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TWO THEOREMS ON THE FOCUS-SHARING ELLIPSES: A THREE-DIMENSIONAL VIEW

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ABSTRACT. Consider three ellipses each two of which share a common focus. The "radical axes" of the pairs of these ellipses are concurrent, and the points of intersection of the common tangents to the pairs of these ellipses are collinear.

We present short synthetical proofs of these facts. Both proofs deal with the prolate spheroids having the given ellipses as axial sections.

1. Introduction

We present synthetical proofs of two theorems on three ellipses sharing common foci. The proofs deal with three-dimensional interpretation of the configuration.

Throughout the paper, we speak on the intersection of lines and planes in a projective sense; thus, all the points of intersection may appear to be ideal.

Both theorems deal with the following configuration. Let $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ be three ellipses such that each two of them share a common focus. We assume that none of the ellipses lies inside the other one. Note that there exist two different configurations satisfying this property: three ellipses with a common focus, and three ellipses with the pairs of foci (A, B), (A, C), and (B, C). Our proofs do not distinguish these cases.

The first theorem was firstly proved by J. H. Neville [2]. There are some synthetical proofs of this theorem, see, for instance, [1].

Theorem 1. *Let A_{ij}, B_{ij} $(1 \leq i < j \leq 3)$ be the points of intersection of ellipses $\mathcal{E}_i$ and $\mathcal{E}_j$. Then the lines $A_{12}B_{12}$, $A_{13}B_{13}$, and $A_{23}B_{23}$ are concurrent.*

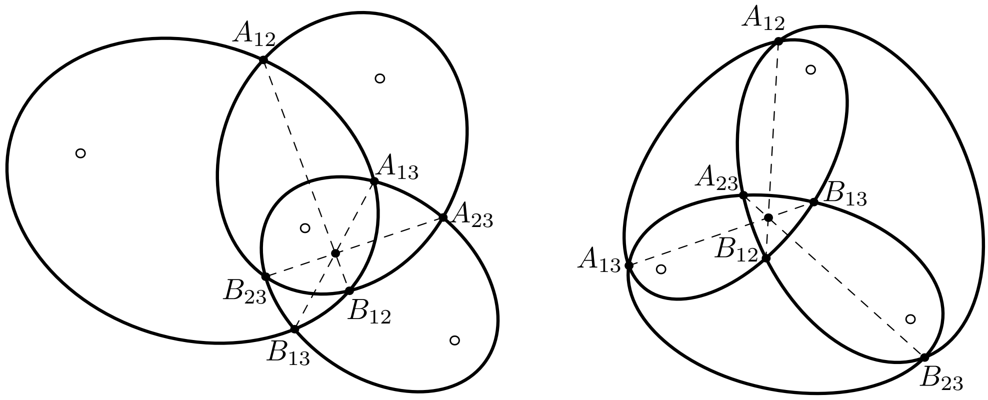

Fig. 1.

The second one was noticed by A. Akopyan; he revealed it to the author in a private communication. In some rough sense, it may be considered as a dual theorem to the previous one.

Theorem 2. *Let a_{ij}, b_{ij} be the two common tangents to $\mathcal{E}_i$ and $\mathcal{E}_j$ $(1 \leq i < j \leq 3)$; denote by K_{ij} the point of their intersection. Then the points K_{12}, K_{13}, and K_{23} are collinear.*

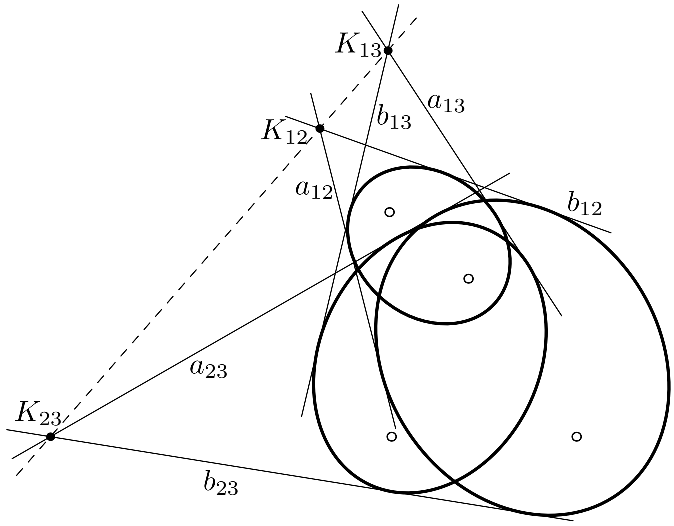

Fig. 2.

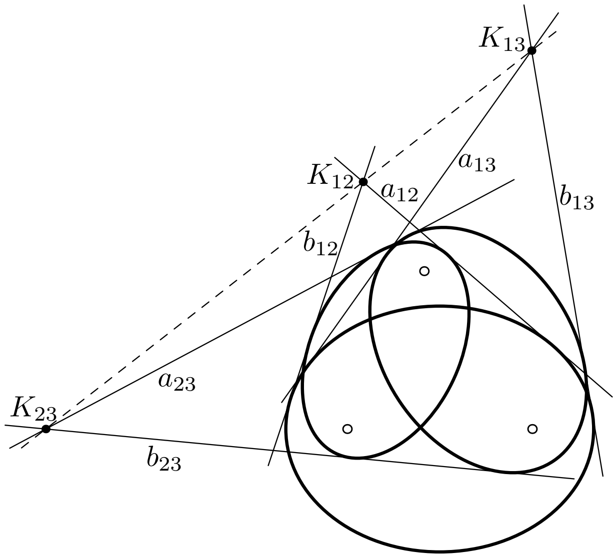

Fig. 3.

Remark 1. If all three ellipses $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ have a common focus F, then they become circles under a duality transform with a center F. Hence in this case both facts allow simpler proofs.

In the proofs of both theorems, we use some geometrical features of ellipses and ellipsoids. Here we recall some basic definitions and properties.

Let F_1, F_2 be two points in the plane, and $a > F_1F_2$ be some real number. An *ellipse* $\mathcal{E}$ with foci F_1, F_2 and major axis a is the locus of the points X in the plane such that $XF_1 + XF_2 = a$. We always assume that $F_1 \neq F_2$.

For each focus F_i of $\mathcal{E}$, there exists a corresponding line d_i which is called a *directrix* such that $\mathcal{E}$ is the locus of points X in the plane such that the ratio $XF_i/\rho(X, d_i)$ is constant. The value ε of this ratio is called the *eccentricity* of ellipse $\mathcal{E}$.

Let ℓ be a line tangent to $\mathcal{E}$ at some point X. Then the segments F_1X and F_2X form equal angles with ℓ. In other words, if F_1' is the point symmetrical to F_1 about ℓ, then the points F_1', X, F_2 are collinear. This property is called *the optical property* of an ellipse.

For every ellipse, we construct a corresponding spheroid in the following way.

Definition 1. Let $\mathcal{E}$ be an ellipse with foci F_1, F_2 and major axis a. Consider a spheroid $\mathcal{D}$ obtained by rotating $\mathcal{E}$ about the axis F_1F_2; we call $\mathcal{D}$ a *spheroid generated by* $\mathcal{E}$. The foci, major axis, and eccentricity of this spheroid are defined as those for the ellipse $\mathcal{E}$.

Remark 2. The spheroid of rotation $\mathcal{D}$ is sometimes called a *prolate spheroid* in contrast to *oblate spheroid* obtained from an ellipse by rotating it about its minor axis.

The next geometrical properties of spheroid $\mathcal{D}$ can be derived easily from the properties of ellipse mentioned above. First, spheroid $\mathcal{D}$ is the locus of points X *in the space* such that $XF_1 + XF_2 = a$. Next, for a focus F_i, define the corresponding *directrix plane* δ_i as the plane perpendicular to the plane of the ellipse and containing the directrix d_i. Then $\mathcal{D}$ is the locus of points X in the space such that the ratio $XF_i/\rho(X, \delta_i) = \varepsilon$ (here, ε is again the eccentricity of ellipse $\mathcal{E}$). Finally, let λ be a plane tangent to $\mathcal{E}$ at some point X, and let F_1' be the point symmetrical to F_1 about λ. Then the points F_1', X, F_2 are collinear.

2. Proof of Theorem 1

We need the following lemma.

Lemma 1. *Let A, B, C be three non-collinear points, and let $\mathcal{D}_1$, $\mathcal{D}_2$ be two prolate spheroids with foci A, B and A, C, respectively. Then all their common points belong to a plane perpendicular to the plane (ABC).*

Proof. Let δ_1, δ_2 be the directrix planes of $\mathcal{D}_1$ and $\mathcal{D}_2$ corresponding to the common focus A, and ε_1, ε_2 be their eccentrities. Consider any common point X of $\mathcal{D}_1$ and $\mathcal{D}_2$. Then it shares the same side of δ_1 as A; analogously, it shares the same side of δ_2 as A. Thus, all such points lie in one dihedral angle defined by δ_1 and δ_2.

Next, from the directrix property we have $AX = \varepsilon_1 \cdot \rho(X, \delta_1) = \varepsilon_2 \cdot \rho(X, \delta_2)$. Hence $\dfrac{\rho(X, \delta_1)}{\rho(X, \delta_2)} = \dfrac{\varepsilon_2}{\varepsilon_1}$. Obviously, all such points in our dihedral angle lie in some plane γ (the planes γ, δ_1, and δ_2 intersect by a line). Finally, from the symmetry about the plane (ABC) we obtain that $\gamma \perp (ABC)$. □

Proof of Theorem 1. Denote by α the plane containing the ellipses $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$. Let $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ be the spheroids generated by $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$.

We consider only the case when these three spheroids share a common point T. By Lemma 1, for every $1 \leq i < j \leq 3$ there exists a plane γ_{ij} containing all the common points of $\mathcal{D}_i$ and $\mathcal{D}_j$, and $\gamma_{ij} \perp \alpha$. Then the projection of γ_{ij} onto α is a line, and it contains the points A_{ij} and B_{ij}; hence this projection is the line $A_{ij}B_{ij}$. On the other hand, point T belongs to all three planes γ_{ij}, hence its projection is the common point of all three lines $A_{ij}B_{ij}$. Thus the theorem is proved. □

Remark 3. To extend this proof to the case when there is no point of intersection of the three spheroids, one may consider them in the complex three-dimensional space $\mathbb{C}^3$. In this space, our three spheroids always share a common point.

3. Proof of Theorem 2

We also need an auxiliary lemma.

Lemma 2. *Let A, B, C be three non-collinear points, and let $\mathcal{D}_1$, $\mathcal{D}_2$ be two prolate spheroids with foci A, B and A, C, respectively. Then all the planes tangent to both spheroids share a common point on the line BC.*

Proof. Let λ be some plane tangent to $\mathcal{D}_1$, $\mathcal{D}_2$ at points X_1, X_2. Denote by b and c the major axes of D_1 and D_2, respectively. Let the point A' be the reflection of A about λ. By the optical property, the points A', X_1, B are collinear (X lies between A' and B), and hence $A'B = A'X_1 + X_1B = AX_1 + X_1B = b$. Analogously, $A'C = c$.

Thus, the point A' belongs to the intersection of two spheres with centers B, C and radii b, c, respectively. This intersection is some circle Γ (independent of λ) with the center lying on the line BC.

Now consider the sphere $\mathcal{S}$ containing Γ and A; let O be its center (this sphere may degenerate to a plane; in this case, O is an ideal point). Since the line BC is perpendicular to Γ and contains its center, the point O is collinear with B and C. Recall that λ is a perpendicular bisector of the segment AA', and both points A and A' belong to $\mathcal{S}$. Hence λ contains a fixed point O. □

Remark 4. It is easy to note that O is an ideal point if and only if $AB^2 - AC^2 = b^2 - c^2$. One may see also that all the common tangents λ touch some cone whose apex is O, and base is some conic.

Proof of Theorem 2. Again, we denote by α the plane containing the ellipses $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$, and consider the spheroids $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ generated by $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$. By Lemma 2, for every $1 \leq i < j \leq 3$ there exists a common point O_{ij} of all the common tangent planes to $\mathcal{D}_i$ and $\mathcal{D}_j$. The lines a_{ij} and b_{ij} belong to some common tangent planes to $\mathcal{D}_i$ and $\mathcal{D}_j$ (these planes are perpendicular to α). Hence we have $K_{ij} = O_{ij}$.

We consider only the case when there exists a plane λ tangent to all three spheroids. Surely, λ is different from α. Hence all three points O_{ij} belong to the line $\alpha \cap \lambda$, therefore they are collinear. □

Remark 5. The extention to the general case is analogous to that in the previous theorem. That is, one may consider any *complex* plane λ touching all three ellipsoids; such plane always exists.

The author is very grateful to A.V. Akopyan for the motivation and valuable remarks.

References

[1] B. Lawrence. 1238. note on focus sharing conics. *The Mathematical Gazette*, 21(243):160–161, 1937.

[2] E. Neville. A focus-sharing set of three conics. *The Mathematical Gazette*, 20(239):182–183, 1936.

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ON CERTAIN TRANSFORMATIONS PRESERVING PERSPECTIVITY OF TRIANGLES

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Abstract. To any pair of perspective triangles we assign a family F of projective transformations such that image of the second triangle under any transformation from F is perspective to the first triangle. This helps us to solve some interesting problems.

All constructions under consideration are on the complex projective plane. We abbreviate "projective transformation" as PT and "affine transformation" as AT. We denote by H_O^c the homothety with center O and factor c.

