

CONJUGATION OF LINES WITH RESPECT TO A TRIANGLE

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ABSTRACT. Isotomic and isogonal conjugate with respect to a triangle is a well-known and well studied map frequently used in classical geometry. In this article we show that there is a reason to study conjugation of lines. This conjugation has many interesting properties and relations to other objects of a triangle.

1. INTRODUCTION

Isotomic conjugation with respect to a triangle ABC is a map which maps any point P with barycentric coordinates (x, y, z) to the point P^* with coordinates $(1/x, 1/y, 1/z)$. *Isogonal conjugation* is defined similarly, but instead barycentric coordinates the trilinear coordinates are used. The following property of isogonal conjugation is important: If P and P^* are isogonal conjugates, then

$$\angle PAB = \angle CAP^*, \quad \angle PBC = \angle ABP^*, \quad \angle PCA = \angle BCP^*.$$

This property is often used as a definition of isogonal conjugation.

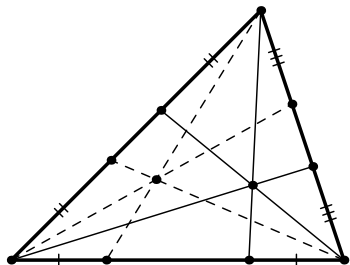


Fig. 1. Isotomic conjugation

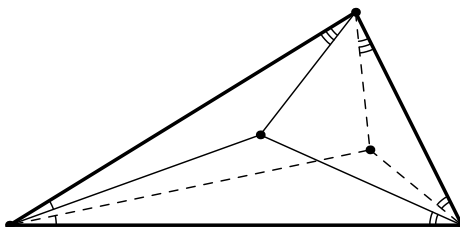


Fig. 2. Isogonal conjugation

Other properties of isogonal and isotomic conjugation and its applications to triangle geometry can be found in [2].

Isotomic and isogonal conjugations are projectively equivalent. It means that there is a projective transformation of the plane which preserves vertices of a triangle and any pair of isotomically conjugate points is mapped to a pair of isogonally conjugate points.

In the general case, define conjugation with respect to a triangle ABC and a point S as follows.

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Definition 1. Let $A'B'C'$ is a cevian triangle of a point S with respect to a triangle ABC . Let P be any point and $A_1B_1C_1$ be its cevian triangle. Choose on the sides of the triangle points $A_2, B_2,$ and C_2 such that

$$(1) \quad \begin{aligned} [A, C_1; B, C'] &= [B, C_2; A, C'], \\ [B, A_1; C, B'] &= [C, A_2; B, A'], \\ [C, B_1; A, C'] &= [A, B_2; C, B'], \end{aligned}$$

where $[X, Y; Z, T]$ means a cross-ratio of four collinear points $X, Y, Z,$ and T . It is easy to see that the cevians $AA_2, BB_2,$ and CC_2 are concurrent. The point of intersection of these three lines called a *conjugate point of the point P with respect to the triangle ABC and the point S* .

Let s be the trilinear polar of the point S with respect to the triangle ABC . Then we can replace the points $A', B',$ and C' in equation (1) by points of intersection of the sides of the triangle and s . In this case we obtain the same map. Call this map by a *conjugation of point this respect to a triangle and a line*.

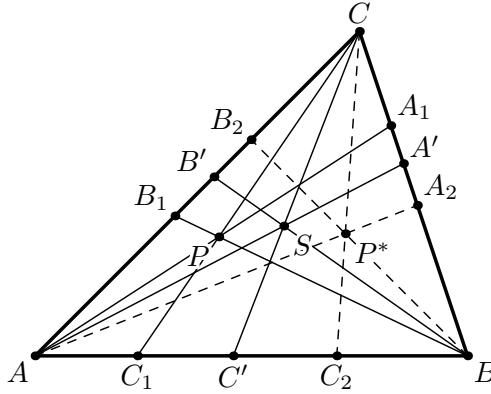


Fig. 3. Conjugation

If point S is the centroid of the triangle then this map is an isotomic conjugation. If S coincides with the incenter of the triangle then this map is an isogonal conjugation.

