A PROOF OF VITTAS' THEOREM AND ITS CONVERSE

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ABSTRACT. We discuss Vittas's theorem, which states that the Euler lines of non-equilateral triangles ABP, BCP, CDP and DAP in a cyclic quadrilateral $ABCD$, whose diagonals AC and BD intersect at P, are concurrent or are pairwise parallel or coincident. We also introduce and prove the converse of these theorems.

1. Introduction

In 2006, *Kostas Vittas* stated without proof an interesting theorem about the concurrence of the Euler lines of triangles PAB, PBC, PCD and PDA in a cyclic quadrilateral $ABCD$ whose two diagonals AC and BD intersect at P. Later, Vladimir Yetti, a Czech physicist, used the properties of conics to prove this theorem for the first time [1]. Then, it is Vittas who gave a synthetic proof of his own theorem [2].

In this article, by Theorems 1 and 2, we will discuss in detail Vittas's theorem. Then, by Theorems 3 and 4, we will introduce the converse of Theorems 1 and 2. It is worth noticing that the proposed approach to the proof of these four theorems is completely original.

The following notations will be used:

The reflection with respect to axis ℓ is denoted by R_ℓ;

The symmetry with respect to point I is denoted by S_I;

The homothety with center P and ratio k is denoted by H_P^k.

For simplification:

If the lines XY and ZT are parallel, it is written $XY \parallel ZT$;

If the lines XY and ZT intersect, it is written $XY \times ZT$;

If the lines XY and ZT are not coincident, it is written $XY \not\equiv ZT$;

If the lines XY, ZT and UV are pairwise parallel, it is written $XY \parallel ZT \parallel UV$;

If the lines XY, ZT and UV are concurrent and pairwise not coincident, it is written $XY \times ZT \times UV$.

For consistency, I denote by P the intersection of the diagonals AC and BD of quadrilateral $ABCD$ in all theorems and their proofs. Accordingly, let O_1, O_2, O_3 and O_4 be the circumcenters of triangles PAB, PBC, PCD and PDA respectively and let H_1, H_2, H_3 and H_4 be the orthocenters of triangles PAB, PBC, PCD and PDA respectively. Obviously, O_1H_1, O_2H_2, O_3H_3 and O_4H_4 are the Euler lines of triangles PAB, PBC, PCD and PDA respectively.

We will use next well-known facts of triangle geometry and transformations theory.

Lemma 1. *In triangle ABC, let $\angle BAC = \alpha < 90°$. A line ℓ contains the internal bisector of $\angle BAC$. Let O, H be the circumcenter and orthocenter of the triangle respectively.*

1. If $\alpha \neq 60°$, then H is the image of O by the opposite similarity $H_A^{2\cos\alpha} R_\ell$.

2. If $\alpha = 60°$, then H is the image of O by the reflection R_ℓ.

Lemma 2. *In triangle ABC, let $\angle BAC = \alpha > 90°$. A line ℓ contains the external bisectors of $\angle BAC$. Let O and H be the circumcenter and orthocenter of the triangle respectively.*

1. If $\alpha \neq 120°$ then H is the image of O by the opposite similarity $H_A^{-2\cos\alpha} R_\ell$.

2. If $\alpha = 120°$ then H is the image of O by the reflection R_ℓ.

Lemma 3. *Every opposite similarity that has similarity ratio other than 1 can be uniquely presented as a dilative reflection i.e. a product of a reflection and a homothety with has a positive ratio and the center lying on the axis of the reflection.*

Lemma 4. *Two pairs of distinct points A, B and A', B' are given in the plane such that the length of segment AB is distinct from the length of segment $A'B'$. There exists a unique dilative reflection transforming A, B into A', B' respectively.*

2. Main Results

Theorem 1. *Let $ABCD$ be a quadrilateral whose diagonals AC and BD intersect at P and form an angle of 60°. If the triangles PAB, PBC, PCD, PDA are all not equilateral, then their Euler lines are pairwise parallel or coincident.*

Proof. Without loss of generality, suppose that $\angle APB = \angle CPD = 120°$; $\angle APD = \angle BPC = 60°$. Let ℓ be the line that contains the internal angle bisectors of $\angle APD$ and $\angle BPC$. According to parts 2 of Lemmas 1 and 2, it can be deduced that H_1, H_2, H_3 and H_4 are the images of O_1, O_2, O_3 and O_4 respectively, by R_ℓ.