Let ABC and $A'B'C'$ be perspective triangles with the perspector O and the perspectrix ℓ. We try to find some good PT taking ABC to $A'B'C'$. For $\ell = \infty$, this is the homothety H_O^c for some complex number c. Now let us generalize this to arbitrary ℓ. A PT is called a *phomothety* if it takes each point P to the point P' such that O, P and P' are collinear and $(P', P, O, K) = c$, where $K = \ell \cap OP$. It is easy to see that, for $\ell = \infty$, the phomothety is the homothety H_O^c. It is clear that $f^{-1} \circ H^c_{f(O),f(\ell)} \circ f = H^c_{O,\ell}$ for any PT f. For a PT f preserving O and taking ℓ to ∞, we have $f^{-1} \circ H_O^c \circ f = H^c_{O,\ell}$.

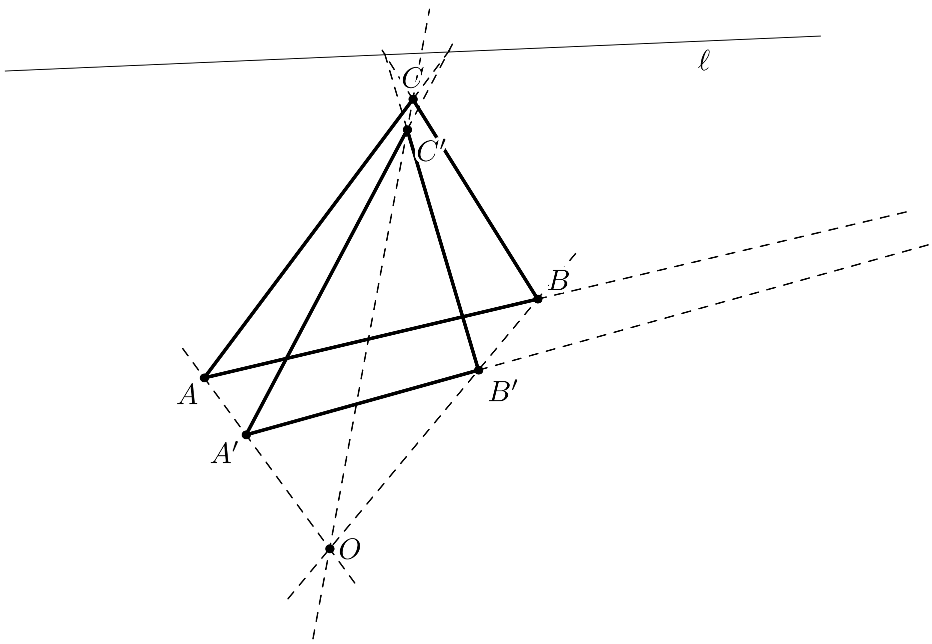

Fig. 1.

We can prove the following properties of phomothety:
1) If $A'B'C' = H^c_{O,\ell}(ABC)$ then $A'B'C'$ and ABC are perspective with perspector O and perspectrix ℓ.

We have $A'B'C' = f^{-1}(H^c_O(f(ABC)))$ and $f(A'B'C') = H^c_O(f(ABC))$, hence $f(ABC)$ and $f(A'B'C')$ are perspective with perspector O and perspectrix ∞, so ABC and $A'B'C'$ are perspective with perspector $f^{-1}(O) = O$ and perspectrix $f^{-1}(\infty) = \ell$.
2) If $A'B'C'$ and ABC are perspective with perspector O and perspectrix ℓ then $A'B'C' = H^c_{O,\ell}(ABC)$ for some complex number c.

The triangles $f(ABC)$ and $f(A'B'C')$ are perspective with perspector $f(O) = O$ and perspectrix $f(\ell) = \infty$, hence for some complex c we have $f(A'B'C') = H^c_O(f(ABC))$ and $A'B'C' = H^c_{O,\ell}(ABC)$.
3) $H^c_{O,\ell} \circ H^d_{O,\ell} = H^{cd}_{O,\ell}$

$H^{cd}_{O,\ell} = f^{-1} \circ H^{cd}_O \circ f = f^{-1} \circ H^c_O \circ H^d_O \circ f = f^{-1} \circ H^c_O \circ f \circ f^{-1} \circ H^d_O \circ f = H^c_{O,\ell} \circ H^d_{O,\ell}$.

Suppose that ω is a conic, A is a point, and $H^c_{O,\ell}$ is a phomothety with ℓ being the polar of O in ω. Let m be the polar of A in ω and m' be the polar of A in $\psi = H^c_{O,\ell}(\omega)$ (ψ is obviously a conic, since $H^c_{O,\ell}$ is a PT). How are m and m' related?

Lemma 1. $m' = H^{c^2}_{O,\ell}(m)$.

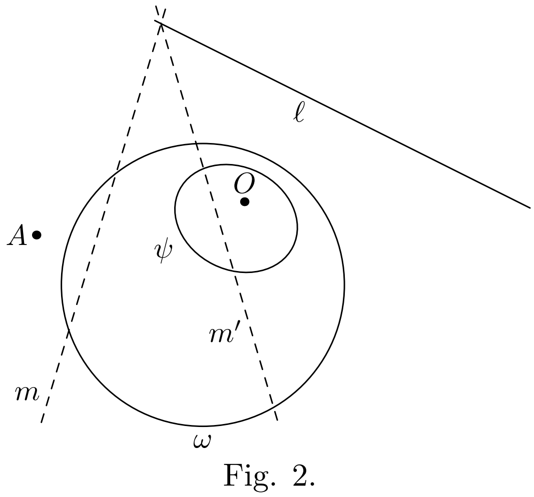

Fig. 2.

Proof. Indeed, find a PT f_1 taking ℓ to ∞, and an AT f_2 making $f_1(\omega)$ a circle, both of them preserving O and A. Denote the composition $f_2 \circ f_1$ by f. We have $f(\ell) = \infty$, $f(O) = O$, $f(O)$ is polar of $f(\ell)$ in $f(\omega)$, therefore O is the center of $f(\omega)$. Since $f(\ell) = \infty$, we have $H^c_{O,\ell} = f^{-1} \circ H^c_O \circ f$, $\psi = H^c_{O,\ell}(\omega)$, hence $f(\psi) = H^c_O(f(\omega))$. Thus $f(\psi)$ and $f(\omega)$ are concentric circles, $f(m)$ is the polar of A in $f(\omega)$, $f(m')$ is the polar of A in $f(\psi)$.

It is clear that $f(m') = H^{c^2}_O(f(m))$, so $m' = H^{c^2}_{O,\ell}(m)$. □

This means that we can easily perform some phomotheties and watch how the polars of some points vary.

The following well-known lemma gives us a pair of perspective triangles in a different way:

Lemma 2. *Let ω be a conic and ABC be a triangle. Denote by A', B', C' the poles of BC, CA, AB in ω. Then ABC and $A'B'C'$ are perspective, and their perspectrix is the polar of their perspector in ω.*

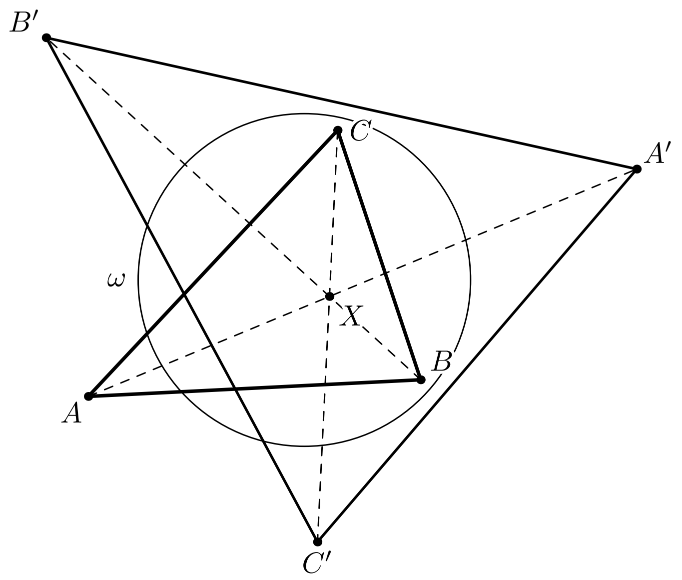

Fig. 3.

Proof. Let $X = AA' \cap BB'$. Applying a PT make ω a circle; after this do another PT preserving ω and taking X to the center of ω. We have $XA' \perp BC$, and X, A, A' are collinear. Similarly, we have $AX \perp BC$, $BX \perp CA$, therefore X is the orthocenter of ABC, hence $CX \perp AB$. Since $XC' \perp AB$, the points C, C', X are collinear. This means that ABC and $A'B'C'$ are perspective, and their perspectrix ∞ is the polar of their perspector X in ω. □

Now we are ready to prove the following theorem:

Theorem 3. *Let ω be a conic and ABC a triangle. Let A', B', C' be the poles of BC, CA, AB in ω. Let $A_1B_1C_1$ be the triangle perspective to $A'B'C'$ with perspector D and perspectrix ℓ, where ℓ is the polar of D in ω. Then ABC and $A_1B_1C_1$ are perspective.*

Proof. From perspectivity of $A_1B_1C_1$ and $A'B'C'$ we have $A_1B_1C_1 = H^c_{D,\ell}(A'B'C')$ for some complex c. Let $\psi = H^{\sqrt{c}}_{D,\ell}(\omega)$. Since $A'B'$, $B'C'$, $C'A'$ are the polars of C, A, B in ω, then from Lemma 1 we have that A_1B_1, B_1C_1, C_1A_1 are the polars of C, A, B in ψ. Therefore from Lemma 2 we conclude that ABC and $A_1B_1C_1$ are perspective. □

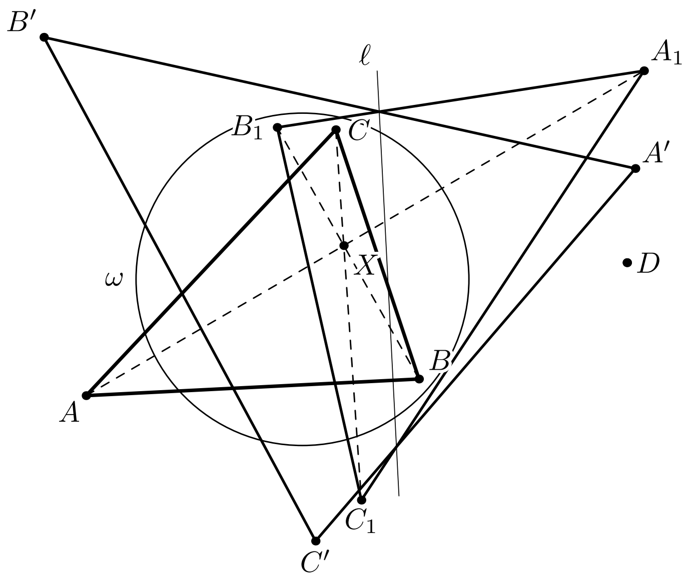

Fig. 4.

Theorem 4. *Suppose that ω is a conic, A_1, A_2, ..., A_n is a collection of points and ℓ_1, ℓ_2, ..., and ℓ_n are their polars in ω. Define the following transformation: choose any pair A_i, ℓ_i and a complex number c; for every j, the new ℓ_j is defined as the image of old ℓ_j under $H^c_{A_i,\ell_i}$, but A_j does not change. Apply this transformation any number of times. Then $A_xA_yA_z$ and triangle formed by ℓ_x, ℓ_y, ℓ_z are perspective for any x, y, z.*

Proof. Let us change ω after every transformation. The new ω is defined as the image of the old ω under $H^{\sqrt{c}}_{A_i,\ell_i}$. Then, as in Theorem 3, the new ℓ_j will be the polar of A_j in the new ω for every j. Thus, after all transformations, ℓ_x, ℓ_y, ℓ_z are the polars of A_x, A_y, A_z in the new ω. Therefore $A_xA_yA_z$ is perspective to the triangle formed by ℓ_x, ℓ_y, ℓ_z. □

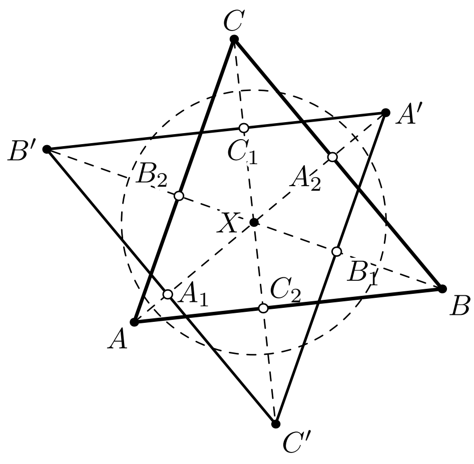

Fig. 5.

Why do we need a conic in Theorem 4? A conic allows us to use Lemma 2. What if we keep only this lemma and forget about the conic? Let us see.

