB in ω. Let $A_1B_1C_1$ be the triangle perspective to $A'B'C'$ with perspector D and perspectrix ℓ, where ℓ is the polar of D in ω. Then ABC and $A_1B_1C_1$ are perspective.*

Proof. From perspectivity of $A_1B_1C_1$ and $A'B'C'$ we have $A_1B_1C_1 = H^{c}_{D,\ell}(A'B'C')$ for some complex c. Let $\psi = H^{\sqrt{c}}_{D,\ell}(\omega)$. Since $A'B'$, $B'C'$, $C'A'$ are the polars of C, A, B in ω, then from Lemma 1 we have that A_1B_1, B_1C_1, C_1A_1 are the polars of C, A, B in ψ. Therefore from Lemma 2 we conclude that ABC and $A_1B_1C_1$ are perspective. □

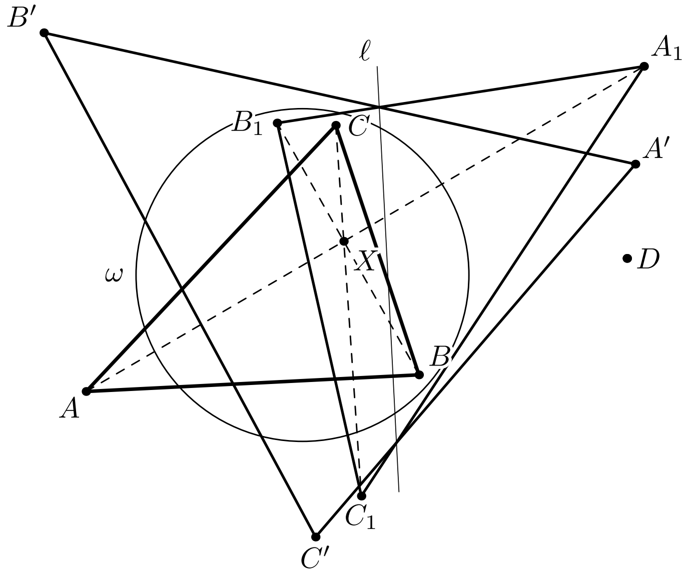

Fig. 4.

Theorem 4. *Suppose that ω is a conic, A_1, A_2, ..., A_n is a collection of points and ℓ_1, ℓ_2, ..., and ℓ_n are their polars in ω. Define the following transformation: choose any pair A_i, ℓ_i and a complex number c; for every j, the new ℓ_j is defined as the image of old ℓ_j under $H^c_{A_i,\ell_i}$, but A_j does not change. Apply this transformation any number of times. Then $A_xA_yA_z$ and triangle formed by ℓ_x, ℓ_y, ℓ_z are perspective for any x, y, z.*

Proof. Let us change ω after every transformation. The new ω is defined as the image of the old ω under $H^{\sqrt{c}}_{A_i,\ell_i}$. Then, as in Theorem 3, the new ℓ_j will be the polar of A_j in the new ω for every j. Thus, after all transformations, ℓ_x, ℓ_y, ℓ_z are the polars of A_x, A_y, A_z in the new ω. Therefore $A_xA_yA_z$ is perspective to the triangle formed by ℓ_x, ℓ_y, ℓ_z. □

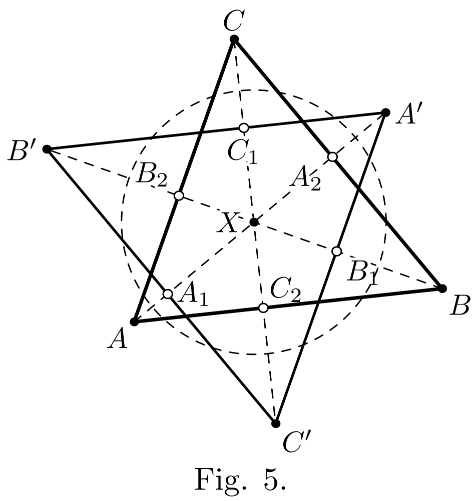

Fig. 5.

