TWO THEOREMS ON THE FOCUS-SHARING ELLIPSES: A THREE-DIMENSIONAL VIEW

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Abstract. Consider three ellipses each two of which share a common focus. The "radical axes" of the pairs of these ellipses are concurrent, and the points of intersection of the common tangents to the pairs of these ellipses are collinear.

We present short synthetical proofs of these facts. Both proofs deal with the prolate spheroids having the given ellipses as axial sections.

1. Introduction

We present synthetical proofs of two theorems on three ellipses sharing common foci. The proofs deal with three-dimensional interpretation of the configuration.

Throughout the paper, we speak on the intersection of lines and planes in a projective sense; thus, all the points of intersection may appear to be ideal.

Both theorems deal with the following configuration. Let $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ be three ellipses such that each two of them share a common focus. We assume that none of the ellipses lies inside the other one. Note that there exist two different configurations satisfying this property: three ellipses with a common focus, and three ellipses with the pairs of foci (A, B), (A, C), and (B, C). Our proofs do not distinguish these cases.

The first theorem was firstly proved by J. H. Neville [2]. There are some synthetical proofs of this theorem, see, for instance, [1].

Theorem 1. *Let A_{ij}, B_{ij} $(1 \leq i < j \leq 3)$ be the points of intersection of ellipses $\mathcal{E}_i$ and $\mathcal{E}_j$. Then the lines $A_{12}B_{12}$, $A_{13}B_{13}$, and $A_{23}B_{23}$ are concurrent.*

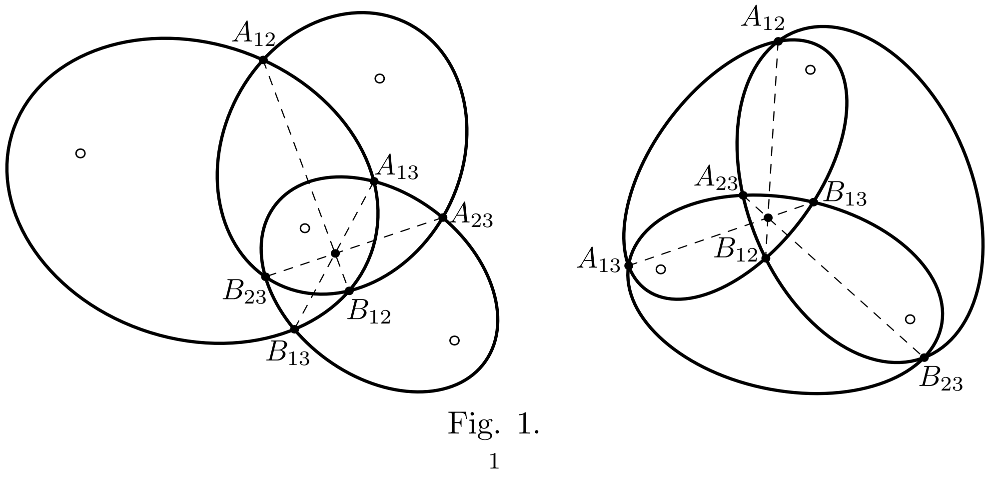

Fig. 1.

The second one was noticed by A. Akopyan; he revealed it to the author in a private communication. In some rough sense, it may be considered as a dual theorem to the previous one.

Theorem 2. *Let a_{ij}, b_{ij} be the two common tangents to $\mathcal{E}_i$ and $\mathcal{E}_j$ ($1 \le i < j \le 3$); denote by K_{ij} the point of their intersection. Then the points K_{12}, K_{13}, and K_{23} are collinear.*

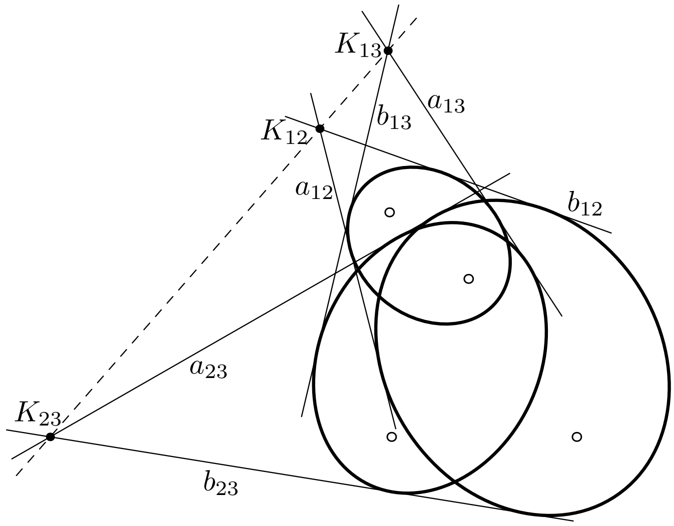

Fig. 2.