Therefore O_1H_1, O_2H_2, O_3H_3 and O_4H_4 are all perpendicular to ℓ. Thus, O_1H_1, O_2H_2, O_3H_3 and O_4H_4 are pairwise parallel or coincident. □

Theorem 2 (Vittas's theorem)**.** *Let $ABCD$ be a quadrilateral with diagonals intersecting at P and forming an angle different from 60°. If $ABCD$ is cyclic then the Euler lines of triangles PAB, PBC, PCD and PDA are concurrent.*

Proof. There are two cases:

The case of perpendicular lines AC and BD is evident. Thus we can suppose that $\angle APD = \alpha < 90°$.

Let ℓ be the line containing the internal bisectors of the angles $\angle APD$ and $\angle BPC$. From parts 1 of Lemmas 1 and 2: H_1, H_2, H_3 and H_4 are the images of O_1, O_2, O_3 and O_4 respectively, by the dilative reflection $H_P^{2\cos\alpha} R_\ell$ (note that $\cos\alpha \neq \frac{1}{2}$).

Let I be the intersection of H_1H_3 and H_2H_4. Note that $H_1H_2H_3H_4$ is a parallelogram, it is deduced that H_3, H_4, H_1 and H_2 are the images of H_1, H_2, H_3 and H_4 respectively, by the symmetry S_I.

Thus, H_3, H_4, H_1 and H_2 are the images of O_1, O_2, O_3 and O_4 respectively, by the dilative reflection $S_I H_P^{2\cos\alpha} R_\ell$.

Let Q be the center of the dilative reflection $S_I H_P^{2\cos\alpha} R_\ell$. I am going to prove that O_1H_1, O_2H_2, O_3H_3 and O_4H_4 are concurrent by proving that O_1H_1, O_2H_2, O_3H_3 and O_4H_4 all contain Q.

Now I will prove that both of the lines O_1H_1 and O_3H_3 contain Q.

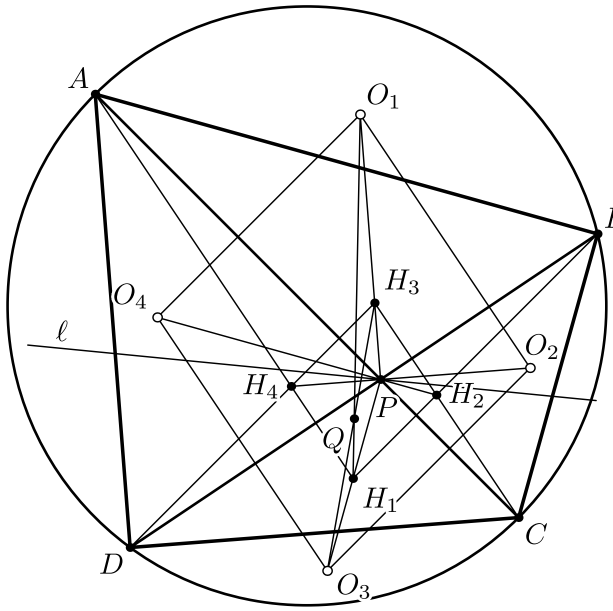

Fig. 1.

There are four possible cases. However, for the sake of simplicity, we only need to consider the main case: $O_1H_1 \not\equiv O_3H_3$; $O_1 \neq H_3$; $O_3 \neq H_1$ (Fig. 1).

Since the dilative reflection $H_P^{2\cos\alpha} R_\ell$ transform H_3 into O_3, the reflection R_ℓ transform the line PH_3 into the line PO_3.

Since quadrilateral $ABCD$ is cyclic, the triangles PAB and PDC are oppositely similar. Hence the dilative reflection $H_P^{\frac{PD}{PA}} R_\ell$ transforms O_1 into O_3. Consequently, the reflection R_ℓ transforms the line PO_1 into the line PO_3.

Thus, the lines PH_3 and PO_1 are coincident. Therefore, P, O_1 and H_3 are collinear. Similarly, P, O_3 and H_1 are collinear.

From this and the fact that triangles PO_1H_1 and PO_3H_3 are oppositely similar, it can be deduced that

$$(H_3O_1,\, H_3O_3) \equiv (H_3P,\, H_3O_3) \equiv (H_1O_1,\, H_1P) \equiv (H_1O_1,\, H_1O_3) \pmod{\pi}.$$

Hence, the four points O_1, H_1, O_3 and H_3 are concyclic.

From this, together with the fact that $O_1O_3 \neq O_1O_3 \cdot 2\cos\alpha = H_1H_3$, the lines O_1H_1 and O_3H_3 intersect.

Let Q' be the intersection of O_1H_1 and O_3H_3.

