CIRCLES TOUCHING SIDES AND THE CIRCUMCIRCLE FOR INSCRIBED QUADRILATERALS

DMITRY S. BABICHEV

ABSTRACT. In an inscribed quadrilateral, four circles touching the circumcircle and two neighboring sides have a radical center.

The main result of the article is the following theorem.

Theorem 1. *Let $ABCD$ be a quadrilateral inscribed to a circle Ω. If Ω_a is the circle touching Ω and segments AB, AD, and circles Ω_b, Ω_c, Ω_d defined similarly (i. e. circles touching Ω and two neighboring sides of $ABCD$), then Ω_a, Ω_b, Ω_c, and Ω_d have a radical center (that is a point having equal powers with respect to Ω_a, Ω_b, Ω_c, and Ω_d).*

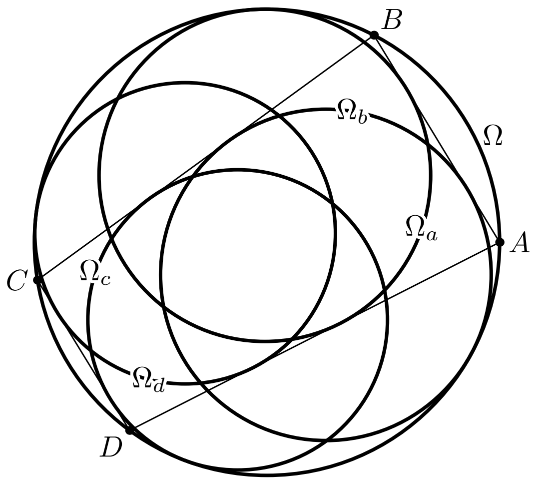

Fig. 1.

We will use the following known Lemmas.

The following lemma is offered on national Russial mathematical olympiad in 2003 year at Number 3 in grade 10 by Berlov.S., Emelyanov.L., Smirnov.A. You can find it in [1].

Lemma 1. *Let $XYZT$ be a quadrilateral with $XZ \perp YT$ inscribed to a circle Ω. Let x, y, z, t be tangents to Ω passing through X, Y, Z, T, respectively. Let $A_1 = t \cap x$, $B_1 = x \cap y$, $C_1 = y \cap z$, $D_1 = z \cap t$. Then the exterior bisectors of quadrilateral $A_1B_1C_1D_1$ form a quadrilateral $X_1Y_1Z_1T_1$ that is homothetic to $XYZT$.*

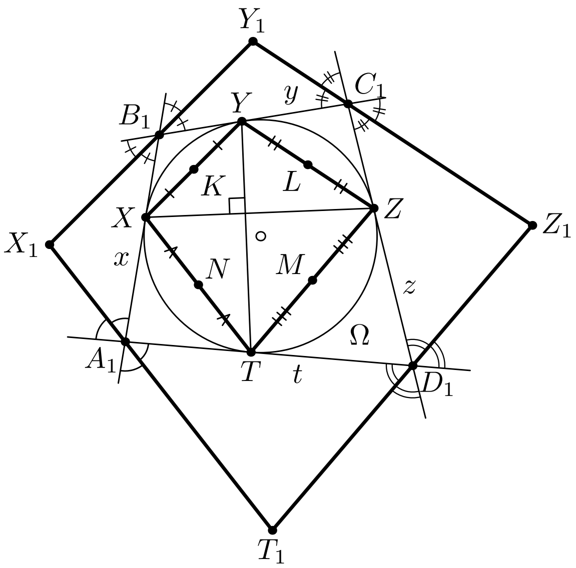

Fig. 2.

Proof. It is obvious that $X_1Y_1 \parallel XY$, $Y_1Z_1 \parallel YZ$, $Z_1T_1 \parallel ZT$, $T_1X_1 \parallel TX$. We will prove that $X_1Z_1 \parallel XZ$. Similarly, $Y_1T_1 \parallel YT$, and the statement of Lemma follows.