Why do we need a conic in Theorem 4? A conic allows us to use Lemma 2. What if we keep only this lemma and forget about the conic? Let us see.

Lemma 5. *If ABC and $A'B'C'$ are perspective, then A', B', C' are the poles of BC, CA, AB with respect to some conic ω.*

Proof. Applying a PT, take the perspectrix of ABC and $A'B'C'$ to ∞. After that, by suitable AT, make $AX \perp BC$ and $BX \perp AC$, where X is the perspector of ABC and $A'B'C'$. Then X is the orthocenter of ABC. Denote $A_1 = AX \cap B'C'$, $B_1 = BX \cap C'A'$, $C_1 = CX \cap A'B'$; $A_2 = AX \cap BC$, $B_2 = BX \cap CA$, $C_2 = CX \cap AB$. The perspectrix of ABC and $A'B'C'$ is ∞, hence $A'B'C' = H_X^c(ABC)$. We know that $\overline{XA}\cdot\overline{XA_2} = \overline{XB}\cdot\overline{XB_2} = \overline{XC}\cdot\overline{XC_2}$. From homothety we have $\overline{XA_1} = c\overline{XA_2}$, $\overline{XB_1} = c\overline{XB_2}$, $\overline{XB_1} = c\overline{XB_2}$, so $\overline{XA} \cdot \overline{XA_1} = \overline{XB} \cdot \overline{XB_1} = \overline{XC} \cdot \overline{XC_1}$. Let ω be the circle with center X and radius $r = \sqrt{\overline{XA} \cdot \overline{XA_1}}$. Then we have $AX \perp B'C'$ and $\overline{XA}\cdot\overline{XA_1} = r^2$, hence $B'C'$ is polar of A in ω. Therefore A', B', C' are poles of BC, CA, AB in ω. Thus, the preimage of ω with respect to the performed transformations is the desired conic. □

For $n > 2$, call a collection of n points $A_1, A_2, \ldots, A_n$ and n lines $\ell_1, \ell_2, \ldots, \ell_n$ *good*, if

1) A_i, A_j, A_k are not collinear for any i, j and k
2) ℓ_i, ℓ_j, ℓ_k are not concurrent for any i, j and k
3) $A_xA_yA_z$ and the triangle formed by ℓ_x, ℓ_y, ℓ_z are perspective for any x, y, z.

Call a collection of n points $A_1, A_2, \ldots, A_n$ and n lines $\ell_1, \ell_2, \ldots, \ell_n$ *good+*, if they are good and $A_i \notin \ell_j$ for any i, j.

Theorem 6. *If n points $A_1, A_2, \ldots, A_n$ and n lines $\ell_1, \ell_2, \ldots, \ell_n$ are good+, then $\ell_1, \ell_2, \ldots, \ell_n$ are the polars of $A_1, A_2, \ldots, A_n$ in some conic ω.*

Proof. For $n = 3$ the theorem is proved. Set $A = A_1$, $\ell_A = \ell_1$, $B = A_2$, $\ell_B = \ell_2$, $C = A_3$, $\ell_C = \ell_3$, D any point different from $\{A_i\}$, and ℓ_D the corresponding line. Let ω be a conic such that A, B, C are the poles of ℓ_A, ℓ_B, ℓ_C in ω. Let ℓ' be the polar of D in ω. We know, that ABD is perspective to the triangle formed by ℓ_A, ℓ_B, ℓ_D; ABD and the triangle formed by ℓ_A, ℓ_B, ℓ' are perspective as well. Both pairs have the same perspectrix ℓ_1: indeed, both perspectrixes pass through $AD \cap \ell_B$ and $BD \cap \ell_A$; or $AD \cap \ell_B = BD \cap \ell_A$. In the first case $AB \neq \ell_1$, else $AD \cap \ell_B \in AB$, $A \in \ell_B$, but this is false; then AB, ℓ_D, ℓ' are concurrent since $AB \cap \ell_1 \in \ell_D$, $AB \cap \ell_1 \in \ell'$; in the second case $\ell_A \cap \ell_B = D$, but $D \notin \ell_A$. Similarly for the triangles BCD and CAD. Thus, $\ell_D \cap \ell'$ have at least three points: $AB \cap \ell_D$, $BC \cap \ell_D$, $CA \cap \ell_D$, therefore $\ell' = \ell_D$, and ω is the needed conic. □