Correctness of the definition 1 could be easily proven by the Ceva's theorem or by means of projective transformation and correctness of the definition of isotomic or isogonal conjugation. We see, that a conjugation with respect to a triangle and a point (or line) is projectively equivalent to isotomic or isogonal conjugation.

There is a well known theorem about isogonal and isotomic conjugation.

Theorem 1. *Under any conjugation with respect to a triangle ABC an image of any line is a conic passing through the vertices of this triangle.*

For example, image of a line at infinity under isogonal conjugation is a circumcircle of a triangle and under isotomic conjugation is a circumscribed Steiner ellipse of the triangle.

2. CONJUGATION OF LINES

Let's apply a dual transformation to the construction in the definition 1. In this case we obtain another map that still makes sense. Let's call this map by a *conjugation of lines with respect to a triangle and a line*.

Let us give a strict definition.

Definition 2. Let a line s intersects the sides of a triangle ABC at points A' , B' and C' . Suppose a line ℓ intersects the sides of the triangle at points A_1 , B_1 , and C_1 . Choose on the sides of the triangle points A_2 , B_2 , and C_2 such that

$$(2) \quad \begin{aligned} [A, C_1; B, C'] &= [B, C_2; A, C'], \\ [B, A_1; C, B'] &= [C, A_2; B, A'], \\ [C, B_1; A, C'] &= [A, B_2; C, B']. \end{aligned}$$

Then the points A_2 , B_2 , and C_2 lie of the same line. Call this line a conjugate line of the line ℓ with respect to the triangle ABC and the line s .

Let S be a trilinear pole of the line s with respect to the triangle ABC . We can replace the points A' , B' , and C' in equation (2) by the vertices of cevian triangle of the point S . In this case we obtain the same map. Call it by a *conjugation of lines with respect to a triangle and a point*.

Correctness of this definition follows from correctness of the definition 1. Also it could be independently proven by the Menelaus theorem.

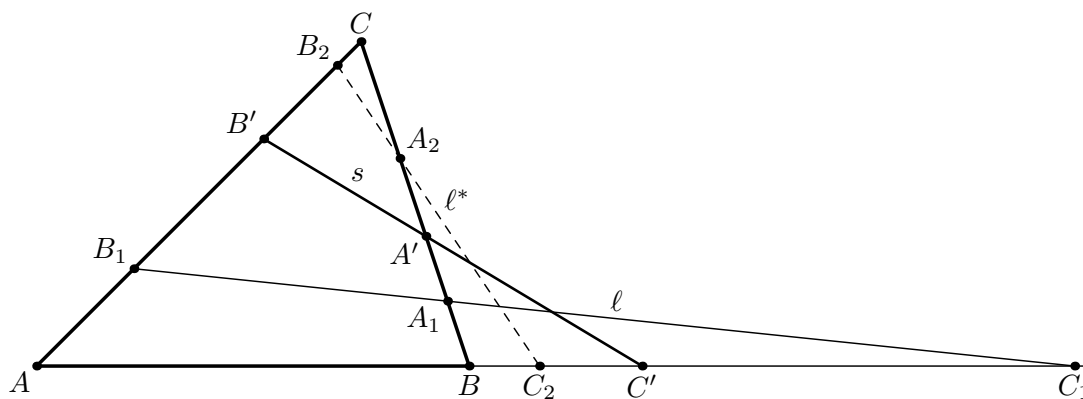


Fig. 4. Conjugation of a line

If a line s is the line at infinity then it is reasonable to call this conjugation *isotomic conjugation of lines*. Also call this map *isogonal conjugation* if s is the line passing through the feet of the three external angle bisectors of the triangle. In these cases this map has a simpler definition (Fig. 5 and 6).