Lemma 5. *If ABC and $A'B'C'$ are perspective, then A', B', C' are the poles of BC, CA, AB with respect to some conic ω.*

Proof. Applying a PT, take the perspectrix of ABC and $A'B'C'$ to ∞. After that, by suitable AT, make $AX \perp BC$ and $BX \perp AC$, where X is the perspector of ABC and $A'B'C'$. Then X is the orthocenter of ABC. Denote $A_1 = AX \cap B'C'$, $B_1 = BX \cap C'A'$, $C_1 = CX \cap A'B'$; $A_2 = AX \cap BC$, $B_2 = BX \cap CA$, $C_2 = CX \cap AB$. The perspectrix of ABC and $A'B'C'$ is ∞, hence $A'B'C' = H_X^c(ABC)$. We know that $\overline{XA} \cdot \overline{XA_2} = \overline{XB} \cdot \overline{XB_2} = \overline{XC} \cdot \overline{XC_2}$. From homothety we have $\overline{XA_1} = c\overline{XA_2}$, $\overline{XB_1} = c\overline{XB_2}$, $\overline{XB_1} = c\overline{XB_2}$, so $\overline{XA} \cdot \overline{XA_1} = \overline{XB} \cdot \overline{XB_1} = \overline{XC} \cdot \overline{XC_1}$. Let ω be the circle with center X and radius $r = \sqrt{\overline{XA} \cdot \overline{XA_1}}$. Then we have $AX \perp B'C'$ and $\overline{XA} \cdot \overline{XA_1} = r^2$, hence $B'C'$ is polar of A in ω. Therefore A', B', C' are poles of BC, CA, AB in ω. Thus, the preimage of ω with respect to the performed transformations is the desired conic. □

For $n > 2$, call a collection of n points $A_1, A_2, \ldots, A_n$ and n lines $\ell_1, \ell_2, \ldots, \ell_n$ *good*, if

1) A_i, A_j, A_k are not collinear for any i, j and k
2) ℓ_i, ℓ_j, ℓ_k are not concurrent for any i, j and k
3) $A_xA_yA_z$ and the triangle formed by ℓ_x, ℓ_y, ℓ_z are perspective for any x, y, z.

Call a collection of n points $A_1, A_2, \ldots, A_n$ and n lines $\ell_1, \ell_2, \ldots, \ell_n$ *good+*, if they are good and $A_i \notin \ell_j$ for any i, j.

Theorem 6. *If n points $A_1, A_2, \ldots, A_n$ and n lines $\ell_1, \ell_2, \ldots, \ell_n$ are good+, then $\ell_1, \ell_2, \ldots, \ell_n$ are the polars of $A_1, A_2, \ldots, A_n$ in some conic ω.*

Proof. For $n = 3$ the theorem is proved. Set $A = A_1$, $\ell_A = \ell_1$, $B = A_2$, $\ell_B = \ell_2$, $C = A_3$, $\ell_C = \ell_3$, D any point different from $\{A_i\}$, and ℓ_D the corresponding line. Let ω be a conic such that A, B, C are the poles of ℓ_A, ℓ_B, ℓ_C in ω. Let ℓ' be the polar of D in ω. We know, that ABD is perspective to the triangle formed by ℓ_A, ℓ_B, ℓ_D; ABD and the triangle formed by ℓ_A, ℓ_B, ℓ' are perspective as well. Both pairs have the same perspectrix ℓ_1: indeed, both perspectrixes pass through $AD \cap \ell_B$ and $BD \cap \ell_A$; or $AD \cap \ell_B = BD \cap \ell_A$. In the first case $AB \neq \ell_1$, else $AD \cap \ell_B \in AB$, $A \in \ell_B$, but this is false; then AB, ℓ_D, ℓ' are concurrent since $AB \cap \ell_1 \in \ell_D$, $AB \cap \ell_1 \in \ell'$; in the second case $\ell_A \cap \ell_B = D$, but $D \notin \ell_A$. Similarly for the triangles BCD and CAD. Thus, $\ell_D \cap \ell'$ have at least three points: $AB \cap \ell_D$, $BC \cap \ell_D$, $CA \cap \ell_D$, therefore $\ell' = \ell_D$, and ω is the needed conic. □

Theorem 7. *For a good+ collection of n points $A_1, A_2, \ldots, A_n$ and n lines $\ell_1, \ell_2, \ldots, \ell_n$, define the following tranformation: choose any pair A_i, ℓ_i and a complex number c; for every j, the new ℓ_j is image of the old ℓ_j under $H^c_{A_i,\ell_i}$, but A_j does not change. Apply this transformation any number of times. Then $A_1, A_2, \ldots, A_n$ and $\ell_1, \ell_2, \ldots, \ell_n$ are good.*

Proof. From Theorem 6 we have a conic ω such that each ℓ_i is the polar of A_i in ω. So, from Theorem 4 we have that, after all transformations, $A_xA_yA_z$ and the triangle formed by ℓ_x, ℓ_y, ℓ_z are perspective. Thus, A_1, A_2, $\ldots$, A_n and ℓ_1, ℓ_2, $\ldots$, ℓ_n are good. □

Now let us explore Theorem 4 for $n = 4$. First, we prove the following lemma:

Lemma 8. *In the notation of Theorem 3, the locus of perspectors of* ABC *and* $A_1B_1C_1$ *with fixed* A, B, C, D *and* ω *and variable* $A_1B_1C_1$ *is the conic* $\psi = \psi(A, B, C, D, \omega)$ *through* A, B, C, D *and the perspector of* ABC *and* $A'B'C'$.

Proof. Let X be the perspector of ABC and $A_1B_1C_1$. We know that $A_1B_1C_1 = H^c_{D,\ell}(A'B'C')$. If $c = 0$ then $A_1 = B_1 = C_1 = D$, hence $X = D$. If $c = 1$, then $A_1 = A'$, $B_1 = B'$, $C_1 = C'$, hence $X = X'$, where X' is the perspector of ABC and $A'B'C'$. So D and X' are in the locus. Apply a PT taking ℓ to ∞, and, after that, an AT making ω a circle. Then D is center of that circle. Let us find a triangle $A_1B_1C_1$ such that ABC and $A_1B_1C_1$ are perspective with perspector A. Set $B'D \cap AB = B_1$, $C'D \cap AC = C_1$. We know $C'D \perp AB$, $B'D \perp AC$, hence $C_1D \perp AB_1$, $B_1D \perp AC_1$, and therefore $AD \perp B_1C_1$. On the other hand, $AD \perp B'C'$, hence $B_1C_1 \parallel B'C'$, and we can find A_1 such that $A_1B_1C_1 = H^c_D(A'B'C')$. Since $BB_1 \cap CC_1 = A$, we conclude that A is the perspector of ABC and $A_1B_1C_1$.

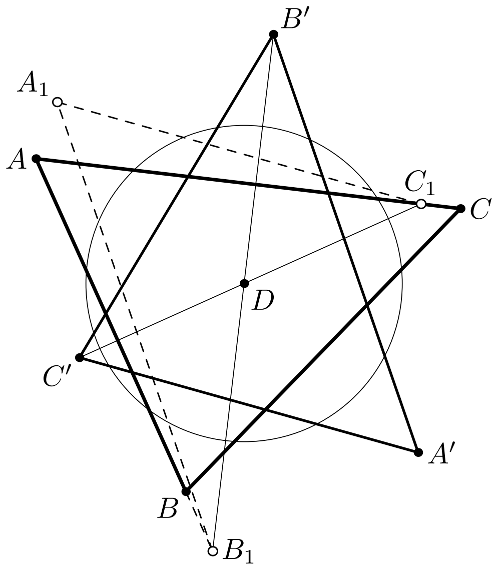

Fig. 6.

Set $X_A = A_1$, $Y_A = B_1$, $Z_A = C_1$. Similarly define X_B, Y_B, Z_B, X_C, Y_C, Z_C. Thus A, B and C are in the locus. Now let us show that the locus is a conic. We have $A_1B_1C_1 = H^c_D(A'B'C')$ for some c. AA_1, BB_1, CC_1 meet ψ second time at A_2, B_2, C_2. (C, D, X', A_2) (this cross-ratio is on ψ) $= (AC, AD, AX', AA_2)$. $(AC, AD, AX', AA_2) = (AX_C, AD, AA', AA_1)$ since the lines coincide. $(AX_C, AD, AA', AA_1) = (X_C, D, A', A_1)$ since these 4 points are collinear.

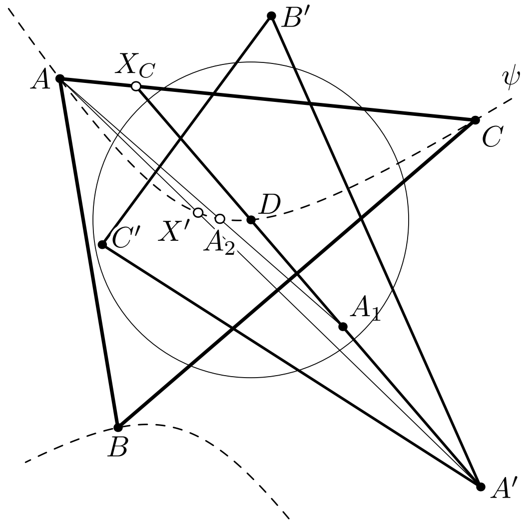

Fig. 7.

Similarly (C, D, X', B_2) (this cross-ratio is on ψ) $= (Y_C, D, B', B_1)$. On the other hand, $X_C Y_C Z_C$, $A'B'C'$ and $A_1B_1C_1$ are pairwise homothetic with the center D, hence $(X_C, D, A', A_1) = (Y_C, D, B', B_1)$ and therefore $(C, D, X', A_2) = (C, D, X', B_2)$. Hence we have $A_2 = B_2$ and, similarly, $A_2 = C_2$, so $A_2 \in AA_1, BB_1, CC_1$, $A_2 = X$, $A_2 \in \psi$, and therefore $X \in \psi$. Note that (X_C, D, A', A_1) can be any complex number, so (C, D, X', X) can be any complex number. Thus the locus of X is ψ. □

Lemma 9. *In the notation of Lemma 8, we have*

$$\psi(A, B, C, D, \omega) = \psi(A, B, C, D, H^c_{D,\ell}(\omega)).$$

Proof. $\psi(A, B, C, D, \omega)$ is the conic through A, B, C, D and X', the perspector of ABC and $A'B'C'$. Clearly, $A, B, C, D \in \psi(A, B, C, D, H^c_{D,\ell}(\omega))$; let us prove that $X' \in \psi(A, B, C, D, H^c_{D,\ell}(\omega))$. From Lemma 1, we have $A_2 = H^{c^2}_{D,\ell}(A')$, $B_2 = H^{c^2}_{D,\ell}(B')$, $C_2 = H^{c^2}_{D,\ell}(C')$ are the poles of BC, CA, AB in $H^c_{D,\ell}(\omega)$. From Lemma 8 we see that the perspector of ABC and $H^{1/c^2}_{D,\ell}(A_2)H^{1/c^2}_{D,\ell}(B_2)H^{1/c^2}_{D,\ell}(C_2)$ are in $\psi(A, B, C, D, H^c_{D,\ell}(\omega))$. But $H^{1/c^2}_{D,\ell}(A_2) = H^{1/c^2}_{D,\ell}(H^{c^2}_{D,\ell}(A')) = A'$ and, similarly, $H^{1/c^2}_{D,\ell}(B_2) = B'$, $H^{1/c^2}_{D,\ell}(C_2) = C'$. Hence, $X' \in \psi(A, B, C, D, H^c_{D,\ell}(\omega))$ and $\psi(A, B, C, D, \omega) = \psi(A, B, C, D, H^c_{D,\ell}(\omega))$. □

Lemma 10. *For a conic ω and points A, B, C, D, let A' be the perspector of BCD and the triangle formed by the poles of CD, DB, BC in ω; the points B', C', D' are defined similarly. Then the points A, B, C, D, A', B', C', D' belong to the same conic.*

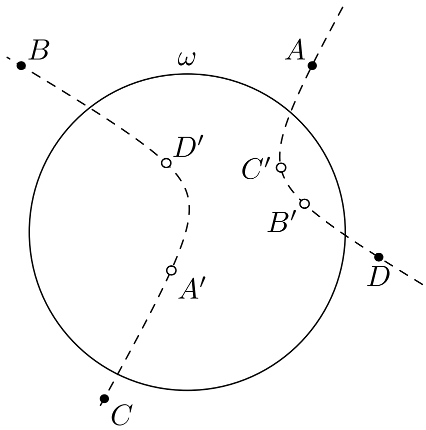

Fig. 8.