Let K, L, M, N be the midpoints of XY, YZ, ZT, TX, respectively. Note that TX is a polar line of A_1 with respect to Ω. Hence T_1X_1 is a polar line of N. Similarly, X_1Y_1 is a polar line of K. Therefore, X_1 is a pole of KN. This means that $OX_1 \perp KN \parallel YT$, where O is the center of Ω. Similarly, $OZ_1 \perp LM \parallel YT$. We get $X_1Z_1 \perp YT$, hence $X_1Z_1 \parallel XZ$. □

Lemma 2. *Let A, B be points on a circle Ω, let X and Y be the midpoints of arcs AB. Suppose that ω is a circle touching the segment AB at P, and touching the arc AYB at Q. Then P, Q, X are collinear, and the power of X with respect to ω equals XA^2.*

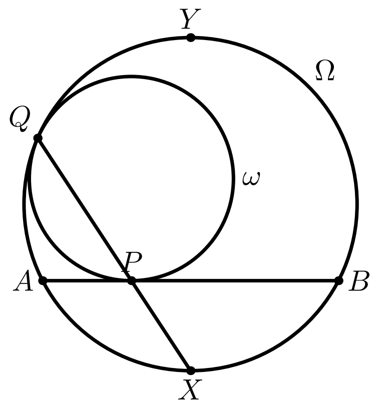

Fig. 3.

Proof. The homothety with center Q taking ω to Ω takes P to X (since the tangent to Ω at X is parallel to AB), hence Q, P, and X are collinear. Note that $\angle AQX = \angle ABX = \angle BAX$, hence triangles XAP and XQA are similar. Therefore, $XP \cdot XQ = XA^2$. □

Lemma 3. *In a projective plane, let $\mathcal{C}$ be a circle (a conic), let ℓ be a line, and let K_1, K_2, K_3, ..., K_{2n-1} be points of ℓ. Consider families of $2n$ points X_1, X_2, ..., $X_{2n} \in \mathcal{C}$ such that $K_i \in X_iX_{i+1}$, for all $i \in \{1, 2, \dots, 2n-1\}$. Then lines $X_{2n}X_1$ pass through a fixed point $K_{2n} \in \ell$.*

Proof. Since the conditions of Lemma are invariant to projective transformations, it is sufficient to consider the following case: $\mathcal{C}$ is a circle, and ℓ is the line at infinity. In this case given one family X_1, X_2, ..., X_{2n} it is easy to obtain the description of all the possible families: for some φ, points X_1, X_3, ..., X_{2n-1} could be rotated over the center of $\mathcal{C}$ by φ clockwise, while points X_2, X_4, ..., X_{2n} rotated by φ counter clockwise. Now it is obvious that the direction of line $X_{2n}X_1$ is invariant, i. e., $X_{2n}X_1$ passes through a fixed point of the line at infinity. □

The following lemma is equivalent to problem 13 from 2002 IMO shortlist suggested by Bulgaria [2].

Lemma 4. *Let ω_1 and ω_2 be two non-intersecting circles with centers O_1 and O_2, respectively; let m, n be common external tangents, and let k be a common internal tangent of ω_1 and ω_2. Let $A = m \cap k$, $B = n \cap k$, $C = \omega_1 \cap k$. Suppose that the circle σ passes through A and B, and touches ω_1 at D, then D, C, and O_2 are collinear.*

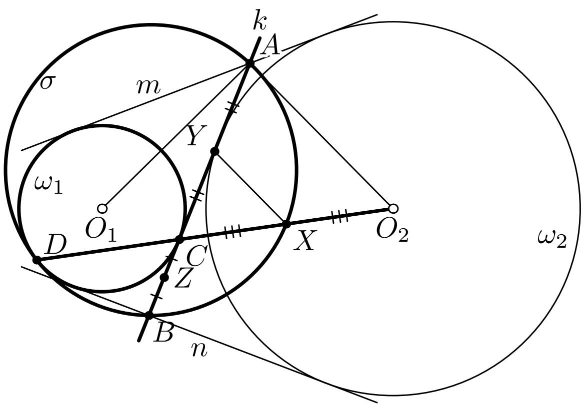

Fig. 4.