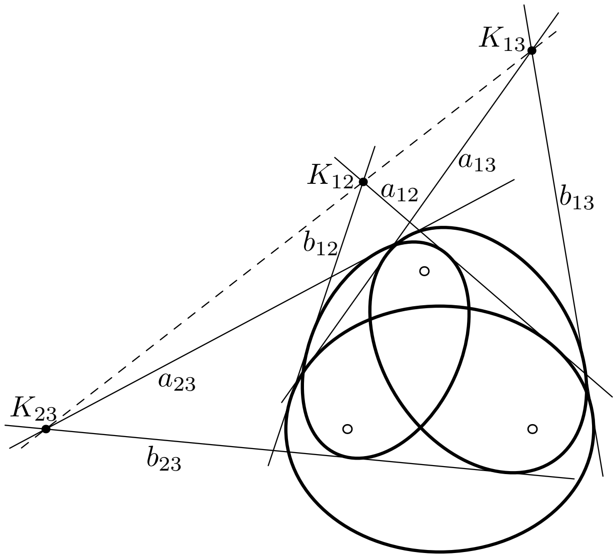

Fig. 3.

Remark 1. If all three ellipses $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ have a common focus F, then they become circles under a duality transform with a center F. Hence in this case both facts allow simpler proofs.

In the proofs of both theorems, we use some geometrical features of ellipses and ellipsoids. Here we recall some basic definitions and properties.

Let F_1, F_2 be two points in the plane, and $a > F_1F_2$ be some real number. An *ellipse* $\mathcal{E}$ with foci F_1, F_2 and major axis a is the locus of the points X in the plane such that $XF_1 + XF_2 = a$. We always assume that $F_1 \neq F_2$.

For each focus F_i of $\mathcal{E}$, there exists a corresponding line d_i which is called a *directrix* such that $\mathcal{E}$ is the locus of points X in the plane such that the ratio $XF_i/\rho(X, d_i)$ is constant. The value ε of this ratio is called the *eccentricity* of ellipse $\mathcal{E}$.

Let ℓ be a line tangent to $\mathcal{E}$ at some point X. Then the segments F_1X and F_2X form equal angles with ℓ. In other words, if F_1' is the point symmetrical to F_1 about ℓ, then the points F_1', X, F_2 are collinear. This property is called *the optical property* of an ellipse.

For every ellipse, we construct a corresponding spheroid in the following way.

Definition 1. Let $\mathcal{E}$ be an ellipse with foci F_1, F_2 and major axis a. Consider a spheroid $\mathcal{D}$ obtained by rotating $\mathcal{E}$ about the axis F_1F_2; we call $\mathcal{D}$ a *spheroid generated by* $\mathcal{E}$. The foci, major axis, and eccentricity of this spheroid are defined as those for the ellipse $\mathcal{E}$.

Remark 2. The spheroid of rotation $\mathcal{D}$ is sometimes called a *prolate spheroid* in contrast to *oblate spheroid* obtained from an ellipse by rotating it about its minor axis.

The next geometrical properties of spheroid $\mathcal{D}$ can be derived easily from the properties of ellipse mentioned above. First, spheroid $\mathcal{D}$ is the locus of points X *in the space* such that $XF_1 + XF_2 = a$. Next, for a focus F_i, define the corresponding *directrix plane* δ_i as the plane perpendicular to the plane of the ellipse and containing the directrix d_i. Then $\mathcal{D}$ is the locus of points X in the space such that the ratio $XF_i/\rho(X, \delta_i) = \varepsilon$ (here, ε is again the eccentricity of ellipse $\mathcal{E}$). Finally, let λ be a plane tangent to $\mathcal{E}$ at some point X, and let F_1' be the point symmetrical to F_1 about λ. Then the points F_1', X, F_2 are collinear.

2. Proof of Theorem 1

We need the following lemma.