Proof. It suffices to check that A, B, C, D, A', B' belong to the same conic. Applying a PT, make ω a circle; after that apply another PT, preserving ω and taking D to the center of ω. Denote by X the pole of CD in ω. D is the center of ω, hence X is point on ∞, and $PX \perp CD$ for any point P. We have $X \in BA'$, hence $BA' \perp CD$. Similarly, $CA' \perp BD$. Thus, A' is the orthocenter of the triangle BCD. Similarly, B' is the orthocenter of the triangle ACD. Let δ be a rectangular hyperbola passing through A, B, C and D. Then it passes through the orthocenters of the triangles BCD and ACD, hence A, B, C, D, A', B' lie on δ. □

Lemma 11. *$\psi(A,B,C,D,\omega)$ does not depend on permutation of A, B, C, D.*

Proof. For a permutation preserving D, the assertion is obvious. Thus, it suffices to prove that $\psi(A,B,C,D,\omega) = \psi(B,C,D,A,\omega)$. Note that both conics in question are the conic from Lemma 10 for A, B, C, D, ω, since both of them pass through A, B, C, D and at least one perspector from Lemma 10. Hence $\psi(A,B,C,D,\omega)$ and $\psi(B,C,D,A,\omega)$ are equal. □

Theorem 12. *Suppose that A_1, A_2, A_3, A_4 are points, ω is a conic, and ℓ_1, ℓ_2, ℓ_3, ℓ_4 are polars of A_1, A_2, A_3, A_4 in ω. Define the transformation: choose any pair A_i, ℓ_i and a complex number c; for every j, the new ℓ_j is the image of the old ℓ_j under $H^c_{A_i,\ell_i}$, but A_j doesn't change. Apply this transformation any number of times. Then the perspector of $A_1A_2A_3$ and the triangle formed by ℓ_1, ℓ_2, ℓ_3 and similar perspectors are on conic through A_1, A_2, A_3, A_4, and this conic does not depend on the transformations.*

Proof. Let us prove that $\psi(A_1,A_2,A_3,A_4,\omega)$ is the desired conic. As in Theorem 4, we change ω after every transformation. From Lemma 9 and Lemma 11 we have that $\psi(A_1,A_2,A_3,A_4,\omega) = \psi(A_1,A_2,A_3,A_4,H^c_{A_i,\ell_i}(\omega))$. Therefore $\psi(A_1,A_2,A_3,A_4,\omega)$ does not change. But all needed perspectors always lie on $\psi(A_1,A_2,A_3,A_4,\omega)$ from Lemma 10, hence the assertion. □

Let us make some conclusions from the theorems and lemmas:

Lemma 13. *Suppose that A, B, C are points, ω is a conic with center O, and A', B', C' are poles of BC, CA, AB in ω. Then ABC and the image of $A'B'C'$ under any homothety with center O are perspective.*

Proof. This is Theorem 4 for the points A, B, C, O and the phomothety $H^c_{O,\infty}$. □

Proposition 14. *Suppose that ABC is a triangle, ω is its inscribed circle with center I, and A', B', C' are points of tangency of ω with BC, CA, AB, respectively. Then ABC and the image of $A'B'C'$ under any homothety with center I are perspective.*

Proof. This is a trivial conclusion from Lemma 13. □

Lemma 15. *Suppose that A, B, C are points, ω is a circle, A', B', C' are poles of BC, CA, AB in ω. Let ℓ_1 and ℓ_2 be a pair of perpendicular lines through the center of ω, suppose that f_1 is a compression in ℓ_1, f_2 is a compression in ℓ_2. Then ABC and the image of $A'B'C'$ under $f_1 \circ f_2$ are perspective.*

Proof. This is Theorem 4 for the points A, B, C, $X = \ell_1 \cap \infty$, $Y = \ell_2 \cap \infty$, the conic ω and two transformations with the point Y and the point X. Indeed, H^c_{X,ℓ_2} is the compression in ℓ_2 with the factor c; similarly, H^c_{Y,ℓ_1} is the compression in ℓ_1 with the factor c. □

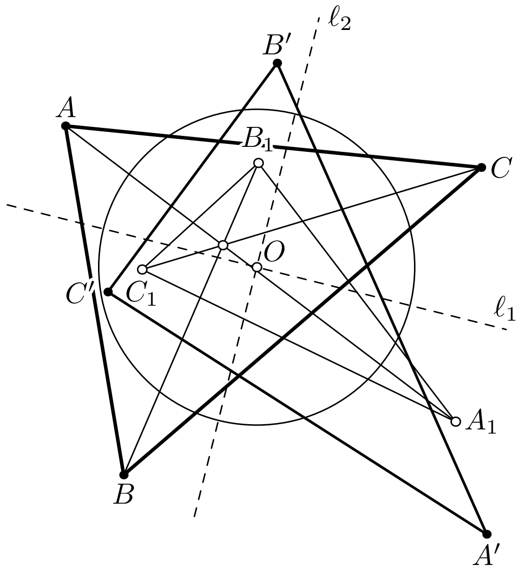

Fig. 9.

Proposition 16. *Let ABC be a triangle and ω be some Tucker's circle of ABC. Suppose that A', B', C' are poles of BC, CA, AB in ω, O and L are the circumcentre and symmedian point of ABC, respectively. Then ABC and the image of $A'B'C'$ under some compression (or just reflection) in OL are perspective.*

Proof. We know that the center of Tucker's circle is belong to OL, hence it is a trivial conclusion from Lemma 15. □

Proposition 17. *Let ABC be a triangle, ω its inscribed circle. Suppose that A', B', C' are points of tangency of ω with BC, CA, AB, respectively; ℓ_1 and ℓ_2 is a pair of perpendicular lines through the center of ω, f_1 is a compression in ℓ_1, f_2 is a compression in ℓ_2. Then ABC and the image of $A'B'C'$ under $f_1 \circ f_2$ are perspective.*

Proof. This is a trivial conclusion from Lemma 15. □

Proposition 18. *Suppose that A, B, C are points, ω is a circle, A', B', C' are poles of BC, CA, AB in ω. Let ℓ be a line through O (the center of ω), c be a number. Then ABC and the image of $A'B'C'$ under the reflection in ℓ and H_O^c are perspective.*

Proof. This is a trivial conclusion from Lemma 15. □

Proposition 19. *(Bulgarian Mathematical Olympiad, 2009). Suppose that ABC is a triangle, ω is its inscribed circle, A', B', C' are points of tangency of ω with BC, CA, AB. Let ℓ be some line through the center of ω. Then ABC and the image of $A'B'C'$ under the reflection in ℓ are perspective.*

Proof. This is a trivial conclusion from Proposition 18. □

The following proposition was proved by N. Beluhov in [1]:

Proposition 20. *Let ABC, $A'B'C'$ be triangles, O be a point. If the relations*

$$\angle(AO, BO) = \angle(A'C', B'C'), \quad \angle(A'O, B'O) = \angle(AC, BC),$$
$$\angle(BO, CO) = \angle(B'A', C'A'), \quad \angle(B'O, C'O) = \angle(BA, CA),$$
$$\angle(CO, AO) = \angle(C'B', A'B'), \quad \angle(C'O, A'O) = \angle(CB, AB),$$

hold, then ABC and $A'B'C'$ are perspective. (here $\angle(\ell_1, \ell_2)$ is the oriented angle from ℓ_1 to ℓ_2)

Proof. This is a trivial conclusion from Proposition 18, since every triangle $A'B'C'$ described in this problem can be obtained by reflection in some line through O and some homothety with center O from $A_0B_0C_0$, where A_0, B_0, C_0 are the poles of BC, CA, AB in the circle with center O and some radius. □

Proposition 21. *Suppose that ABC is a triangle, ω is its inscribed circle, A', B', C' are points of tangency of ω with BC, CA, AB. Let P be a point on the plane. Let $A'P$, $B'P$, $C'P$ meet ω second time at A_1, B_1, C_1, respectively. Then ABC and $A_1B_1C_1$ are perspective.*

Proof. This is Theorem 4 for the points A, B, C, P, the conic ω and the phomothety $H_{P,\ell}^{-1}$, where ℓ is the polar of P in ω. □

Proposition 22. *Let ABC be a triangle and AL be the bisector of the angle $\angle BAC$, $L \in BC$. Suppose that B', C' are foots of the perpendiculars from L to AB and to AC, respectively. Then $BC' \cap CB'$ is on the altitude of ABC from A.*

Proof. AL is the bisector of the angle $\angle BAC$, so $LB' = LC'$. Let ω be a circle with center L and radius LB' and h be the altitude of ABC from A. Then $h \cap \infty$, B', C' are the poles of BC, AC, AB in ω. Then from Lemma 2 we conclude that $BC' \cap CB'$, A, $h \cap \infty$ are collinear, hence $BC' \cap CB' \in h$. □

Proposition 23. *Let ABC be a triangle and AL be the bisector of the angle $\angle BAC$, $L \in BC$. Suppose that B', C' are foots of perpendiculars from L to AB and to AC, respectively. Suppose that B_1 is on LB' and C_1 is on LC', $B_1C_1 \parallel B'C'$. Then $BC_1 \cap CB_1$ is on the altitude of ABC from A.*

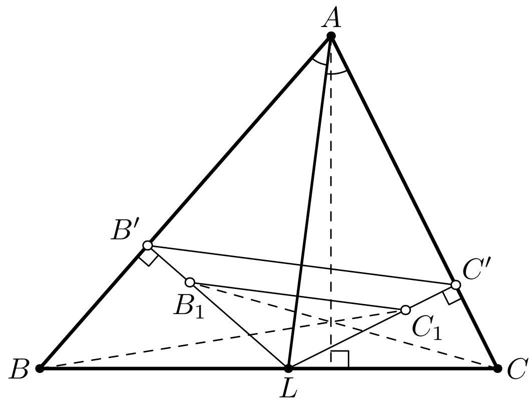

Fig. 10.

Proof. In the notation of the proof of Proposition 22, $h \cap \infty$, B', C' are the poles of BC, AC, AB in ω. Hence from Lemma 13 we see that $BC_1 \cap CB_1$, $h \cap \infty$, A are collinear, therefore $BC_1 \cap CB_1 \in h$. □

References

[1] N. I. Beluhov. On some pairs of perspective triangles. *Matematicheskoe Prosveschenie. Tret'ya Seriya.*, (14):256–269, 2010.

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ISOTOMIC SIMILARITY

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Abstract. Let A_1, B_1, C_1 be points chosen on the sidelines BC, CA, BA of a triangle ABC, respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2 respectively. We prove that triangle $A_2B_2C_2$ is similar to triangle $A_3B_3C_3$, where A_3, B_3, C_3 are symmetric to A_1, B_1,C_1 with respect to the midpoints of the sides BC, CA, BA respectively.

Theorem 1. *Let A_1, B_1, C_1 be points chosen on the sidelines BC, CA, BA of a triangle ABC, respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2 respectively. Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, BA respectively. Then the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.*

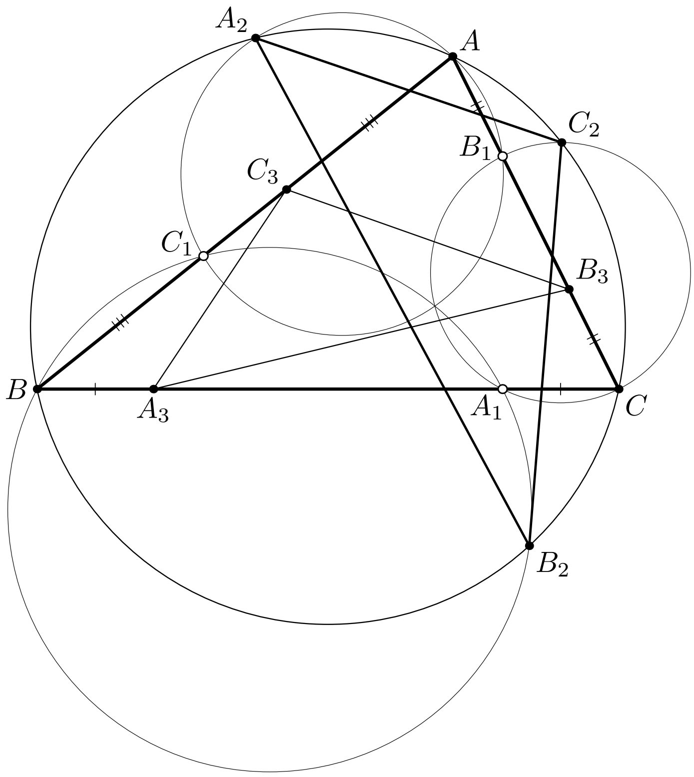

Fig. 1.

Preliminary. Let us introduce some notions and formulate known lemmas that we use in the proof.

We will work with oriented angles between lines. For two straight lines ℓ, m in the plane, $\angle(\ell, m)$ denotes the angle of counterclockwise rotation which transform line ℓ into a line parallel to m (the choice of the rotation centre is irrelevant). This is signed quantity; values differing by a multiple of π are identified, so that

$$\angle(\ell, m) = -\angle(m, \ell), \qquad \angle(\ell, m) + \angle(m, n) = \angle(\ell, n).$$

If ℓ is the line through the points K, L and m is the line the M, N, one writes $\angle(KL, MN)$ for $\angle(\ell, m)$; the characters K, L are freely interchangeable; and so are M, N. The counterpart of the classical theorem about cyclic quadrilaterals is the following:

Lemma 1. *Four non-collinear points K, L, M, N are concyclic if and only if*

$$\angle(KM, LM) = \angle(KN, LN). \tag{1}$$

Further we use (1) without explicit reference.

Lemma 2. *Suppose that A_1, B_1, C_1 are points on the sidelines BC, CA, BA of a triangle ABC, respectively; then the three circles (AB_1C_1), (BC_1A_1), (CA_1B_1) have a common point.*

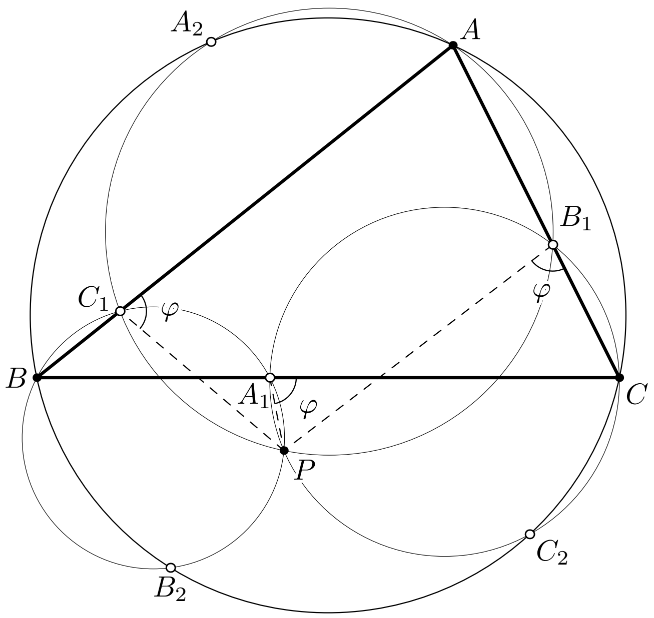

Fig. 2.

