ISOTOMIC SIMILARITY

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Abstract. Let A_1, B_1, C_1 be points chosen on the sidelines BC, CA, BA of a triangle ABC, respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2 respectively. We prove that triangle $A_2B_2C_2$ is similar to triangle $A_3B_3C_3$, where A_3, B_3, C_3 are symmetric to A_1, B_1,C_1 with respect to the midpoints of the sides BC, CA, BA respectively.

Theorem 1. *Let A_1, B_1, C_1 be points chosen on the sidelines BC, CA, BA of a triangle ABC, respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2 respectively. Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, BA respectively. Then the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.*

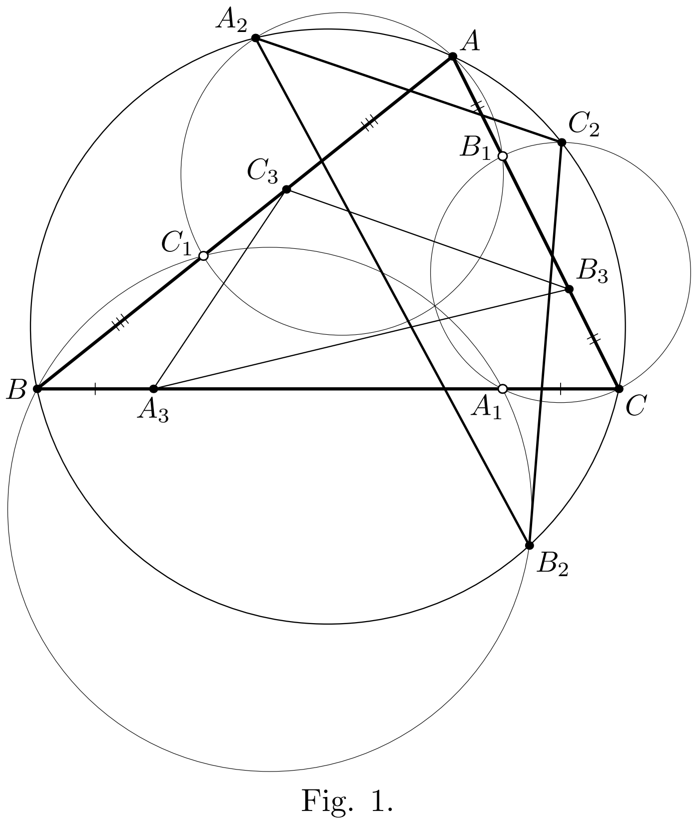

Fig. 1.

Preliminary. Let us introduce some notions and formulate known lemmas that we use in the proof.

We will work with oriented angles between lines. For two straight lines ℓ, m in the plane, $\angle(\ell, m)$ denotes the angle of counterclockwise rotation which transform line ℓ into a line parallel to m (the choice of the rotation centre is irrelevant). This is signed quantity; values differing by a multiple of π are identified, so that

$$\angle(\ell, m) = -\angle(m, \ell), \qquad \angle(\ell, m) + \angle(m, n) = \angle(\ell, n).$$

If ℓ is the line through the points K, L and m is the line the M, N, one writes $\angle(KL, MN)$ for $\angle(\ell, m)$; the characters K, L are freely interchangeable; and so are M, N. The counterpart of the classical theorem about cyclic quadrilaterals is the following:

Lemma 1. *Four non-collinear points K, L, M, N are concyclic if and only if*

$$\angle(KM, LM) = \angle(KN, LN). \tag{1}$$

Further we use (1) without explicit reference.