Proof. Let X, Y, and Z be the midpoints of CO_2, CA , and CB, respectively. Note that Y has equal powers with respect to ω_1 and A (here A is considered as a circle of radius 0), and $XY \parallel AO_2 \perp AO_1$. Hence XY is the radical axis of ω_1 and A. Similarly, XZ is the radical axis of ω_1 and B. Therefore, X is the radical center of ω_1, A, and B. We obtain that the power of X with respect to ω equals $AX^2 = BX^2$. Using the converse to the statement of Lemma 2 we obtain that X lies on σ (X is the midpoint of the arc AB of σ).

By Lemma 2, D, C, and X are collinear. From that it follows the statement of Lemma. □

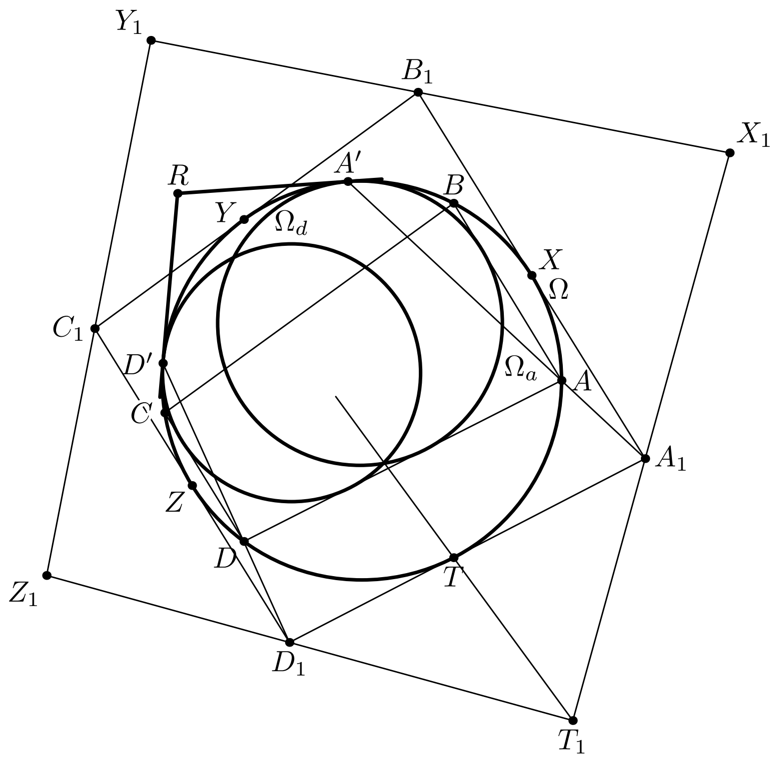

Fig. 5.

Now proceed to the proof of Theorem 1. Let X, Y, Z, T be midpoints of arcs AB, BC, CD, DA containing no other vertices of $ABCD$ (except the endpoints of arcs). Let A_1, B_1, C_1, D_1, X_1, Y_1, Z_1, T_1 be points defined in Lemma 1. Let us fix Ω and quadrilaterals $XYZT$, $X_1Y_1Z_1T_1$, $A_1B_1C_1D_1$; in this construction one can consider a family $\mathcal{F}$ of corresponding quadrilaterals $ABCD$ (starting with any $A \in \Omega$ one can obtain $B \in \Omega$ such that $AB \parallel A_1B_1$, then obtain $C \in \Omega$ such that $BC \parallel B_1C_1$, then obtain $D \in \Omega$ such that $CD \parallel C_1D_1$; hence it is easy to see that $DA \parallel D_1A_1$).

Quadrilateral $XYZT$ has perpendicular diagonals, so by Lemma 1, there exists the center S of homothety that takes $XYZT$ to $X_1Y_1Z_1T_1$. Thus S is a common point of lines XX_1, YY_1, ZZ_1, and TT_1. We will prove that TT_1 is the radical axis of Ω_d and Ω_a (and similarly, XX_1, YY_1, and ZZ_1 are radical axes for pairs Ω_a and Ω_b, Ω_b and Ω_c, Ω_c and Ω_d). From this it follows that S is the radical center of Ω_a, Ω_b, Ω_c, and Ω_d.