Proof. Let (AB_1C_1) and (BC_1A_1) intersect at C_1 and P. Then

$$\angle(PA_1, CA_1) = \angle(PA_1, BA_1) = \angle(PC_1, BC_1) =$$
$$= \angle(PC_1, AC_1) = \angle(PB_1, AB_1) = \angle(PB_1, CB_1).$$

The equality between the outer terms shows that the points A_1, B_1, P, C are concyclic. Thus P is the common point of the three mentioned circles. $\square$

Lemma 3. *Let A_1, B_1, C_1 be points on the sidelines BC, CA, BA of a triangle ABC, respectively; and the circles (AB_1C_1), (BC_1A_1), (CA_1B_1) meet at P. Suppose that the lines AP, PB, CP meet the circle (ABC) again at A', B', C', respectively; then triangles $A_1B_1C_1$ and $A'B'C'$ are similar. (In particular, the pedal triangle of P is similar to $A'B'C'$.)*

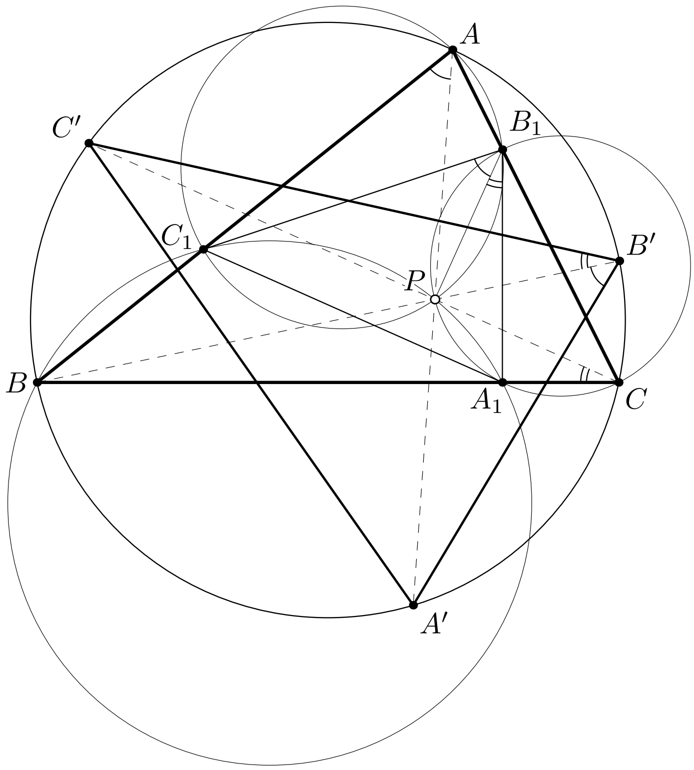

Fig. 3.

Proof. We have

(2)
$$\begin{aligned}\angle(A_1B_1, C_1B_1) &= \angle(A_1B_1, PB_1) + \angle(PB_1, C_1B_1) = \\ &= \angle(A_1C, PC) + \angle(PA, C_1A).\end{aligned}$$

On the other hand,

(3)
$$\begin{aligned}\angle(A'B', C'B') &= \angle(A'B', BB') + \angle(BB', C'B') = \\ &= \angle(AA', BA) + \angle(BC, C'C).\end{aligned}$$

But the lines $A'A$, BA, BC, $C'C$ coincide respectively with PA, C_1A, A_1C, PC. So the sums on the right-hand of (2) and (3) are equal, that leads to $\angle(A_1B_1, C_1B_1) = \angle(A'B', C'B')$. Hence (by cyclic shift, once more) also

$$\angle(B_1C_1, A_1C_1) = \angle(B'C', A'C') \text{ and } \angle(C_1A_1, B_1A_1) = \angle(C'A', B'A').$$

This means that triangles $A_1B_1C_1$ and $A'B'C'$ are similar. □

Proof of the Theorem. Let the circles (AB_1C_1), (BC_1A_1), (CA_1B_1) meet at P (see Lemma 2), and let

(4)
$$\varphi = \angle(PA_1, BC) = \angle(PB_1, CA) = \angle(PC_1, AB).$$

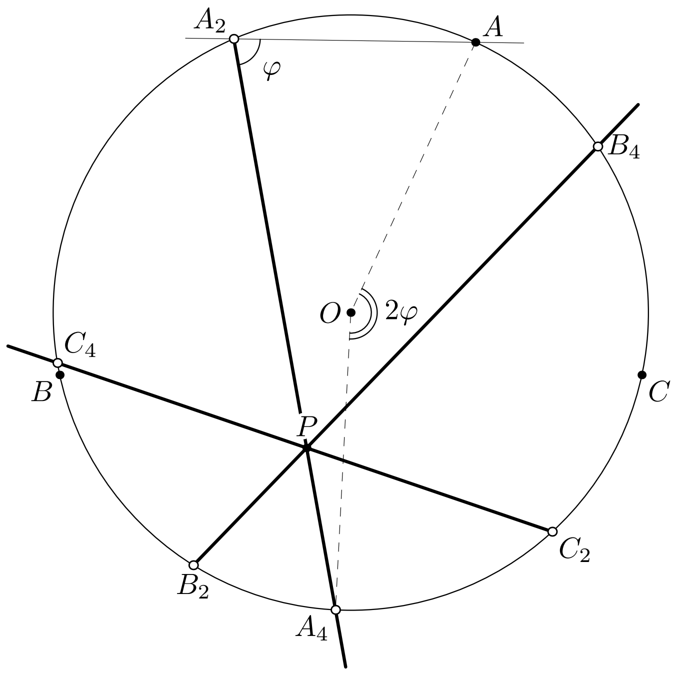

Fig. 4.

Let lines A_2P, B_2P, C_2P meet the circle (ABC) again at A_4, B_4, C_4, respectively. Since

$$\angle(A_4A_2, AA_2) = \angle(PA_2, AA_2) = \angle(PC_1, AC_1) = \angle(PC_1, AB) = \varphi,$$

we have $\angle(OA_4, OA) = 2\varphi$ (here O is the center of (ABC)). Hence A is the image of A_4 under rotation by 2φ about O. The same rotation takes B_4 to B, and C_4 to C. Thus triangle ABC is the image of $A_4B_4C_4$ under this rotation, therefore

$$\angle(A_4B_4, AB) = \angle(B_4C_4, BC) = \angle(C_4A_4, CA) = 2\varphi. \tag{5}$$

Further, we have $\angle(AB_4, AB) = \angle(B_2B_4, B_2B) = \varphi$. Hence by (4)

$$\angle(AB_4, PC_1) = \angle(AB_4, AB) + \angle(AB, PC_1) = \varphi + (-\varphi) = 0,$$

which means that $AB_4 \parallel PC_1$.

Let C_5 be the intersection of lines PC_1 and A_4B_4; define A_5, B_5 analogously. So $AB_4 \parallel C_1C_5$ and, by (5) and (4),

$$\angle(A_4B_4, PC_1) = \angle(A_4B_4, AB) + \angle(AB, PC_1) = 2\varphi + (-\varphi) = \varphi; \tag{6}$$

i.e., $\angle(B_4C_5, C_5C_1) = \varphi$. This combined with $\angle(C_5C_1, C_1A) = \angle(PC_1, AB) = \varphi$ (see (4)) proves that the quadrilateral $AB_4C_5C_1$ is an isosceles trapezoid with $AC_1 = B_4C_5$.

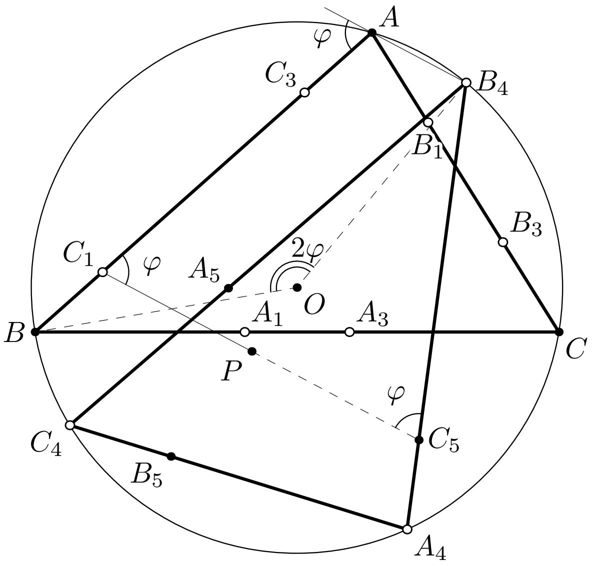

Fig. 5.

Suppose $\overrightarrow{AC_3} = \lambda\overrightarrow{AB}$; then $\overrightarrow{BC_1} = \lambda\overrightarrow{BA}$, and $\overrightarrow{A_4C_5} = \lambda\overrightarrow{A_4B_4}$. In other words, the rotation which maps triangle $A_4B_4C_4$ onto ABC carries C_5 onto C_3. Likewise, it takes A_5 to A_3, and B_5 to B_3. So the triangles $A_3B_3C_3$ and $A_5B_5C_5$ are congruent.

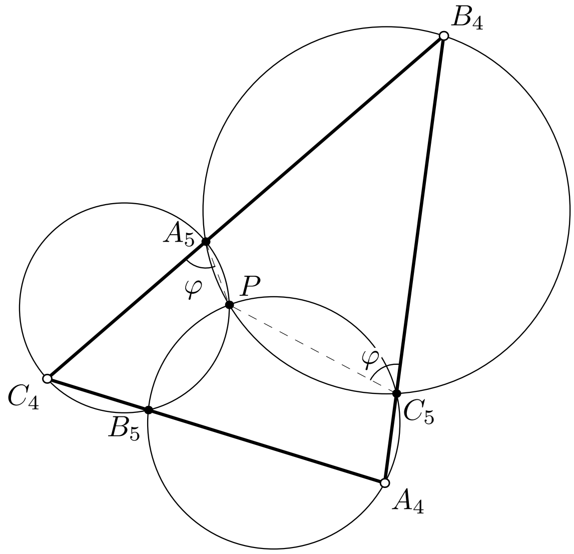

Fig. 6.

Lines B_4C_5 and PC_5 coincide respectively with A_4B_4 and PC_1. Thus by (6)

$$\angle(B_4C_5, PC_5) = \varphi.$$

Analogously (by cyclic shift) $\varphi = \angle(C_4A_5, PA_5)$, which rewrites as

$$\varphi = \angle(B_4A_5, PA_5).$$

These relations imply that the points P, B_4, C_5, A_5 are concyclic. Analogously P, C_4, A_5, B_5 and P, A_4, B_5, C_5 are concyclic quadruples.

Now it is sufficient to apply Lemma 3 for triangle $A_4B_4C_4$ and points A_5, B_5, C_5. It provides similarity of triangles $A_2B_2C_2$ and $A_5B_5C_5$. This ends the proof of Theorem. □

References

[1] A. V. Akopyan and A. A. Zaslavsky. *Geometry of conics*, volume 26 of *Mathematical World*. American Mathematical Society, Providence, RI, 2007.

[2] H. S. M. Coxeter and S. L. Greitzer. *Geometry revisited*, volume 19. The Mathematical Association of America, 1967.

[3] V. V. Prasolov. *Problems in Plane Geometry*. M.:MCCME, 2006.

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CONJUGATION OF LINES WITH RESPECT TO A TRIANGLE

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ABSTRACT. Isotomic and isogonal conjugate with respect to a triangle is a well-known and well studied map frequently used in classical geometry. In this article we show that there is a reason to study conjugation of lines. This conjugation has many interesting properties and relations to other objects of a triangle.

1. Introduction

Isotomic conjugation with respect to a triangle ABC is a map which maps any point P with barycentric coordinates (x, y, z) to the point P^* with coordinates $(1/x, 1/y, 1/z)$. *Isogonal conjugation* is defined similarly, but instead barycentric coordinates the trilinear coordinates are used. The following property of isogonal conjugation is important: If P and P^* are isogonal conjugates, then

$$\angle PAB = \angle CAP^*, \quad \angle PBC = \angle ABP^*, \quad \angle PCA = \angle BCP^*.$$

This property is often used as a definition of isogonal conjugation.

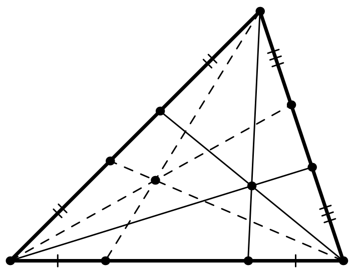

Fig. 1. Isotomic conjugation

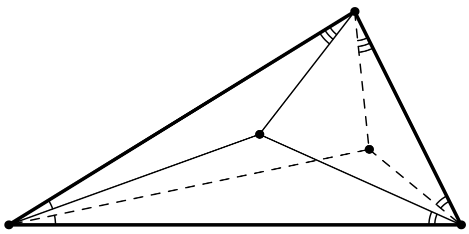

Fig. 2. Isogonal conjugation

Other properties of isogonal and isotomic conjugation and its applications to triangle geometry can be found in [2].

Isotomic and isogonal conjugations are projectively equivalent. It means that there is a projective transformation of the plane which preserves vertices of a triangle and any pair of isotomically conjugate points is mapped to a pair of isogonally conjugate points.

In the general case, define conjugation with respect to a triangle ABC and a point S as follows.