Theorem 7. *For a good+ collection of n points $A_1, A_2, \ldots, A_n$ and n lines $\ell_1, \ell_2, \ldots, \ell_n$, define the following tranformation: choose any pair A_i, ℓ_i and a complex number c; for every j, the new ℓ_j is image of the old ℓ_j under $H^c_{A_i,\ell_i}$, but A_j does not change. Apply this transformation any number of times. Then $A_1, A_2, \ldots, A_n$ and $\ell_1, \ell_2, \ldots, \ell_n$ are good.*

Proof. From Theorem 6 we have a conic ω such that each ℓ_i is the polar of A_i in ω. So, from Theorem 4 we have that, after all transformations, $A_xA_yA_z$ and the triangle formed by ℓ_x, ℓ_y, ℓ_z are perspective. Thus, A_1, A_2, ..., A_n and ℓ_1, ℓ_2, ..., ℓ_n are good. □

Now let us explore Theorem 4 for $n = 4$. First, we prove the following lemma:

Lemma 8. *In the notation of Theorem 3, the locus of perspectors of ABC and $A_1B_1C_1$ with fixed A, B, C, D and ω and variable $A_1B_1C_1$ is the conic $\psi = \psi(A, B, C, D, \omega)$ through A, B, C, D and the perspector of ABC and $A'B'C'$.*

Proof. Let X be the perspector of ABC and $A_1B_1C_1$. We know that $A_1B_1C_1 = H^c_{D,\ell}(A'B'C')$. If $c = 0$ then $A_1 = B_1 = C_1 = D$, hence $X = D$. If $c = 1$, then $A_1 = A'$, $B_1 = B'$, $C_1 = C'$, hence $X = X'$, where X' is the perspector of ABC and $A'B'C'$. So D and X' are in the locus. Apply a PT taking ℓ to ∞, and, after that, an AT making ω a circle. Then D is center of that circle. Let us find a triangle $A_1B_1C_1$ such that ABC and $A_1B_1C_1$ are perspective with perspector A. Set $B'D \cap AB = B_1$, $C'D \cap AC = C_1$. We know $C'D \perp AB$, $B'D \perp AC$, hence $C_1D \perp AB_1$, $B_1D \perp AC_1$, and therefore $AD \perp B_1C_1$. On the other hand, $AD \perp B'C'$, hence $B_1C_1 \parallel B'C'$, and we can find A_1 such that $A_1B_1C_1 = H^c_D(A'B'C')$. Since $BB_1 \cap CC_1 = A$, we conclude that A is the perspector of ABC and $A_1B_1C_1$.

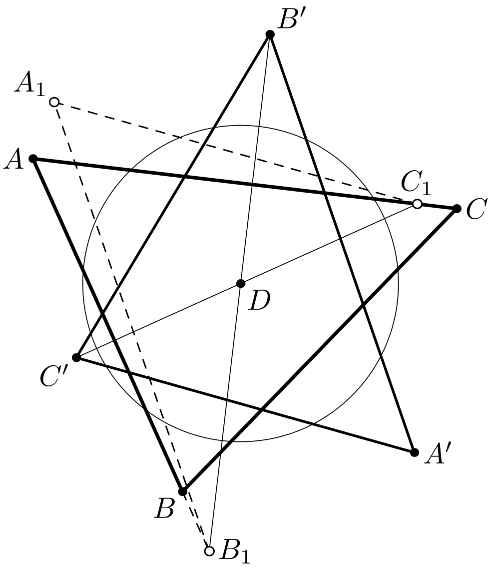

Fig. 6.