Note that O_1, H_1, O_3 and H_3 are concyclic, it could be seen that $Q'O_1O_3$ and $Q'H_3H_1$ are oppositely similar.

Therefore, Q' is the center of dilative reflection transforming O_1 and O_3 into H_3 and H_1respectively.

From this and Lemmas 3 and 4, it is seen that Q' coincides with Q.

Thus, both lines O_1H_1, O_3H_3 contain Q.

Similarly, both lines O_2H_2 and O_4H_4 contain Q.

In short, the lines O_1H_1, O_2H_2, O_3H_3 and O_4H_4 are concurrent (point Q). □

Theorem 3. *Let $ABCD$ be a quadrilateral with diagonals intersecting at P. If all four triangles PAB, PBC, PCD, PDA are not equilateral and the Euler lines of three out of these four triangles are pairwise parallel or coincident, then the angle formed by AC and BD is* $60°$.

To prove the above theorem, the following lemma is needed.

Lemma 5. *Let ABC and $A'B'C'$ be two oppositely similar triangle such that $AB \parallel A'C'$ and $AC \parallel A'B'$. If $AA' \parallel BB' \parallel CC'$ then the triangles ABC and $A'B'C'$ are oppositely congruent.*

Proof. Let E be the intersection of $A'C'$ and BB'; let F be the intersection of $A'B'$ and CC' (Fig. 2).

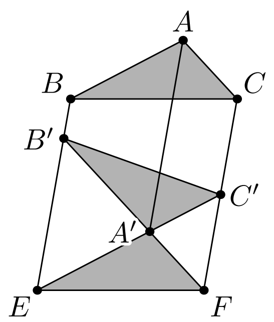

Fig. 2.

Obviously, $ABEA'$ and $ACFA'$ are parallelograms.

Therefore, the triangle $A'EF$ is the image of triangle ABC under the translation by the vector $\overrightarrow{AA'}$.

Hence, triangles ABC and $A'EF$ are directly congruent.

From this as well as the fact that triangles ABC and $A'B'C'$ are oppositely similar, we have

$$(B'C', B'F) \equiv (B'C', B'A') \equiv (BA, BC) \equiv (EA', EF) \equiv (EC', EF) \pmod{\pi}.$$

Therefore, the four points B', C', E and F belong to the same circle.

In addition, $EB' \parallel FC'$. As such, B', C', E, F are four vertices of an isosceles trapezoid with bases EB' and FC'.

So the triangles $A'EF$ and $A'B'C'$ are oppositely congruent.

In short, the triangles ABC and $A'B'C'$ are oppositely congruent. □

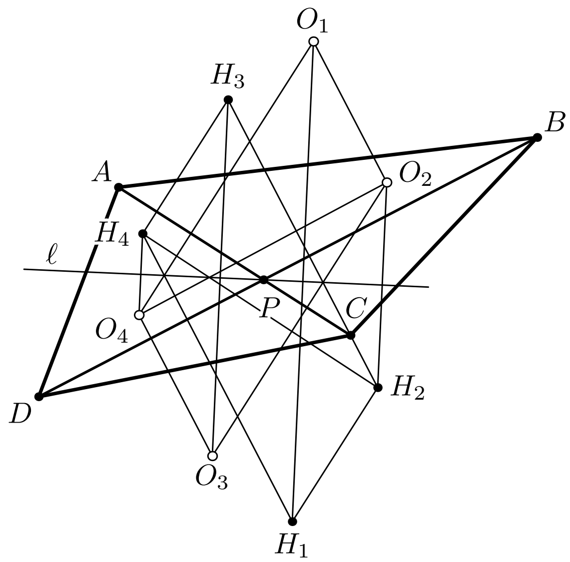

Fig. 3.

Now I am going to prove Theorem 3.

Proof of Theorem 3. Notice that the angle between AC and BD is not $90°$ (observe case 1 of the proof of Theorem 2).

Let $\angle APD = \angle BPC = \alpha$. Without loss of generality, suppose that $\alpha < 90°$.

There are six possible cases. However, for the sake of simplicity, we only need to consider the main case: $O_1H_1 \parallel O_2H_2 \parallel O_4H_4$ (Fig. 3).

Let ℓ be the line containing the bisectors of the angles $\angle APD$ and $\angle BPC$. From Lemmas 1 and 2, it can be learnt that H_1, H_2, H_3 and H_4 are the images of O_1, O_2, O_3 and O_4 respectively, by the opposite similarity $H_P^{2\cos\alpha} R_\ell$ (note that the value of $2\cos\alpha$ is not known yet).