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Definition 1. Let $A'B'C'$ is a cevian triangle of a point S with respect to a triangle ABC. Let P be any point and $A_1B_1C_1$ be its cevian triangle. Choose on the sides of the triangle points A_2, B_2, and C_2 such that

$$\begin{aligned} &[A, C_1; B, C'] = [B, C_2; A, C'],\\ &[B, A_1; C, B'] = [C, A_2; B, A'],\\ &[C, B_1; A, C'] = [A, B_2; C, B'], \end{aligned} \tag{1}$$

where $[X, Y; Z, T]$ means a cross-ratio of four collinear points X, Y, Z, and T. It is easy to see that the cevians AA_2, BB_2, and CC_2 are concurrent. The point of intersection of these three lines called a *conjugate point of the point P with respect to the triangle ABC and the point S.*

Let s be the trilinear polar of the point S with respect to the triangle ABC. Then we can replace the points A', B', and C' in equation (1) by points of intersection of the sides of the triangle and s. In this case we obtain the same map. Call this map by a *conjugation of point this respect to a triangle and a line.*

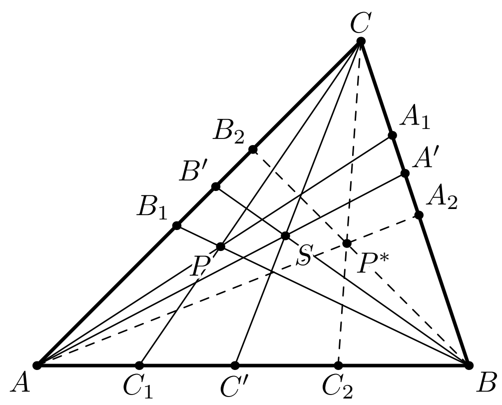

Fig. 3. Conjugation

If point S is the centroid of the triangle then this map is an isotomic conjugation. If S coincides with the incenter of the triangle then this map is an isogonal conjugation.

Correctness of the definition 1 could be easily proven by the Ceva's theorem or by means of projective transformation and correctness of the definition of isotomic or isogonal conjugation. We see, that a conjugation with respect to a triangle an a point (or line) is projectively equivalent to isotomic or isogonal conjugation.

There is a well known theorem about isogonal and isotomic conjugation.

Theorem 1. *Under any conjugation with respect to a triangle ABC an image of any line is a conic passing through the vertices of this triangle.*

For example, image of a line at infinity under isogonal conjugation is a circumcircle of a triangle and under isotomic conjugation is a circumscribed Steiner ellipse of the triangle.

2. Conjugation of lines

Let's apply a dual transformation to the construction in the definition 1. In this case we obtain another map that still makes sense. Let's call this map by *a conjugation of lines with respect to a triangle and a line.*

Let us give a strict definition.

Definition 2. Let a line s intersects the sides of a triangle ABC at points A', B' and C'. Suppose a line ℓ intersects the sides of the triangle at points A_1, B_1, and C_1. Choose on the sides of the triangle points A_2, B_2, and C_2 such that

$$
\begin{aligned}
&[A, C_1; B, C'] = [B, C_2; A, C'],\\
&[B, A_1; C, B'] = [C, A_2; B, A'],\\
&[C, B_1; A, C'] = [A, B_2; C, B'].
\end{aligned}
\tag{2}
$$

Then the points A_2, B_2, and C_2 lie of the same line. Call this line a conjugate line of the line ℓ with respect to the triangle ABC and the line s.

Let S be a trilinear pole of the line s with respect to the triangle ABC. We can replace the points A', B', and C' in equation (2) by the vertices of cevian triangle of the point S. In this case we obtain the same map. Call it by a*conjugation of lines with respect to a triangle and a point.*

Correctness of this definition follows from correctness of the definition 1. Also it could be independently proven by the Menelaus theorem.

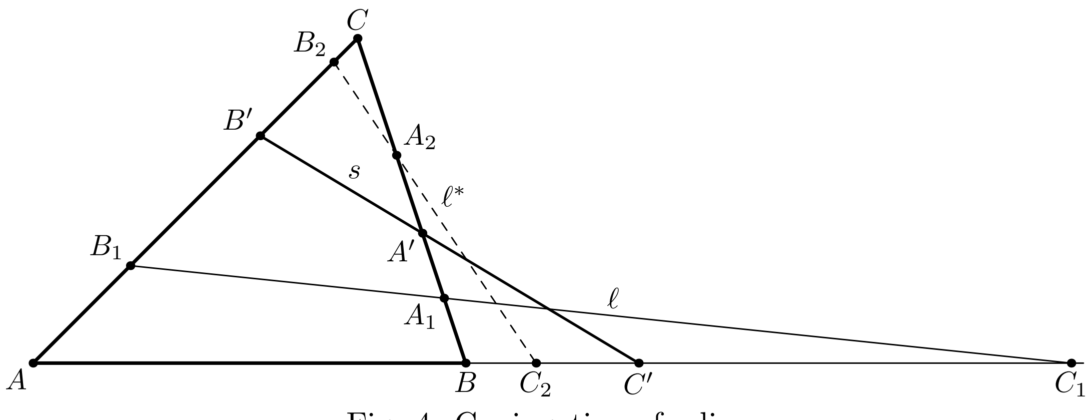

Fig. 4. Conjugation of a line

If a line s is the line at infinity then it is reasonable to call this conjugation *isotomic conjugation of lines.* Also call this map *isogonal conjugation* if s is the line passing through the feet of the three external angle bisectors of the triangle. In these cases this map has a simpler definition (Fig. 5 and 6).

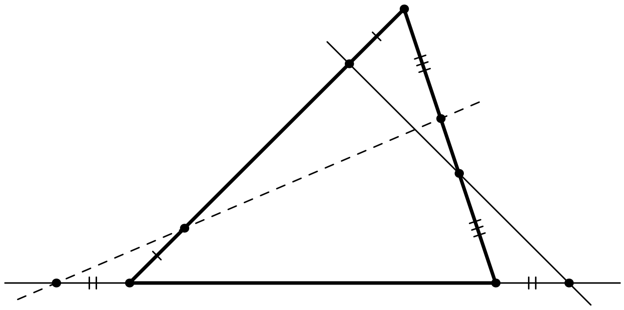

Fig. 5. Isotomic conjugation of a line

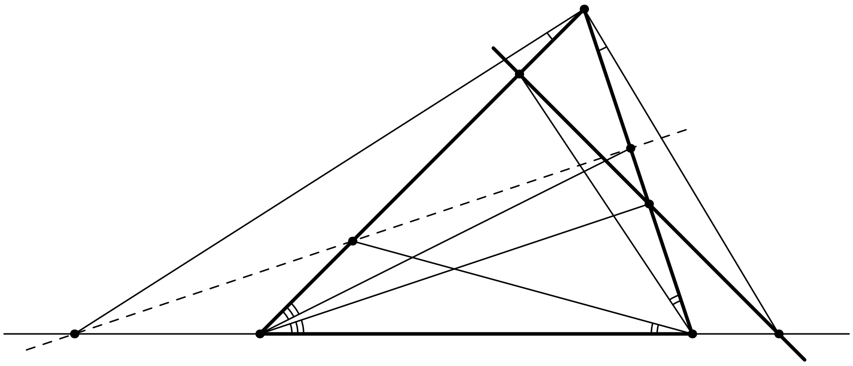

Fig. 6. Isogonal conjugation of a line

The dual theorem to Theorem 1 is the following.

Theorem 2. *The set of lines passing through a fixed point is conjugate with respect to $\triangle ABC$ to the set of tangent lines to some inscribed conic.*

Considering "critical lines" of the conjugation we can see that perspector of the conic which correspond to the point P is the conjugate point P^*.

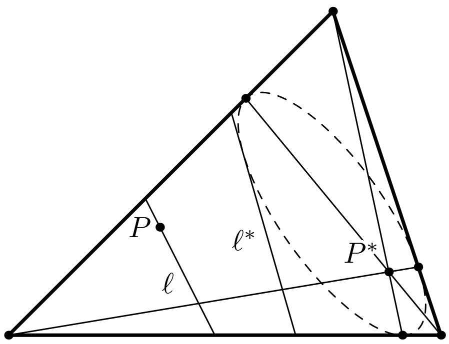

Fig. 7. Conjugation of a pencil of lines

Theorem 2 has the following corollaries.

Corollary 3. *Isotomically (or isogonally) conjugate line to any line passing through the center of an equilateral triangle is a tangent line to the inscribed circle of the triangle.*

Corollary 4. *Isotomically conjugate line to any line passing through the Nagel point of a triangle is a tangent line to the inscribed circle of the triangle.*

The proofs follow from the fact that Gergonne point and Nagel point are isotomically conjugate.

Corollary 5. *Isotomically conjugate line to any line passing through the centroid of a triangle is a tangent line to the inscribed Steiner ellipse of the triangle.*

Corollary 6. *Isogonally conjugate line to any line passing through the internal similitude center of the incircle and the circumcircle of a triangle is a tangent line to the inscribed circle of the triangle.*

The proof follows from the fact that the Gergonne point and the internal similitude center are isogonally conjugate.

Corollary 7. *Isogonally conjugate line to any line passing through the Lemoine point of a triangle is a tangent line to the inscribed Steiner ellipse of the triangle.*

The proof follows from the fact that the Lemoine point and the centroid of the triangle are isogonally conjugate. Also, since the Brocard ellipse is tangent to the sides of a triangle in the feet of its semidians we obtain the following.

Corollary 8. *Isogonally conjugate line to any line passing through the centroid of a triangle is a tangent line to the Brocard ellipse.*

3. Concrete lines

From corollaries of the previous section we can formulate some statements about conjugate lines of well-known lines of a triangle. Let us recall that Nagel point, centroid and incenter of a triangle lie on the same line and this line is called a *Nagel line* of a triangle.

Theorem 9.
(1) *Isotomically conjugate line of the Euler line is tangent to the inscribed Steiner ellipse.*
(2) *Isogonally conjugate line of the Euler line is tangent to the Brocard ellipse.*
(3) *Isogonally conjugate line of the line IO (line through the incenter and the circumcenter of a triangle) is tangent to the incribed circle of a triangle.*
(4) *Isotomically conjugate line of the Nagel line is tangent to the inscribed Steiner ellipse and the incircle of a triangle.*
(5) *Isogonally conjugate line of the Nagel line is tangent to the Brocard ellipse.*

It is possible to continue this list for other lines and centers of a triangle.

Let us prove, that the isotomically conjugate line of the Nagel line possesses another interesting property.

Theorem 10. *Isotomically conjugate line to the Nagel line of a triangle is tangent to the inscribed circle at the Feuerbach point.*

Because of Theorem 9 it is sufficient to prove the following lemma.

Lemma 11. *Common tangent line to the incircle and the inscribed Steiner ellipse of a triangle (which doesn't coincide with any side of this triangle) is tangent to the inscribed circle at the Feuerbach point.*

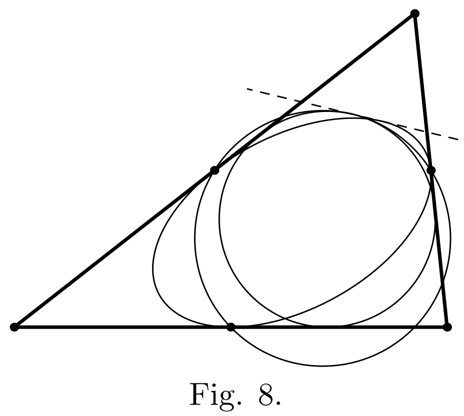

Fig. 8.

Proof. Denote vertices of a triangle by A, B, C and this common tangent line by ℓ. Let A_1, B_1, and C_1 be points of intersection of ℓ and the sides of $\triangle ABC$. For the proof of the lemma it is sufficient to prove that the points A_1, B_1, and C_1 are on the radical axis of the inscribed circle and the nine-point circle of the triangle ABC. We calculate powers of these points with respect to both circles.

Let the incircle be tangent to the sides of the triangle at points A_2, B_2, and C_2. Denote midpoints of the triangle by A_3, B_3, and C_3 and feet of altitudes by A_4, B_4, and C_4. Lengths of the sides are denoted by a, b, and c.

Suppose P is the point of intersection of lines AA_1 and BB_1. From the Brianchon theorem it follows that the point P lies on the lines A_2B_2 and A_3B_3, therefore it is a point of intersection of these lines.

It is known that the second point of intersection of the line C_2P and the incircle is the Feuerbach point (See [3], Problem 22). From the Brianchon theorem it follows that this point is the point of tangent line A_1B_1 and incircle. So, this line is tangent to the incircle in the Feuerbach point.

Since the proof of the mentioned property of the Feuerbach point is little bit complicated and based on other properties of Feuerbach point, we will give another independent proof.

Let us show that the points A_1, B_1, and C_1 lie on the radical axis of incircle and nine-point circle of the triangle.

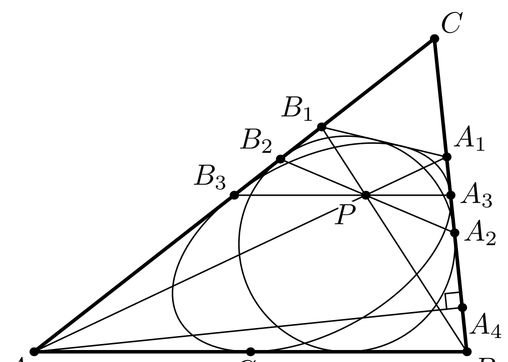

Fig. 9.