Set $X_A = A_1$, $Y_A = B_1$, $Z_A = C_1$. Similarly define X_B, Y_B, Z_B, X_C, Y_C, Z_C. Thus A, B and C are in the locus. Now let us show that the locus is a conic. We have $A_1B_1C_1 = H^c_D(A'B'C')$ for some c. AA_1, BB_1, CC_1 meet ψ second time at A_2, B_2, C_2. (C, D, X', A_2) (this cross-ratio is on ψ) $=$ (AC, AD, AX', AA_2). $(AC, AD, AX', AA_2) = (AX_C, AD, AA', AA_1)$ since the lines coincide. $(AX_C, AD, AA', AA_1) = (X_C, D, A', A_1)$ since these 4 points are collinear.

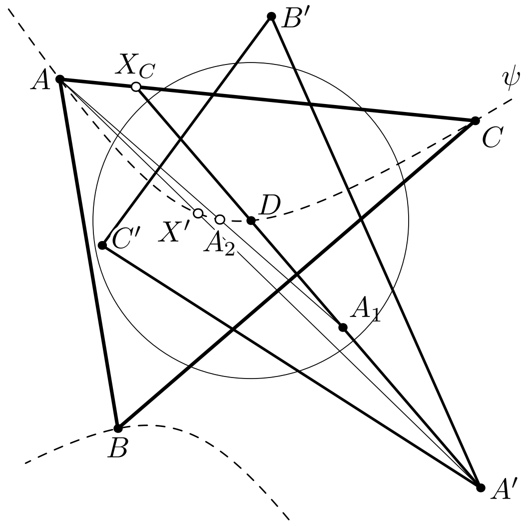

Fig. 7.

Similarly (C, D, X', B_2) (this cross-ratio is on ψ) $= (Y_C, D, B', B_1)$. On the other hand, $X_C Y_C Z_C$, $A'B'C'$ and $A_1B_1C_1$ are pairwise homothetic with the center D, hence $(X_C, D, A', A_1) = (Y_C, D, B', B_1)$ and therefore $(C, D, X', A_2) = (C, D, X', B_2)$. Hence we have $A_2 = B_2$ and, similarly, $A_2 = C_2$, so $A_2 \in AA_1, BB_1, CC_1$, $A_2 = X$, $A_2 \in \psi$, and therefore $X \in \psi$. Note that (X_C, D, A', A_1) can be any complex number, so (C, D, X', X) can be any complex number. Thus the locus of X is ψ. $\square$

Lemma 9. *In the notation of Lemma 8, we have*

$$\psi(A, B, C, D, \omega) = \psi(A, B, C, D, H^c_{D,\ell}(\omega)).$$

Proof. $\psi(A, B, C, D, \omega)$ is the conic through A, B, C, D and X', the perspector of ABC and $A'B'C'$. Clearly, $A, B, C, D \in \psi(A, B, C, D, H^c_{D,\ell}(\omega))$; let us prove that $X' \in \psi(A, B, C, D, H^c_{D,\ell}(\omega))$. From Lemma 1, we have $A_2 = H^{c^2}_{D,\ell}(A')$, $B_2 = H^{c^2}_{D,\ell}(B')$, $C_2 = H^{c^2}_{D,\ell}(C')$ are the poles of BC, CA, AB in $H^c_{D,\ell}(\omega)$. From Lemma 8 we see that the perspector of ABC and $H^{1/c^2}_{D,\ell}(A_2)H^{1/c^2}_{D,\ell}(B_2)H^{1/c^2}_{D,\ell}(C_2)$ are in $\psi(A, B, C, D, H^c_{D,\ell}(\omega))$. But $H^{1/c^2}_{D,\ell}(A_2) = H^{1/c^2}_{D,\ell}(H^{c^2}_{D,\ell}(A')) = A'$ and, similarly, $H^{1/c^2}_{D,\ell}(B_2) = B'$, $H^{1/c^2}_{D,\ell}(C_2) = C'$. Hence, $X' \in \psi(A, B, C, D, H^c_{D,\ell}(\omega))$ and $\psi(A, B, C, D, \omega) = \psi(A, B, C, D, H^c_{D,\ell}(\omega))$. $\square$