Therefore, $H_1H_2H_4$ and $O_1O_2O_4$ are oppositely similar.

Moreover, H_1H_2 and H_1H_4 are parallel to O_1O_4 and O_1O_2 respectively (note that H_1H_2 and O_1O_4 are perpendicular to AC; H_1H_4 and O_1O_2 are perpendicular to BD).

Furthermore, by hypothesis, $O_1H_1 \parallel O_2H_2 \parallel O_4H_4$.

Thus, by Lemma 5, the triangles $H_1H_2H_4$ and $O_1O_2O_4$ are oppositely congruent. Hence, $O_2O_4 = H_2H_4$.

On the other hand, because H_2 and H_4 are the images of O_2 and O_4 respectively, by the opposite similarity $H_P^{2\cos\alpha} R_\ell$, $O_2O_4 = H_2H_4 \cdot 2\cos\alpha$.

Thus, $H_2H_4 = O_2O_4 = H_2H_4 \cdot 2\cos\alpha$.

So, $\cos\alpha = \frac{1}{2}$.

As such, $\alpha = 60°$.

In other words, the angle between AC and BD is $60°$. □

Theorem 4. *Let $ABCD$ be a quadrilateral with diagonals intersecting at P and forming an angle different from $90°$. If the triangles PAB, PBC, PCD and PDA are not equilateral and the Euler lines of three out of these four triangles are concurrent, then the quadrilateral $ABCD$ is cyclic.*

In order to prove Theorem 4, the following lemma is needed.

Lemma 6. *Let ABC and $A'B'C'$ be two oppositely similar triangle such that $AB \parallel A'C'$ and $AC \parallel A'B'$. If $AA' \times BB' \times CC'$ then B, C, B' and C' are on the same circle.*

Proof. Omit the simple case where $A' = BB' \cap CC'$.

Let Q be the point of concurrency of AA', BB' and CC'.

As $AB \parallel A'C'$ and $AC \parallel A'B'$,

$$\begin{cases} (B'C', AC) \equiv (B'C', A'B') \not\equiv 0 \pmod{\pi} \\ (B'C', AB) \equiv (B'C', A'C') \not\equiv 0 \pmod{\pi}. \end{cases}$$

Hence, $B'C' \times AC$ and $B'C' \times AB$.

Let $B'C'$ meet AC, AB at X, Y respectively.

As $A' \neq BB' \cap CC'$, $A' \in CC'$ and $A' \in BB'$.

Hence, $\begin{cases} (CC', A'C') \not\equiv 0 \pmod{\pi} \\ (BB', A'B') \not\equiv 0 \pmod{\pi}. \end{cases}$

Then, noting that $AB \parallel A'C'$ and $AC \parallel A'B'$, deduce that

$$\begin{cases} (CC', AB) \not\equiv 0 \pmod{\pi} \\ (BB', AC) \not\equiv 0 \pmod{\pi}. \end{cases}$$

This means that $CC' \times AB$ and $BB' \times AC$.

Let CC' meet AB at Z, and BB' meet AC at T (Fig. 4).

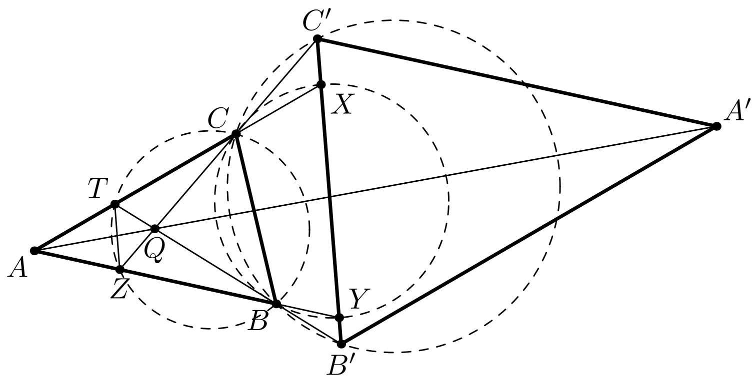

Fig. 4.

Since $AC \parallel A'B'$ and the triangles ABC and $A'B'C'$ are oppositely similar,

$$(XY, XC) \equiv (B'C', CA) \equiv (B'C', B'A') \equiv (BA, BC) \equiv (BY, BC) \pmod{\pi}.$$

As such, the points B, C, X and Y belong to the same circle.