Lemma 2. *Suppose that A_1, B_1, C_1 are points on the sidelines BC, CA, BA of a triangle ABC, respectively; then the three circles (AB_1C_1), (BC_1A_1), (CA_1B_1) have a common point.*

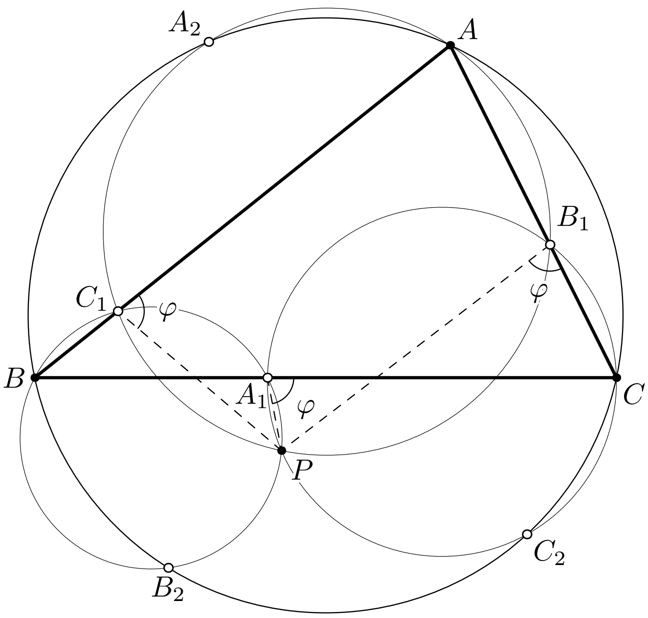

Fig. 2.

Proof. Let (AB_1C_1) and (BC_1A_1) intersect at C_1 and P. Then

$$\begin{aligned}\angle(PA_1, CA_1) = \angle(PA_1, BA_1) = \angle(PC_1, BC_1) = \\ = \angle(PC_1, AC_1) = \angle(PB_1, AB_1) = \angle(PB_1, CB_1).\end{aligned}$$

The equality between the outer terms shows that the points A_1, B_1, P, C are concyclic. Thus P is the common point of the three mentioned circles. $\square$

Lemma 3. *Let A_1, B_1, C_1 be points on the sidelines BC, CA, BA of a triangle ABC, respectively; and the circles (AB_1C_1), (BC_1A_1), (CA_1B_1) meet at P. Suppose that the lines AP, PB, CP meet the circle (ABC) again at A', B', C', respectively; then triangles $A_1B_1C_1$ and $A'B'C'$ are similar. (In particular, the pedal triangle of P is similar to $A'B'C'$.)*

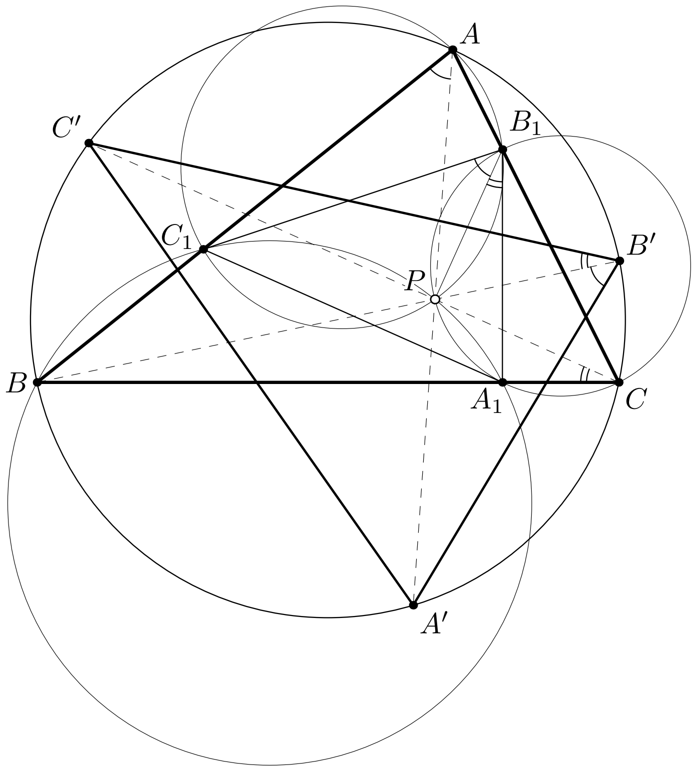

Fig. 3.