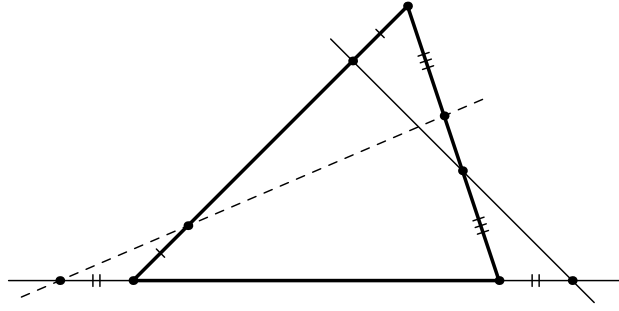


Fig. 5. Isotomic conjugation of a line

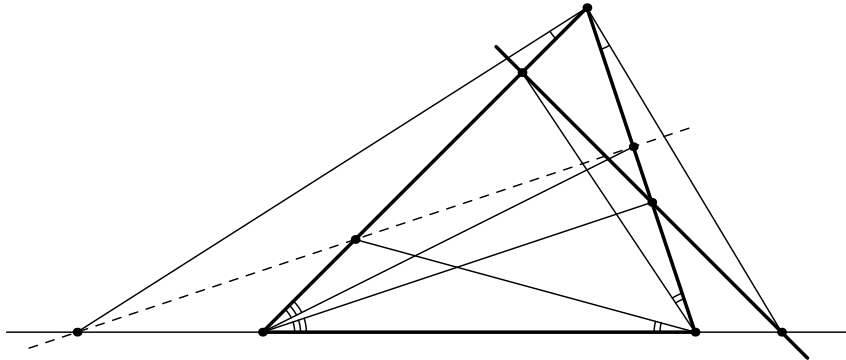


Fig. 6. Isogonal conjugation of a line

The dual theorem to Theorem 1 is the following.

Theorem 2. *The set of lines passing through a fixed point is conjugate with respect to $\triangle ABC$ to the set of tangent lines to some inscribed conic.*

Considering “critical lines” of the conjugation we can see that perspector of the conic which correspond to the point P is the conjugate point P^* .

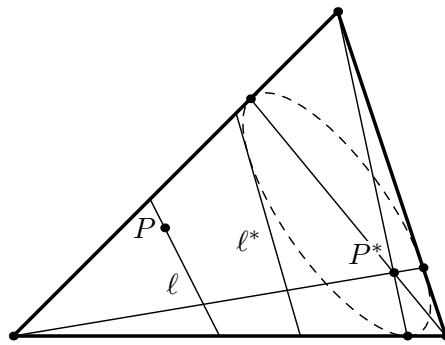


Fig. 7. Conjugation of a pencil of lines

Theorem 2 has the following corollaries.

Corollary 3. *Isotomically (or isogonally) conjugate line to any line passing through the center of an equilateral triangle is a tangent line to the inscribed circle of the triangle.*

Corollary 4. *Isotomically conjugate line to any line passing through the Nagel point of a triangle is a tangent line to the inscribed circle of the triangle.*

The proofs follow from the fact that Gergonne point and Nagel point are isotomically conjugate.

Corollary 5. *Isotomically conjugate line to any line passing through the centroid of a triangle is a tangent line to the inscribed Steiner ellipse of the triangle.*

Corollary 6. *Isogonally conjugate line to any line passing through the internal similitude center of the incircle and the circumcircle of a triangle is a tangent line to the inscribed circle of the triangle.*

The proof follows from the fact that the Gergonne point and the internal similitude center are isogonally conjugate.

Corollary 7. *Isogonally conjugate line to any line passing through the Lemoine point of a triangle is a tangent line to the inscribed Steiner ellipse of the triangle.*

The proof follows from the fact that the Lemoine point and the centroid of the triangle are isogonally conjugate. Also, since the Brocard ellipse is tangent to the sides of a triangle in the feet of its semidians we obtain the following.