Lemma 10. *For a conic ω and points A, B, C, D, let A' be the perspector of BCD and the triangle formed by the poles of CD, DB, BC in ω; the points B', C', D' are defined similarly. Then the points A, B, C, D, A', B', C', D' belong to the same conic.*

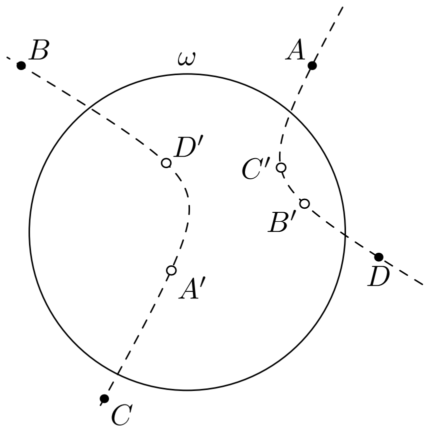

Fig. 8.

Proof. It suffices to check that A, B, C, D, A', B' belong to the same conic. Applying a PT, make ω a circle; after that apply another PT, preserving ω and taking D to the center of ω. Denote by X the pole of CD in ω. D is the center of ω, hence X is point on ∞, and $PX \perp CD$ for any point P. We have $X \in BA'$, hence $BA' \perp CD$. Similarly, $CA' \perp BD$. Thus, A' is the orthocenter of the triangle BCD. Similarly, B' is the orthocenter of the triangle ACD. Let δ be a rectangular hyperbola passing through A, B, C and D. Then it passes through the orthocenters of the triangles BCD and ACD, hence A, B, C, D, A', B' lie on δ. □

Lemma 11. *$\psi(A, B, C, D, \omega)$ does not depend on permutation of A, B, C, D.*

Proof. For a permutation preserving D, the assertion is obvious. Thus, it suffices to prove that $\psi(A, B, C, D, \omega) = \psi(B, C, D, A, \omega)$. Note that both conics in question are the conic from Lemma 10 for A, B, C, D, ω, since both of them pass through A, B, C, D and at least one perspector from Lemma 10. Hence $\psi(A, B, C, D, \omega)$ and $\psi(B, C, D, A, \omega)$ are equal. □

Theorem 12. *Suppose that A_1, A_2, A_3, A_4 are points, ω is a conic, and ℓ_1, ℓ_2, ℓ_3, ℓ_4 are polars of A_1, A_2, A_3, A_4 in ω. Define the transformation: choose any pair A_i, ℓ_i and a complex number c; for every j, the new ℓ_j is the image of the old ℓ_j under $H^c_{A_i,\ell_i}$, but A_j doesn't change. Apply this transformation any number of times. Then the perspector of $A_1A_2A_3$ and the triangle formed by ℓ_1, ℓ_2, ℓ_3 and similar perspectors are on conic through A_1, A_2, A_3, A_4, and this conic does not depend on the transformations.*

Proof. Let us prove that $\psi(A_1, A_2, A_3, A_4, \omega)$ is the desired conic. As in Theorem 4, we change ω after every transformation. From Lemma 9 and Lemma 11 we have that $\psi(A_1, A_2, A_3, A_4, \omega) = \psi(A_1, A_2, A_3, A_4, H^c_{A_i,\ell_i}(\omega))$. Therefore $\psi(A_1, A_2, A_3, A_4, \omega)$ does not change. But all needed perspectors always lie on $\psi(A_1, A_2, A_3, A_4, \omega)$ from Lemma 10, hence the assertion. □