Lemma 1. *Let A, B, C be three non-collinear points, and let $\mathcal{D}_1$, $\mathcal{D}_2$ be two prolate spheroids with foci A, B and A, C, respectively. Then all their common points belong to a plane perpendicular to the plane (ABC).*

Proof. Let δ_1, δ_2 be the directrix planes of $\mathcal{D}_1$ and $\mathcal{D}_2$ corresponding to the common focus A, and ε_1, ε_2 be their eccentrities. Consider any common point X of $\mathcal{D}_1$ and $\mathcal{D}_2$. Then it shares the same side of δ_1 as A; analogously, it shares the same side of δ_2 as A. Thus, all such points lie in one dihedral angle defined by δ_1 and δ_2.

Next, from the directrix property we have $AX = \varepsilon_1 \cdot \rho(X, \delta_1) = \varepsilon_2 \cdot \rho(X, \delta_2)$. Hence $\dfrac{\rho(X, \delta_1)}{\rho(X, \delta_2)} = \dfrac{\varepsilon_2}{\varepsilon_1}$. Obviously, all such points in our dihedral angle lie in some plane γ (the planes γ, δ_1, and δ_2 intersect by a line). Finally, from the symmetry about the plane (ABC) we obtain that $\gamma \perp (ABC)$. □

Proof of Theorem 1. Denote by α the plane containing the ellipses $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$. Let $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ be the spheroids generated by $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$.

We consider only the case when these three spheroids share a common point T. By Lemma 1, for every $1 \leq i < j \leq 3$ there exists a plane γ_{ij} containing all the common points of $\mathcal{D}_i$ and $\mathcal{D}_j$, and $\gamma_{ij} \perp \alpha$. Then the projection of γ_{ij} onto α is a line, and it contains the points A_{ij} and B_{ij}; hence this projection is the line $A_{ij}B_{ij}$. On the other hand, point T belongs to all three planes γ_{ij}, hence its projection is the common point of all three lines $A_{ij}B_{ij}$. Thus the theorem is proved. □

Remark 3. To extend this proof to the case when there is no point of intersection of the three spheroids, one may consider them in the complex three-dimensional space $\mathbb{C}^3$. In this space, our three spheroids always share a common point.

3. Proof of Theorem 2

We also need an auxiliary lemma.

Lemma 2. *Let A, B, C be three non-collinear points, and let $\mathcal{D}_1$, $\mathcal{D}_2$ be two prolate spheroids with foci A, B and A, C, respectively. Then all the planes tangent to both spheroids share a common point on the line BC.*

Proof. Let λ be some plane tangent to $\mathcal{D}_1$, $\mathcal{D}_2$ at points X_1, X_2. Denote by b and c the major axes of D_1 and D_2, respectively. Let the point A' be the reflection of A about λ. By the optical property, the points A', X_1, B are collinear (X lies between A' and B), and hence $A'B = A'X_1 + X_1B = AX_1 + X_1B = b$. Analogously, $A'C = c$.

Thus, the point A' belongs to the intersection of two spheres with centers B, C and radii b, c, respectively. This intersection is some circle Γ (independent of λ) with the center lying on the line BC.

Now consider the sphere $\mathcal{S}$ containing Γ and A; let O be its center (this sphere may degenerate to a plane; in this case, O is an ideal point). Since the line BC is perpendicular to Γ and contains its center, the point O is collinear with B and C. Recall that λ is a perpendicular bisector of the segment AA', and both points A and A' belong to $\mathcal{S}$. Hence λ contains a fixed point O. □

Remark 4. It is easy to note that O is an ideal point if and only if $AB^2 - AC^2 = b^2 - c^2$. One may see also that all the common tangents λ touch some cone whose apex is O, and base is some conic.

Proof of Theorem 2. Again, we denote by α the plane containing the ellipses $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$, and consider the spheroids $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ generated by $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$. By Lemma 2, for every $1 \leq i < j \leq 3$ there exists a common point O_{ij} of all the common tangent planes to $\mathcal{D}_i$ and $\mathcal{D}_j$. The lines a_{ij} and b_{ij} belong to some common tangent planes to $\mathcal{D}_i$ and $\mathcal{D}_j$ (these planes are perpendicular to α). Hence we have $K_{ij} = O_{ij}$.

We consider only the case when there exists a plane λ tangent to all three spheroids. Surely, λ is different from α. Hence all three points O_{ij} belong to the line $\alpha \cap \lambda$, therefore they are collinear. □

Remark 5. The extention to the general case is analogous to that in the previous theorem. That is, one may consider any *complex* plane λ touching all three ellipsoids; such plane always exists.

The author is very grateful to A.V. Akopyan for the motivation and valuable remarks.

References

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[2] E. Neville. A focus-sharing set of three conics. *The Mathematical Gazette*, 20(239):182–183, 1936.

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