Corollary 8. *Isogonally conjugate line to any line passing through the centroid of a triangle is a tangent line to the Brocard ellipse.*

3. CONCRETE LINES

From corollaries of the previous section we can formulate some statements about conjugate lines of well-known lines of a triangle. Let us recall that Nagel point, centroid and incenter of a triangle lie on the same line and this line is called a *Nagel line* of a triangle.

- Theorem 9.**
- (1) *Isotomically conjugate line of the Euler line is tangent to the inscribed Steiner ellipse.*
 - (2) *Isogonally conjugate line of the Euler line is tangent to the Brocard ellipse.*
 - (3) *Isogonally conjugate line of the line IO (line through the incenter and the circumcenter of a triangle) is tangent to the inscribed circle of a triangle.*
 - (4) *Isotomically conjugate line of the Nagel line is tangent to the inscribed Steiner ellipse and the incircle of a triangle.*
 - (5) *Isogonally conjugate line of the Nagel line is tangent to the Brocard ellipse.*

It is possible to continue this list for other lines and centers of a triangle.

Let us prove, that the isotomically conjugate line of the Nagel line possesses another interesting property.

Theorem 10. *Isotomically conjugate line to the Nagel line of a triangle is tangent to the inscribed circle at the Feuerbach point.*

Because of Theorem 9 it is sufficient to prove the following lemma.

Lemma 11. *Common tangent line to the incircle and the inscribed Steiner ellipse of a triangle (which doesn't coincide with any side of this triangle) is tangent to the inscribed circle at the Feuerbach point.*

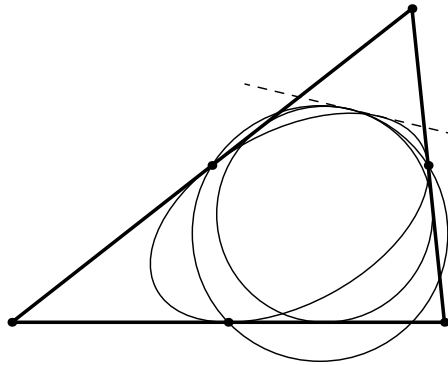


Fig. 8.

Proof. Denote vertices of a triangle by A, B, C and this common tangent line by ℓ . Let A_1, B_1 , and C_1 be points of intersection of ℓ and the sides of $\triangle ABC$. For the proof of the lemma it is sufficient to prove that the points A_1, B_1 , and C_1 are on the radical axis of the inscribed circle and the nine-point circle of the triangle ABC . We calculate powers of these points with respect to both circles.

Let the incircle be tangent to the sides of the triangle at points A_2, B_2 , and C_2 . Denote midpoints of the triangle by A_3, B_3 , and C_3 and feet of altitudes by A_4, B_4 , and C_4 . Lengths of the sides are denoted by a, b , and c .

Suppose P is the point of intersection of lines AA_1 and BB_1 . From the Brianchon theorem it follows that the point P lies on the lines A_2B_2 and A_3B_3 , therefore it is a point of intersection of these lines.

It is known that the second point of intersection of the line C_2P and the incircle is the Feuerbach point (See [3], Problem 22). From the Brianchon theorem it follows that this point is the point of tangent line A_1B_1 and incircle. So, this line is tangent to the incircle in the Feuerbach point.

Since the proof of the mentioned property of the Feuerbach point is little bit complicated and based on other properties of Feuerbach point, we will give another independent proof.

Let us show that the points A_1, B_1 , and C_1 lie on the radical axis of incircle and nine-point circle of the triangle.

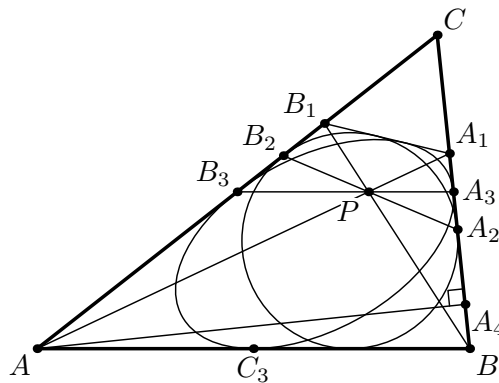


Fig. 9.