Let us make some conclusions from the theorems and lemmas:

Lemma 13. *Suppose that A, B, C are points, ω is a conic with center O, and A', B', C' are poles of BC, CA, AB in ω. Then ABC and the image of $A'B'C'$ under any homothety with center O are perspective.*

Proof. This is Theorem 4 for the points A, B, C, O and the phomothety $H^c_{O,\infty}$. □

Proposition 14. *Suppose that ABC is a triangle, ω is its inscribed circle with center I, and A', B', C' are points of tangency of ω with BC, CA, AB, respectively. Then ABC and the image of $A'B'C'$ under any homothety with center I are perspective.*

Proof. This is a trivial conclusion from Lemma 13. □

Lemma 15. *Suppose that A, B, C are points, ω is a circle, A', B', C' are poles of BC, CA, AB in ω. Let ℓ_1 and ℓ_2 be a pair of perpendicular lines through the center of ω, suppose that f_1 is a compression in ℓ_1, f_2 is a compression in ℓ_2. Then ABC and the image of $A'B'C'$ under $f_1 \circ f_2$ are perspective.*

Proof. This is Theorem 4 for the points A, B, C, $X = \ell_1 \cap \infty$, $Y = \ell_2 \cap \infty$, the conic ω and two transformations with the point Y and the point X. Indeed, H^c_{X,ℓ_2} is the compression in ℓ_2 with the factor c; similarly, H^c_{Y,ℓ_1} is the compression in ℓ_1 with the factor c. □

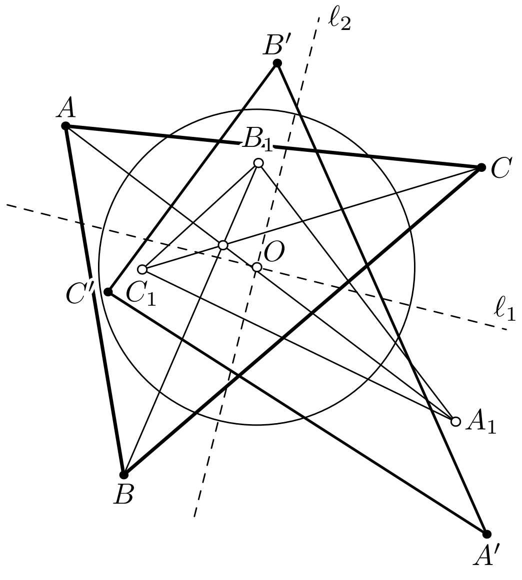

Fig. 9.

Proposition 16. *Let ABC be a triangle and ω be some Tucker's circle of ABC. Suppose that A', B', C' are poles of BC, CA, AB in ω, O and L are the circumcentre and symmedian point of ABC, respectively. Then ABC and the image of $A'B'C'$ under some compression (or just reflection) in OL are perspective.*

Proof. We know that the center of Tucker's circle is belong to OL, hence it is a trivial conclusion from Lemma 15. □

Proposition 17. *Let ABC be a triangle, ω its inscribed circle. Suppose that A', B', C' are points of tangency of ω with BC, CA, AB, respectively; ℓ_1 and ℓ_2 is a pair of perpendicular lines through the center of ω, f_1 is a compression in ℓ_1, f_2 is a compression in ℓ_2. Then ABC and the image of $A'B'C'$ under $f_1 \circ f_2$ are perspective.*

Proof. This is a trivial conclusion from Lemma 15. □

Proposition 18. *Suppose that A, B, C are points, ω is a circle, A', B', C' are poles of BC, CA, AB in ω. Let ℓ be a line through O (the center of ω), c be a number. Then ABC and the image of $A'B'C'$ under the reflection in ℓ and H_O^c are perspective.*

Proof. This is a trivial conclusion from Lemma 15. □

Proposition 19. *(Bulgarian Mathematical Olympiad, 2009). Suppose that ABC is a triangle, ω is its inscribed circle, A', B', C' are points of tangency of ω with BC, CA, AB. Let ℓ be some line through the center of ω. Then ABC and the image of $A'B'C'$ under the reflection in ℓ are perspective.*