Let's apply the Menelaus theorem for the line A_2B_2 and the triangle CA_3B_3.

$$\text{(3)} \qquad \frac{A_3P}{B_3P} = \frac{A_3A_2}{A_2C} \cdot \frac{CB_2}{B_2B_3} = \frac{c-b}{a+b-c} \cdot \frac{a+b-c}{c-a} = \frac{c-b}{a-c}.$$

Now apply the Menelaus theorem for the line AA_1 and the triangle CA_3B_3.

$$\text{(4)} \qquad \frac{A_1A_3}{A_1C} = \frac{A_3P}{B_3P} \cdot \frac{B_3A}{AC} = \frac{c-b}{2(a-c)}.$$

Since $CA_3 = a/2$ we have

$$\text{(5)} \qquad A_1C = \frac{2(a-c)}{c-b+2a-2c} \cdot \frac{a}{2} = \frac{a(a-c)}{2a-b-c}.$$

Now it is easy to find lengths of the segments A_1A_2, A_1A_3, and A_1A_4.

$$\text{(6)} \qquad A_1A_2 = \frac{a+b-c}{2} - \frac{a(a-c)}{2a-b-c},$$

$$\text{(7)} \qquad A_1A_3 = \frac{a}{2} - \frac{a(a-c)}{2a-b-c},$$

$$\text{(8)} \qquad A_1A_4 = \frac{a^2+b^2-c^2}{2b} - \frac{a(a-c)}{2a-b-c}.$$

We check the equation $A_1A_2^2 = A_1A_3 \cdot A_1A_4$.

$$\text{(9)} \quad A_1A_2^2 - A_1A_3 \cdot A_1A_4 = \left(\frac{a+b-c}{2}\right)^2 - 2\frac{a+b-c}{2} \cdot \frac{a(a-c)}{2a-b-c} - $$
$$- \frac{a}{2} \cdot \frac{a^2+b^2-c^2}{2b} + \frac{a^2+b^2-c^2+ab}{2b} \cdot \frac{a(a-c)}{2a-b-c} =$$
$$= \frac{c-a-b}{4} \cdot \frac{2a^2-ab-ac+b^2-c^2}{2a-b-c} + \frac{2a^3+a^2b-3a^2c+b^3-b^2c-c^2b+c^3}{4(2a-b-c)} = 0.$$

We obtain that the point A_1 lies on the radical axis of the inscribed circle and the nine-point circle of the triangle. The same argument works for the points B_1 and C_1.

Note that we also proved the Feuerbach theorem, because we showed that the radical axis of the incirbed circle and the nine-point circle is tangent to the first one. □

The following theorem follows from Lemma 11.

Theorem 12. *Isogonally conjugate line to the line passing through the Lemoine point and the internal similitude center of the incircle and the circumcircle of a triangle is tangent to the inscribed circle at the Feuerbach point.*

4. Related results

We mention another result related to construction on the Fig. 9.

Theorem 13. *Suppose triangles* $A_1B_1C_1$ *and* $A_2B_2C_2$ *are cevian triangles in a triangle* ABC*. Correspondent sides of these triangles intersect each other at points* A_3*,* B_3*, and* C_3*. Then*

1) The lines A_1A_3*,* B_1B_3*, and* C_1C_3 *are concurrent. Denote the point of intersection of these lines by* P*.*

The lines A_2A_3*,* B_2B_3*, and* C_2C_3 *are concurrent. Denote the point of intersection of these lines by* Q*.*

2) The following triples of points are collinear: (A*,* B_3*,* C_3*), (*B*,* A_3*,* C_3*) and (*C*,* A_3*,* B_3*). Denote intersection of this lines and the correspondent sides of triangle by* A_4*,* B_4*, and* C_4*.*

3) The points A_4*,* B_4 *and* C_4 *lie on the line* PQ*.*

Proof. Inscribe into the triangle ABC two conics α_1 and α_2, which touch the sides of ABC in the vertices of the triangles $A_1B_1C_1$ and $A_2B_2C_2$. Then fourth common tangent of α_1 and α_2 will be line from Item 3 of the Theorem. The points of tangent of these conics and the line are the points P and Q. Items 1 and 2 follows from Brianchon theorem. □

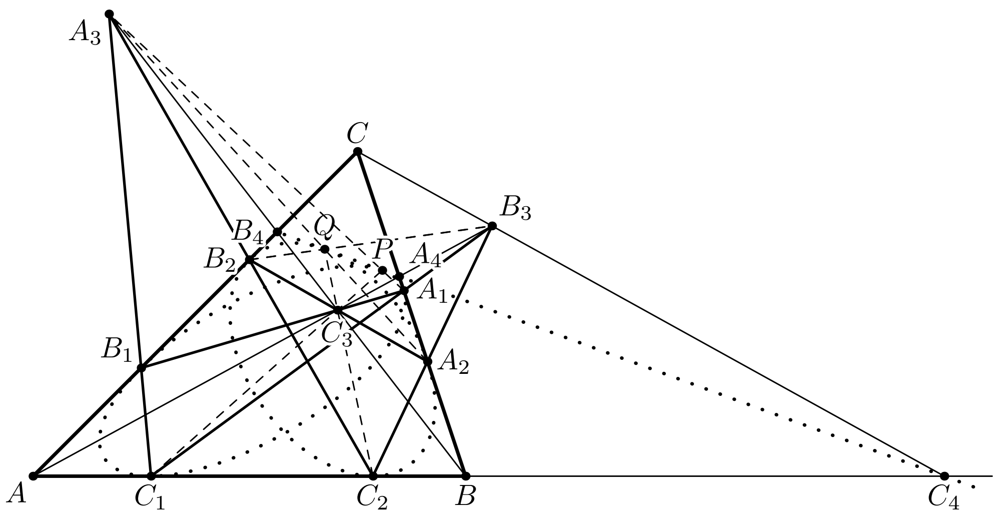

Fig. 10.

The Nagel triangle is a triangle with vertices in points of tangency of excircles and correspondent sides of a triangle ABC. The ellipse tangent to sides of ABC at these points is called *the Nagel ellipse.*

The recent result F. Ivlev [4] states that if $A_1B_1C_1$ (in terms of Theorem 13) is a medial triangle and $A_2B_2C_2$ is Nagel triangle then Q is the Feuerbach point of the triangle ABC. Reformulating Ivlev's theorem in terms of this article we obtain the following.

Theorem 14. *The Nagel ellipse of a triangle passes through the Feuerbach point. The tangent line to the Nagel ellipse at this point is also tangent to the inscribed Steiner ellipse.*

Corollary 15. *Isotomically conjugate line to the line passing through centroid of a triangle and the Gergonne point passes through the Feuerbach point.*

Corollary 16. *Isogonally conjugate line to the line passing through the Lemoine point and the external similitude center of the incircle and the circumcircle of a triangle passes through the Feuerbach point.*

Ivlev's proof is based only on the fact that the Nagel point and the Gergonne point are isotomically conjugate (actually his theorem is more general). So, it is possible to generalize Theorem 14.

Theorem 17. *Points P and Q are conjugate with respect to the point T in the triangle ABC. Conjugation of the lines is given. Let α_1, α_2, and α_0 three inscribed in the triangle ABC conics which correspond to the points P, Q, and T (see Theorem 2). Then fourth common tangents to the pairs of conics α_0, α_1 and α_0, α_2 intersect at one of the points of intersection α_1 and α_2.*

Note that for any conjugation there are four points such that this conjugation can be defined with respect to any of these four points (for example isogonal conjugation is also a conjugation with respect to any excenter of a triangle). For each of these points there is a different point of intersection of α_1 and α_2 in Theorem 17.

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References

[1] A. V. Akopyan and A. A. Zaslavsky. Different views on the isogonal conjugation. *Matematicheskoe Prosveschenie. Tret'ya Seriya.*, (11):61–78, 2007.

[2] A. V. Akopyan and A. A. Zaslavsky. *Geometry of conics*, volume 26 of *Mathematical World*. American Mathematical Society, Providence, RI, 2007.

[3] L. A. Emelyanov and T. L. Emelyanova. Feuerbach family. In *The Tournament of Towns Summer Conference*. 2001.

[4] F. Ivlev. Some straight lines passing through the feuerbach point. *Matematicheskoe Prosveschenie. Tret'ya Seriya.*, (15):219–228, 2011.

[5] C. Kimberling. Encyclopedia of triangle centers, 2000.

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A PROOF OF VITTAS' THEOREM AND ITS CONVERSE

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Abstract. We discuss Vittas's theorem, which states that the Euler lines of non-equilateral triangles ABP, BCP, CDP and DAP in a cyclic quadrilateral $ABCD$, whose diagonals AC and BD intersect at P, are concurrent or are pairwise parallel or coincident. We also introduce and prove the converse of these theorems.

1. Introduction

In 2006, *Kostas Vittas* stated without proof an interesting theorem about the concurrence of the Euler lines of triangles PAB, PBC, PCD and PDA in a cyclic quadrilateral $ABCD$ whose two diagonals AC and BD intersect at P. Later, Vladimir Yetti, a Czech physicist, used the properties of conics to prove this theorem for the first time [1]. Then, it is Vittas who gave a synthetic proof of his own theorem [2].

In this article, by Theorems 1 and 2, we will discuss in detail Vittas's theorem. Then, by Theorems 3 and 4, we will introduce the converse of Theorems 1 and 2. It is worth noticing that the proposed approach to the proof of these four theorems is completely original.

The following notations will be used:

The reflection with respect to axis ℓ is denoted by R_ℓ;

The symmetry with respect to point I is denoted by S_I;

The homothety with center P and ratio k is denoted by H_P^k.

For simplification:

If the lines XY and ZT are parallel, it is written $XY \parallel ZT$;

If the lines XY and ZT intersect, it is written $XY \times ZT$;

If the lines XY and ZT are not coincident, it is written $XY \not\equiv ZT$;

If the lines XY, ZT and UV are pairwise parallel, it is written $XY \parallel ZT \parallel UV$;

If the lines XY, ZT and UV are concurrent and pairwise not coincident, it is written $XY \times ZT \times UV$.

For consistency, I denote by P the intersection of the diagonals AC and BD of quadrilateral $ABCD$ in all theorems and their proofs. Accordingly, let O_1, O_2, O_3 and O_4 be the circumcenters of triangles PAB, PBC, PCD and PDA respectively and let H_1, H_2, H_3 and H_4 be the orthocenters of triangles PAB, PBC, PCD and PDA respectively. Obviously, O_1H_1, O_2H_2, O_3H_3 and O_4H_4 are the Euler lines of triangles PAB, PBC, PCD and PDA respectively.

We will use next well-known facts of triangle geometry and transformations theory.

Lemma 1. *In triangle ABC, let $\angle BAC = \alpha < 90°$. A line ℓ contains the internal bisector of $\angle BAC$. Let O, H be the circumcenter and orthocenter of the triangle respectively.*

1. If $\alpha \neq 60°$, then H is the image of O by the opposite similarity $H_A^{2\cos\alpha} R_\ell$.

2. If $\alpha = 60°$, then H is the image of O by the reflection R_ℓ.

Lemma 2. *In triangle ABC, let $\angle BAC = \alpha > 90°$. A line ℓ contains the external bisectors of $\angle BAC$. Let O and H be the circumcenter and orthocenter of the triangle respectively.*

1. If $\alpha \neq 120°$ then H is the image of O by the opposite similarity $H_A^{-2\cos\alpha} R_\ell$.

2. If $\alpha = 120°$ then H is the image of O by the reflection R_ℓ.

Lemma 3. *Every opposite similarity that has similarity ratio other than* 1 *can be uniquely presented as a dilative reflection i.e. a product of a reflection and a homothety with has a positive ratio and the center lying on the axis of the reflection.*

Lemma 4. *Two pairs of distinct points A, B and A', B' are given in the plane such that the length of segment AB is distinct from the length of segment $A'B'$. There exists a unique dilative reflection transforming A, B into A', B' respectively.*

2. Main Results

Theorem 1. *Let $ABCD$ be a quadrilateral whose diagonals AC and BD intersect at P and form an angle of* 60°. *If the triangles PAB, PBC, PCD, PDA are all not equilateral, then their Euler lines are pairwise parallel or coincident.*

Proof. Without loss of generality, suppose that $\angle APB = \angle CPD = 120°$; $\angle APD = \angle BPC = 60°$. Let ℓ be the line that contains the internal angle bisectors of $\angle APD$ and $\angle BPC$. According to parts 2 of Lemmas 1 and 2, it can be deduced that H_1, H_2, H_3 and H_4 are the images of O_1, O_2, O_3 and O_4 respectively, by R_ℓ.

Therefore O_1H_1, O_2H_2, O_3H_3 and O_4H_4 are all perpendicular to ℓ. Thus, O_1H_1, O_2H_2, O_3H_3 and O_4H_4 are pairwise parallel or coincident. □

Theorem 2 (Vittas's theorem)**.** *Let $ABCD$ be a quadrilateral with diagonals intersecting at P and forming an angle different from* 60°. *If $ABCD$ is cyclic then the Euler lines of triangles PAB, PBC, PCD and PDA are concurrent.*

Proof. There are two cases:

The case of perpendicular lines AC and BD is evident. Thus we can suppose that $\angle APD = \alpha < 90°$.

Let ℓ be the line containing the internal bisectors of the angles $\angle APD$ and $\angle BPC$. From parts 1 of Lemmas 1 and 2: H_1, H_2, H_3 and H_4 are the images of O_1, O_2, O_3 and O_4 respectively, by the dilative reflection $H_P^{2\cos\alpha} R_\ell$ (note that $\cos\alpha \neq \frac{1}{2}$).

Let I be the intersection of H_1H_3 and H_2H_4. Note that $H_1H_2H_3H_4$ is a parallelogram, it is deduced that H_3, H_4, H_1 and H_2 are the images of H_1, H_2, H_3 and H_4 respectively, by the symmetry S_I.

Thus, H_3, H_4, H_1 and H_2 are the images of O_1, O_2, O_3 and O_4 respectively, by the dilative reflection $S_I H_P^{2\cos\alpha} R_\ell$.