Proof. We have

$$\angle(A_1B_1, C_1B_1) = \angle(A_1B_1, PB_1) + \angle(PB_1, C_1B_1) = \angle(A_1C, PC) + \angle(PA, C_1A). \tag{2}$$

On the other hand,

$$\angle(A'B', C'B') = \angle(A'B', BB') + \angle(BB', C'B') = \angle(AA', BA) + \angle(BC, C'C). \tag{3}$$

But the lines $A'A$, BA, BC, $C'C$ coincide respectively with PA, C_1A, A_1C, PC. So the sums on the right-hand of (2) and (3) are equal, that leads to $\angle(A_1B_1, C_1B_1) = \angle(A'B', C'B')$. Hence (by cyclic shift, once more) also

$$\angle(B_1C_1, A_1C_1) = \angle(B'C', A'C') \text{ and } \angle(C_1A_1, B_1A_1) = \angle(C'A', B'A').$$

This means that triangles $A_1B_1C_1$ and $A'B'C'$ are similar. □

Proof of the Theorem. Let the circles (AB_1C_1), (BC_1A_1), (CA_1B_1) meet at P (see Lemma 2), and let

$$\varphi = \angle(PA_1, BC) = \angle(PB_1, CA) = \angle(PC_1, AB). \tag{4}$$

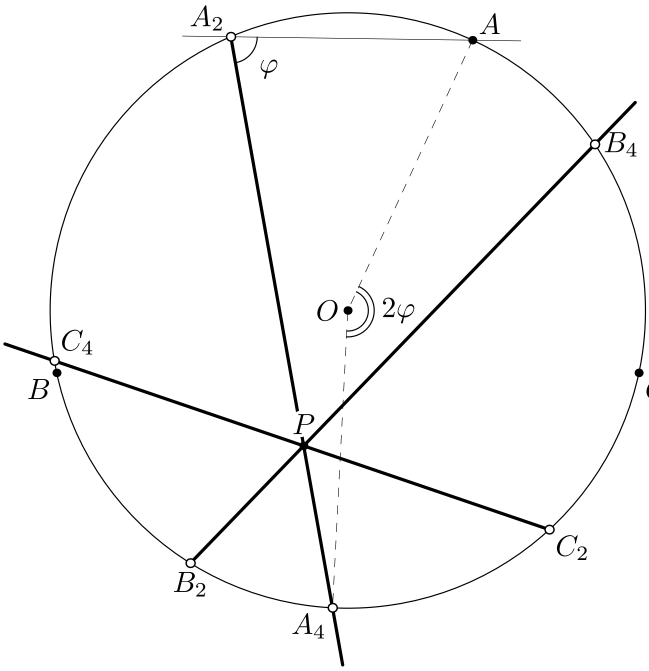

Fig. 4.

Let lines A_2P, B_2P, C_2P meet the circle (ABC) again at A_4, B_4, C_4, respectively. Since

$$\angle(A_4A_2, AA_2) = \angle(PA_2, AA_2) = \angle(PC_1, AC_1) = \angle(PC_1, AB) = \varphi,$$

we have $\angle(OA_4, OA) = 2\varphi$ (here O is the center of (ABC)). Hence A is the image of A_4 under rotation by 2φ about O. The same rotation takes B_4 to B, and C_4 to C. Thus triangle ABC is the image of $A_4B_4C_4$ under this rotation, therefore

$$\angle(A_4B_4, AB) = \angle(B_4C_4, BC) = \angle(C_4A_4, CA) = 2\varphi. \tag{5}$$

Further, we have $\angle(AB_4, AB) = \angle(B_2B_4, B_2B) = \varphi$. Hence by (4)

$$\angle(AB_4, PC_1) = \angle(AB_4, AB) + \angle(AB, PC_1) = \varphi + (-\varphi) = 0,$$

which means that $AB_4 \parallel PC_1$.