Proof. This is a trivial conclusion from Proposition 18. □

The following proposition was proved by N. Beluhov in [1]:

Proposition 20. *Let ABC, $A'B'C'$ be triangles, O be a point. If the relations*

$$\begin{aligned} \angle(AO, BO) &= \angle(A'C', B'C'), \quad & \angle(A'O, B'O) &= \angle(AC, BC), \\ \angle(BO, CO) &= \angle(B'A', C'A'), \quad & \angle(B'O, C'O) &= \angle(BA, CA), \\ \angle(CO, AO) &= \angle(C'B', A'B'), \quad & \angle(C'O, A'O) &= \angle(CB, AB), \end{aligned}$$

hold, then ABC and $A'B'C'$ are perspective. (here $\angle(\ell_1, \ell_2)$ is the oriented angle from ℓ_1 to ℓ_2)

Proof. This is a trivial conclusion from Proposition 18, since every triangle $A'B'C'$ described in this problem can be obtained by reflection in some line through O and some homothety with center O from $A_0B_0C_0$, where A_0, B_0, C_0 are the poles of BC, CA, AB in the circle with center O and some radius. □

Proposition 21. *Suppose that ABC is a triangle, ω is its inscribed circle, A', B', C' are points of tangency of ω with BC, CA, AB. Let P be a point on the plane. Let $A'P$, $B'P$, $C'P$ meet ω second time at A_1, B_1, C_1, respectively. Then ABC and $A_1B_1C_1$ are perspective.*

Proof. This is Theorem 4 for the points A, B, C, P, the conic ω and the phomothety $H_{P,\ell}^{-1}$, where ℓ is the polar of P in ω. □

Proposition 22. *Let ABC be a triangle and AL be the bisector of the angle $\angle BAC$, $L \in BC$. Suppose that B', C' are foots of the perpendiculars from L to AB and to AC, respectively. Then $BC' \cap CB'$ is on the altitude of ABC from A.*

Proof. AL is the bisector of the angle $\angle BAC$, so $LB' = LC'$. Let ω be a circle with center L and radius LB' and h be the altitude of ABC from A. Then $h \cap \infty$, B', C' are the poles of BC, AC, AB in ω. Then from Lemma 2 we conclude that $BC' \cap CB'$, A, $h \cap \infty$ are collinear, hence $BC' \cap CB' \in h$. □

Proposition 23. *Let ABC be a triangle and AL be the bisector of the angle $\angle BAC$, $L \in BC$. Suppose that B', C' are foots of perpendiculars from L to AB and to AC, respectively. Suppose that B_1 is on LB' and C_1 is on LC', $B_1C_1 \parallel B'C'$. Then $BC_1 \cap CB_1$ is on the altitude of ABC from A.*

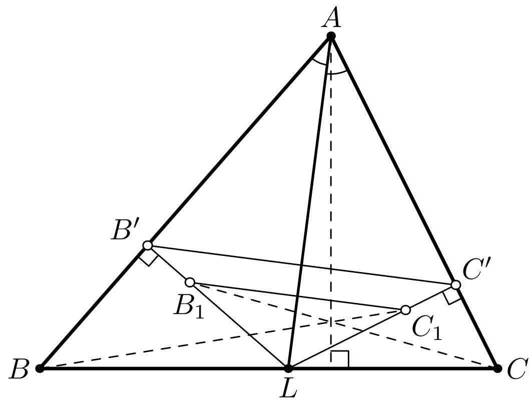

Fig. 10.

Proof. In the notation of the proof of Proposition 22, $h \cap \infty$, B', C' are the poles of BC, AC, AB in ω. Hence from Lemma 13 we see that $BC_1 \cap CB_1$, $h \cap \infty$, A are collinear, therefore $BC_1 \cap CB_1 \in h$. □

References

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