On the other hand, by hypothesis $AB \parallel A'C'$ and $AC \parallel A'B'$, it can be deduced in respect to the Thales theorem that:

$$\frac{\overrightarrow{ZQ}}{\overrightarrow{C'Q}} = \frac{\overrightarrow{AQ}}{\overrightarrow{A'Q}} = \frac{\overrightarrow{TQ}}{\overrightarrow{B'Q}}.$$

So, ZT is parallel to $B'C'$. This means that ZT is parallel to XY.

By the concyclicity of B, C, X and Y, together with the fact that $ZT \parallel XY$, we learn that B, C, Z and T are concyclic.

From this and the fact that $ZT \parallel B'C'$, it can be seen that B, C, B' and C' are on the same circle. □

Now I am going to prove Theorem 4.

Proof of Theorem 4. Let $\angle APD = \angle BPC = \alpha$. Without loss of generality, suppose that $\alpha < 90°$.

From Theorem 1, $\alpha \neq 60°$.

Let ℓ be the line containing the bisectors of the angles $\angle APD$ and $\angle BPC$.

There are ten possible cases. However, for the sake of simplicity, we only need to consider the main case: $O_1H_1 \times O_2H_2 \times O_4H_4$ (Fig. 5).

As above, it is easy to see that the triangles $H_1H_2H_4$ and $O_1O_2O_4$ are oppositely similar and H_1H_2 and H_1H_4 are parallel to O_1O_4 and O_1O_2 respectively.

Furthermore, by hypothesis, $O_1H_1 \times O_2H_2 \times O_4H_4$.

Thus, by Lemma 6, H_2, H_4, O_2 and O_4 are on the same circle.

From this and the fact that $O_2O_4 \neq O_2O_4 2\cos\alpha = H_2H_4$, it could be learnt that O_2H_4 and O_4H_2 intersect.

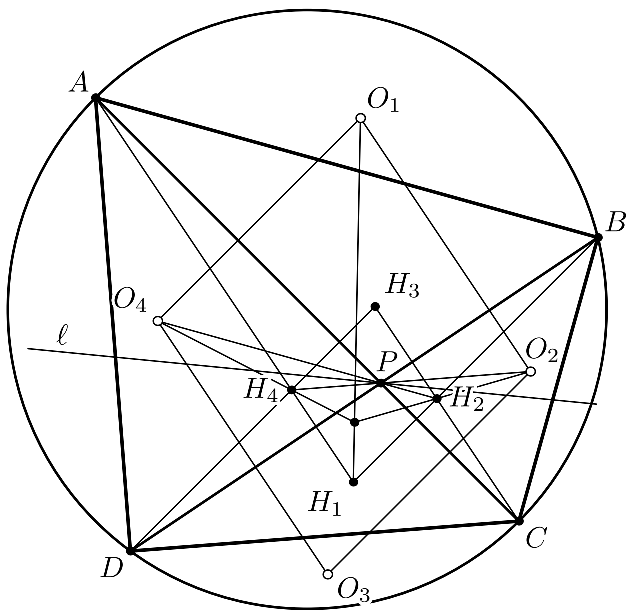

Fig. 5.

Let P' be the intersection of O_2H_4 and O_4H_2.

Then we could see that triangles $P'O_2H_2$ and $P'O_4H_4$ are oppositely similar.

This means that P' is the center of the dilative reflection transforming O_2 and H_2 into O_4 and H_4 respectively.

On the other hand, as mentioned before, the dilative reflection with center P, $H_P^{2\cos\alpha}R_\ell$, transforms O_2 and H_2 into O_4 and H_4 respectively.

This implies that, by Lemmas 3 and 4, P' coincides with P.

So, P lies on the lines O_2H_4 and O_4H_2.

From the result that H_1, H_2, H_3 and H_4 are the images of O_1, O_2, O_3 and O_4 respectively, under the dilative reflection $H_P^{2\cos\alpha}R_\ell$, we learn that the reflection R_ℓ transforms the line PO_2 into PH_2.

As ℓ contains the bisectors of $\angle APD$ and $\angle BPC$, the reflection R_ℓ transforms the line AC into BD.

Thus,

$$\begin{aligned}(AD,\, AC) &\equiv (AD,\, PH_4) + (PO_2,\, AC) \equiv \\ &\equiv \tfrac{\pi}{2} + (BD,\, PH_2) \equiv \tfrac{\pi}{2} + (BD,\, BC) + (BC,\, PH_2) \equiv \\ &\equiv \tfrac{\pi}{2} + (BD,\, BC) + \tfrac{\pi}{2} \equiv (BD,\, BC) \pmod{\pi}.\end{aligned}$$

Hence, $ABCD$ is a concyclic quadrilateral. □

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