Let C_5 be the intersection of lines PC_1 and A_4B_4; define A_5, B_5 analogously. So $AB_4 \parallel C_1C_5$ and, by (5) and (4),

$$\angle(A_4B_4, PC_1) = \angle(A_4B_4, AB) + \angle(AB, PC_1) = 2\varphi + (-\varphi) = \varphi; \tag{6}$$

i.e., $\angle(B_4C_5, C_5C_1) = \varphi$. This combined with $\angle(C_5C_1, C_1A) = \angle(PC_1, AB) = \varphi$ (see (4)) proves that the quadrilateral $AB_4C_5C_1$ is an isosceles trapezoid with $AC_1 = B_4C_5$.

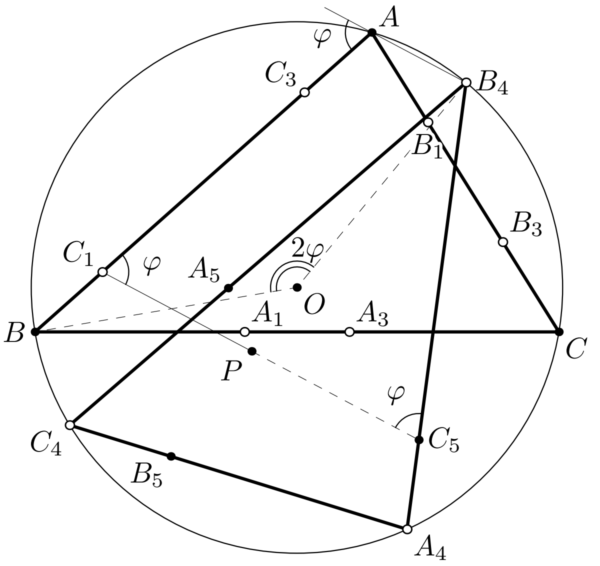

Fig. 5.

Suppose $\overrightarrow{AC_3} = \lambda\overrightarrow{AB}$; then $\overrightarrow{BC_1} = \lambda\overrightarrow{BA}$, and $\overrightarrow{A_4C_5} = \lambda\overrightarrow{A_4B_4}$. In other words, the rotation which maps triangle $A_4B_4C_4$ onto ABC carries C_5 onto C_3. Likewise, it takes A_5 to A_3, and B_5 to B_3. So the triangles $A_3B_3C_3$ and $A_5B_5C_5$ are congruent.

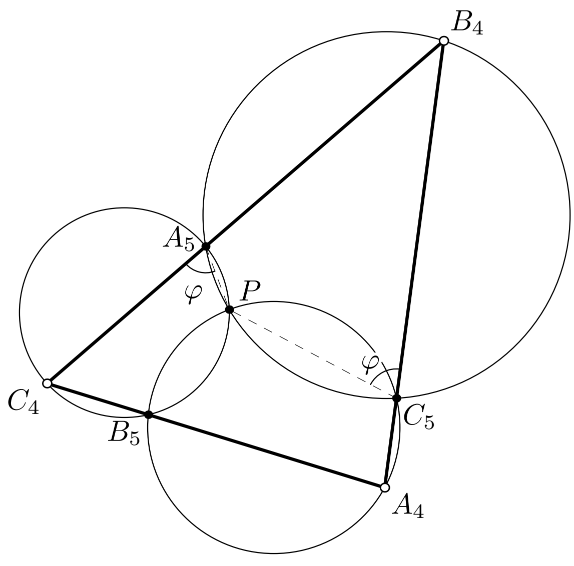

Fig. 6.

Lines B_4C_5 and PC_5 coincide respectively with A_4B_4 and PC_1. Thus by (6)

$$\angle(B_4C_5, PC_5) = \varphi.$$

Analogously (by cyclic shift) $\varphi = \angle(C_4A_5, PA_5)$, which rewrites as

$$\varphi = \angle(B_4A_5, PA_5).$$

These relations imply that the points P, B_4, C_5, A_5 are concyclic. Analogously P, C_4, A_5, B_5 and P, A_4, B_5, C_5 are concyclic quadruples.

Now it is sufficient to apply Lemma 3 for triangle $A_4B_4C_4$ and points A_5, B_5, C_5. It provides similarity of triangles $A_2B_2C_2$ and $A_5B_5C_5$. This ends the proof of Theorem. $\square$

References

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