Let's apply the Menelaus theorem for the line A_2B_2 and the triangle CA_3B_3 .

$$(3) \quad \frac{A_3P}{B_3P} = \frac{A_3A_2}{A_2C} \cdot \frac{CB_2}{B_2B_3} = \frac{c-b}{a+b-c} \cdot \frac{a+b-c}{c-a} = \frac{c-b}{a-c}.$$

Now apply the Menelaus theorem for the line AA_1 and the triangle CA_3B_3 .

$$(4) \quad \frac{A_1A_3}{A_1C} = \frac{A_3P}{B_3P} \cdot \frac{B_3A}{AC} = \frac{c-b}{2(a-c)}.$$

Since $CA_3 = a/2$ we have

$$(5) \quad A_1C = \frac{2(a-c)}{c-b+2a-2c} \cdot \frac{a}{2} = \frac{a(a-c)}{2a-b-c}.$$

Now it is easy to find lengths of the segments A_1A_2 , A_1A_3 , and A_1A_4 .

$$(6) \quad A_1A_2 = \frac{a+b-c}{2} - \frac{a(a-c)}{2a-b-c},$$

$$(7) \quad A_1A_3 = \frac{a}{2} - \frac{a(a-c)}{2a-b-c},$$

$$(8) \quad A_1A_4 = \frac{a^2+b^2-c^2}{2b} - \frac{a(a-c)}{2a-b-c}.$$

We check the equation $A_1A_2^2 = A_1A_3 \cdot A_1A_4$.

$$(9) \quad \begin{aligned} A_1A_2^2 - A_1A_3 \cdot A_1A_4 &= \left(\frac{a+b-c}{2} \right)^2 - 2 \frac{a+b-c}{2} \cdot \frac{a(a-c)}{2a-b-c} - \\ &\quad - \frac{a}{2} \cdot \frac{a^2+b^2-c^2}{2b} + \frac{a^2+b^2-c^2+ab}{2b} \cdot \frac{a(a-c)}{2a-b-c} = \\ &= \frac{c-a-b}{4} \cdot \frac{2a^2-ab-ac+b^2-c^2}{2a-b-c} + \frac{2a^3+a^2b-3a^2c+b^3-b^2c-c^2b+c^3}{4(2a-b-c)} = 0. \end{aligned}$$

We obtain that the point A_1 lies on the radical axis of the inscribed circle and the nine-point circle of the triangle. The same argument works for the points B_1 and C_1 .

Note that we also proved the Feuerbach theorem, because we showed that the radical axis of the incircle and the nine-point circle is tangent to the first one. \square

The following theorem follows from Lemma 11.

Theorem 12. *Isogonally conjugate line to the line passing through the Lemoine point and the internal similitude center of the incircle and the circumcircle of a triangle is tangent to the inscribed circle at the Feuerbach point.*

4. RELATED RESULTS

We mention another result related to construction on the Fig. 9.

Theorem 13. *Suppose triangles $A_1B_1C_1$ and $A_2B_2C_2$ are cevian triangles in a triangle ABC . Correspondent sides of these triangles intersect each other at points $A_3, B_3,$ and C_3 . Then*

1) *The lines $A_1A_3, B_1B_3,$ and C_1C_3 are concurrent. Denote the point of intersection of these lines by P .*

The lines $A_2A_3, B_2B_3,$ and C_2C_3 are concurrent. Denote the point of intersection of these lines by Q .