By Lemma 2, T has equal powers with respect to Ω_a and Ω_d.

Suppose that Ω_a and Ω_d touch Ω at A' and D', respectively. Tangents to Ω passing through A' and D' intersect at R that is the radical center of Ω, Ω_a, and Ω_d. Note that $A'D'$ is a polar line of R with respect to Ω.

Now it is sufficient to prove that $R \in TT_1$.

Considering homothety with center A' taking Ω_a to Ω we obtain that A_1, A, A' are collinear. Similarly, D_1, D, D' are collinear.

Applying Lemma 3 for $X_1 = A'$, $X_2 = A$, $X_3 = D$, and $X_4 = D'$ ($A'A$ and $D'D$ pass through A_1 and D_1, respectively; AD is parallel to A_1D_1), we obtain that for all $ABCD \in \mathcal{F}$ line $A'D'$ passes through a fixed point of A_1D_1 (or parallel to A_1D_1). Since T is the pole of A_1D_1, this means that R lies on a fixed line passing through T. Thus it is sufficient to prove that $R \in TT_1$ for some particular $ABCD \in \mathcal{F}$.

Now consider $ABCD \in \mathcal{F}$ such that $A' = D' = R$ (thus $\Omega_a = \Omega_d$). Let M be the touch point of Ω_a and AD; let N be the intersection point of external bisectors of angles BAD and CDA. By Lemma 4 (take $\omega_1 = \Omega_a$, $\sigma = \Omega$), we obtain that R, M, and N are collinear. Consider the homothety with center R that takes Ω_a to Ω. This homothety takes M to T, and N to T_1. This means that in the particular case $R \in TT_1$. This completes the proof.

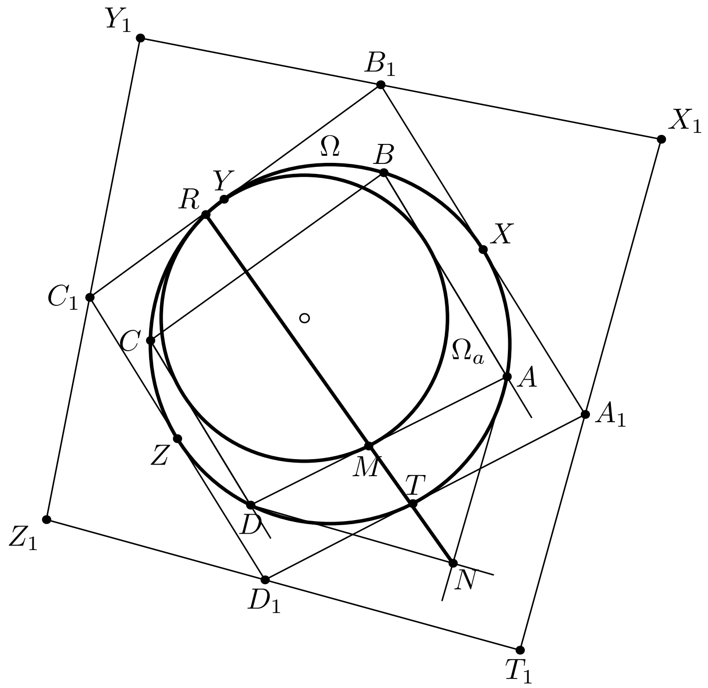

Fig. 6.

Remark. Note that the statement of Theorem 1 is equivalent to the following statement. There exists a circle ω, $\omega \neq \Omega$ such that ω touches each of the circles Ω_a, Ω_b, Ω_c, and Ω_d.

Acknowledgement. The author is grateful to P.A. Kozhevnikov for some remarks that helped to shorten the proof.

References

[1] N. Kh. Agakhanov, I. I. Bogdanov, P. A. Kozhevnikov, O. K. Podlipsky, and D. A. Tereshin. *All-russian mathematical olympiad 1993–2009.* M.:MCCME, 2007.

[2] D. Djukić, V. Janković, I. Matić, and N. Petrović. *The IMO compendium.* Springer, 2006.

E-mail address: dimitry008@gmail.com

Moscow Institute of Physics and Technology