2) *The following triples of points are collinear: $(A, B_3, C_3), (B, A_3, C_3)$ and (C, A_3, B_3) . Denote intersection of this lines and the correspondent sides of triangle by $A_4, B_4,$ and C_4 .*

3) *The points A_4, B_4 and C_4 lie on the line PQ .*

Proof. Inscribe into the triangle ABC two conics α_1 and α_2 , which touch the sides of ABC in the vertices of the triangles $A_1B_1C_1$ and $A_2B_2C_2$. Then fourth common tangent of α_1 and α_2 will be line from Item 3 of the Theorem. The points of tangent of these conics and the line are the points P and Q . Items 1 and 2 follows from Brianchon theorem. \square

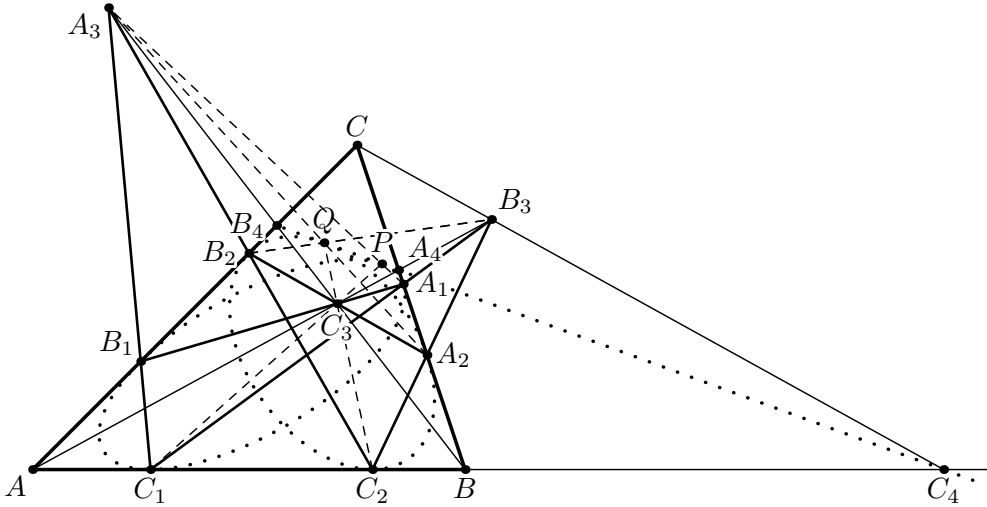


Fig. 10.

The Nagel triangle is a triangle with vertices in points of tangency of excircles and correspondent sides of a triangle ABC . The ellipse tangent to sides of ABC at these points is called *the Nagel ellipse*.

The recent result F. Ivlev [4] states that if $A_1B_1C_1$ (in terms of Theorem 13) is a medial triangle and $A_2B_2C_2$ is Nagel triangle then Q is the Feuerbach point of the triangle ABC . Reformulating Ivlev's theorem in terms of this article we obtain the following.

Theorem 14. *The Nagel ellipse of a triangle passes through the Feuerbach point. The tangent line to the Nagel ellipse at this point is also tangent to the inscribed Steiner ellipse.*

Corollary 15. *Isotomically conjugate line to the line passing through centroid of a triangle and the Gergonne point passes through the Feuerbach point.*

Corollary 16. *Isogonally conjugate line to the line passing through the Lemoine point and the external similitude center of the incircle and the circumcircle of a triangle passes through the Feuerbach point.*

Ivlev's proof is based only on the fact that the Nagel point and the Gergonne point are isotomically conjugate (actually his theorem is more general). So, it is possible to generalize Theorem 14.

Theorem 17. *Points P and Q are conjugate with respect to the point T in the triangle ABC . Conjugation of the lines is given. Let α_1 , α_2 , and α_0 three inscribed in the triangle ABC conics which correspond to the points P , Q , and T (see Theorem 2). Then fourth common tangents to the pairs of conics α_0 , α_1 and α_0 , α_2 intersect at one of the points of intersection α_1 and α_2 .*

Note that for any conjugation there are four points such that this conjugation can be defined with respect to any of these four points (for example isogonal conjugation is also a conjugation with respect to any excenter of a triangle). For each of these points there is a different point of intersection of α_1 and α_2 in Theorem 17.

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