Let Q be the center of the dilative reflection $S_I H_P^{2\cos\alpha} R_\ell$. I am going to prove that O_1H_1, O_2H_2, O_3H_3 and O_4H_4 are concurrent by proving that O_1H_1, O_2H_2, O_3H_3 and O_4H_4 all contain Q.

Now I will prove that both of the lines O_1H_1 and O_3H_3 contain Q.

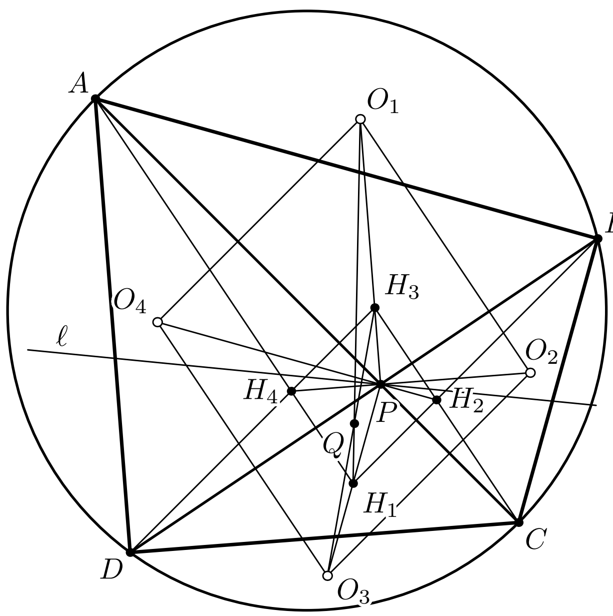

Fig. 1.

There are four possible cases. However, for the sake of simplicity, we only need to consider the main case: $O_1H_1 \not\equiv O_3H_3$; $O_1 \neq H_3$; $O_3 \neq H_1$ (Fig. 1).

Since the dilative reflection $H_P^{2\cos\alpha} R_\ell$ transform H_3 into O_3, the reflection R_ℓ transform the line PH_3 into the line PO_3.

Since quadrilateral $ABCD$ is cyclic, the triangles PAB and PDC are oppositely similar. Hence the dilative reflection $H_P^{\frac{PD}{PA}} R_\ell$ transforms O_1 into O_3. Consequently, the reflection R_ℓ transforms the line PO_1 into the line PO_3.

Thus, the lines PH_3 and PO_1 are coincident. Therefore, P, O_1 and H_3 are collinear. Similarly, P, O_3 and H_1 are collinear.

From this and the fact that triangles PO_1H_1 and PO_3H_3 are oppositely similar, it can be deduced that

$$(H_3O_1,\, H_3O_3) \equiv (H_3P,\, H_3O_3) \equiv (H_1O_1,\, H_1P) \equiv (H_1O_1,\, H_1O_3) \pmod{\pi}.$$

Hence, the four points O_1, H_1, O_3 and H_3 are concyclic.

From this, together with the fact that $O_1O_3 \neq O_1O_3 \cdot 2\cos\alpha = H_1H_3$, the lines O_1H_1 and O_3H_3 intersect.

Let Q' be the intersection of O_1H_1 and O_3H_3.

Note that O_1, H_1, O_3 and H_3 are concyclic, it could be seen that $Q'O_1O_3$ and $Q'H_3H_1$ are oppositely similar.

Therefore, Q' is the center of dilative reflection transforming O_1 and O_3 into H_3 and H_1respectively.

From this and Lemmas 3 and 4, it is seen that Q' coincides with Q.

Thus, both lines O_1H_1, O_3H_3 contain Q.

Similarly, both lines O_2H_2 and O_4H_4 contain Q.

In short, the lines O_1H_1, O_2H_2, O_3H_3 and O_4H_4 are concurrent (point Q). □

Theorem 3. *Let $ABCD$ be a quadrilateral with diagonals intersecting at P. If all four triangles PAB, PBC, PCD, PDA are not equilateral and the Euler lines of three out of these four triangles are pairwise parallel or coincident, then the angle formed by AC and BD is* $60°$.

To prove the above theorem, the following lemma is needed.

Lemma 5. *Let ABC and $A'B'C'$ be two oppositely similar triangle such that $AB \parallel A'C'$ and $AC \parallel A'B'$. If $AA' \parallel BB' \parallel CC'$ then the triangles ABC and $A'B'C'$ are oppositely congruent.*

Proof. Let E be the intersection of $A'C'$ and BB'; let F be the intersection of $A'B'$ and CC' (Fig. 2).

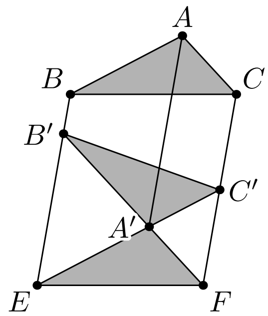

Fig. 2.

Obviously, $ABEA'$ and $ACFA'$ are parallelograms.

Therefore, the triangle $A'EF$ is the image of triangle ABC under the translation by the vector $\overrightarrow{AA'}$.

Hence, triangles ABC and $A'EF$ are directly congruent.

From this as well as the fact that triangles ABC and $A'B'C'$ are oppositely similar, we have

$$(B'C', B'F) \equiv (B'C', B'A') \equiv (BA, BC) \equiv (EA', EF) \equiv (EC', EF) \pmod{\pi}.$$

Therefore, the four points B', C', E and F belong to the same circle.

In addition, $EB' \parallel FC'$. As such, B', C', E, F are four vertices of an isosceles trapezoid with bases EB' and FC'.

So the triangles $A'EF$ and $A'B'C'$ are oppositely congruent.

In short, the triangles ABC and $A'B'C'$ are oppositely congruent. □

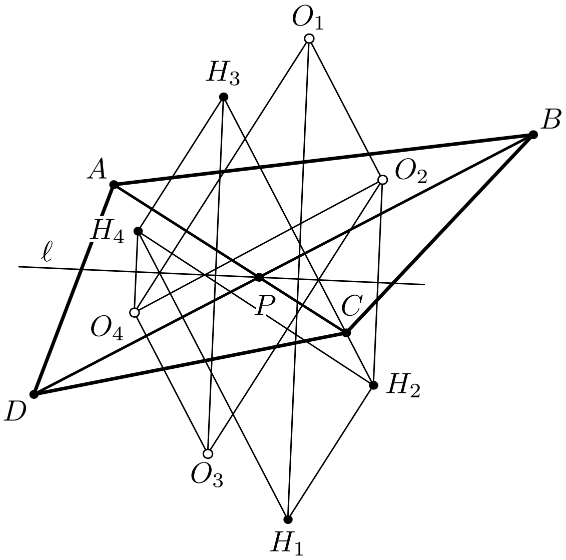

Fig. 3.

Now I am going to prove Theorem 3.

Proof of Theorem 3. Notice that the angle between AC and BD is not $90°$ (observe case 1 of the proof of Theorem 2).

Let $\angle APD = \angle BPC = \alpha$. Without loss of generality, suppose that $\alpha < 90°$.

There are six possible cases. However, for the sake of simplicity, we only need to consider the main case: $O_1H_1 \parallel O_2H_2 \parallel O_4H_4$ (Fig. 3).

Let ℓ be the line containing the bisectors of the angles $\angle APD$ and $\angle BPC$. From Lemmas 1 and 2, it can be learnt that H_1, H_2, H_3 and H_4 are the images of O_1, O_2, O_3 and O_4 respectively, by the opposite similarity $H_P^{2\cos\alpha}R_\ell$ (note that the value of $2\cos\alpha$ is not known yet).

Therefore, $H_1H_2H_4$ and $O_1O_2O_4$ are oppositely similar.

Moreover, H_1H_2 and H_1H_4 are parallel to O_1O_4 and O_1O_2 respectively (note that H_1H_2 and O_1O_4 are perpendicular to AC; H_1H_4 and O_1O_2 are perpendicular to BD).

Furthermore, by hypothesis, $O_1H_1 \parallel O_2H_2 \parallel O_4H_4$.

Thus, by Lemma 5, the triangles $H_1H_2H_4$ and $O_1O_2O_4$ are oppositely congruent. Hence, $O_2O_4 = H_2H_4$.

On the other hand, because H_2 and H_4 are the images of O_2 and O_4 respectively, by the opposite similarity $H_P^{2\cos\alpha}R_\ell$, $O_2O_4 = H_2H_4 \cdot 2\cos\alpha$.

Thus, $H_2H_4 = O_2O_4 = H_2H_4 \cdot 2\cos\alpha$.

So, $\cos\alpha = \frac{1}{2}$.

As such, $\alpha = 60°$.

In other words, the angle between AC and BD is $60°$. □

Theorem 4. *Let $ABCD$ be a quadrilateral with diagonals intersecting at P and forming an angle different from $90°$. If the triangles PAB, PBC, PCD and PDA are not equilateral and the Euler lines of three out of these four triangles are concurrent, then the quadrilateral $ABCD$ is cyclic.*

In order to prove Theorem 4, the following lemma is needed.

Lemma 6. *Let ABC and $A'B'C'$ be two oppositely similar triangle such that $AB \parallel A'C'$ and $AC \parallel A'B'$. If $AA' \times BB' \times CC'$ then B, C, B' and C' are on the same circle.*

Proof. Omit the simple case where $A' = BB' \cap CC'$.

Let Q be the point of concurrency of AA', BB' and CC'.

As $AB \parallel A'C'$ and $AC \parallel A'B'$,

$$\begin{cases} (B'C', AC) \equiv (B'C', A'B') \not\equiv 0 \pmod{\pi} \\ (B'C', AB) \equiv (B'C', A'C') \not\equiv 0 \pmod{\pi}. \end{cases}$$

Hence, $B'C' \times AC$ and $B'C' \times AB$.

Let $B'C'$ meet AC, AB at X, Y respectively.

As $A' \neq BB' \cap CC'$, $A' \in CC'$ and $A' \in BB'$.

Hence, $\begin{cases} (CC', A'C') \not\equiv 0 \pmod{\pi} \\ (BB', A'B') \not\equiv 0 \pmod{\pi}. \end{cases}$

Then, noting that $AB \parallel A'C'$ and $AC \parallel A'B'$, deduce that

$$\begin{cases} (CC', AB) \not\equiv 0 \pmod{\pi} \\ (BB', AC) \not\equiv 0 \pmod{\pi}. \end{cases}$$

This means that $CC' \times AB$ and $BB' \times AC$.

Let CC' meet AB at Z, and BB' meet AC at T (Fig. 4).

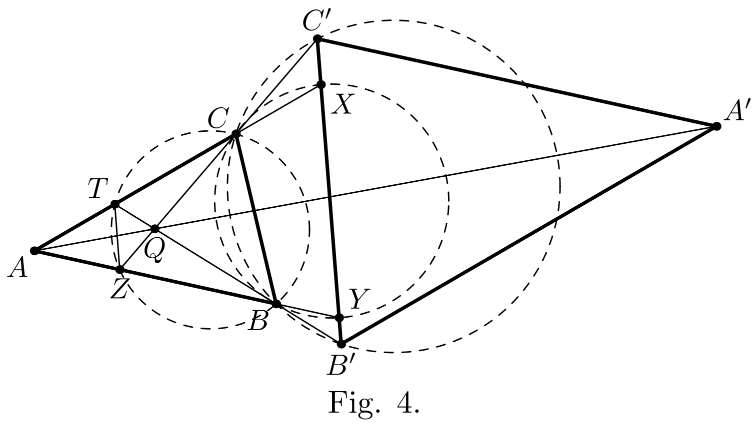

Fig. 4.

Since $AC \parallel A'B'$ and the triangles ABC and $A'B'C'$ are oppositely similar,

$$(XY, XC) \equiv (B'C', CA) \equiv (B'C', B'A') \equiv (BA, BC) \equiv (BY, BC) \pmod{\pi}.$$

As such, the points B, C, X and Y belong to the same circle.

On the other hand, by hypothesis $AB \parallel A'C'$ and $AC \parallel A'B'$, it can be deduced in respect to the Thales theorem that:

$$\frac{\overrightarrow{ZQ}}{\overrightarrow{C'Q}} = \frac{\overrightarrow{AQ}}{\overrightarrow{A'Q}} = \frac{\overrightarrow{TQ}}{\overrightarrow{B'Q}}.$$

So, ZT is parallel to $B'C'$. This means that ZT is parallel to XY.

By the concyclicity of B, C, X and Y, together with the fact that $ZT \parallel XY$, we learn that B, C, Z and T are concyclic.

From this and the fact that $ZT \parallel B'C'$, it can be seen that B, C, B' and C' are on the same circle. □

Now I am going to prove Theorem 4.

Proof of Theorem 4. Let $\angle APD = \angle BPC = \alpha$. Without loss of generality, suppose that $\alpha < 90°$.

From Theorem 1, $\alpha \neq 60°$.

Let ℓ be the line containing the bisectors of the angles $\angle APD$ and $\angle BPC$.

There are ten possible cases. However, for the sake of simplicity, we only need to consider the main case: $O_1H_1 \times O_2H_2 \times O_4H_4$ (Fig. 5).

As above, it is easy to see that the triangles $H_1H_2H_4$ and $O_1O_2O_4$ are oppositely similar and H_1H_2 and H_1H_4 are parallel to O_1O_4 and O_1O_2 respectively.

Furthermore, by hypothesis, $O_1H_1 \times O_2H_2 \times O_4H_4$.

Thus, by Lemma 6, H_2, H_4, O_2 and O_4 are on the same circle.

From this and the fact that $O_2O_4 \neq O_2O_4 2\cos\alpha = H_2H_4$, it could be learnt that O_2H_4 and O_4H_2 intersect.

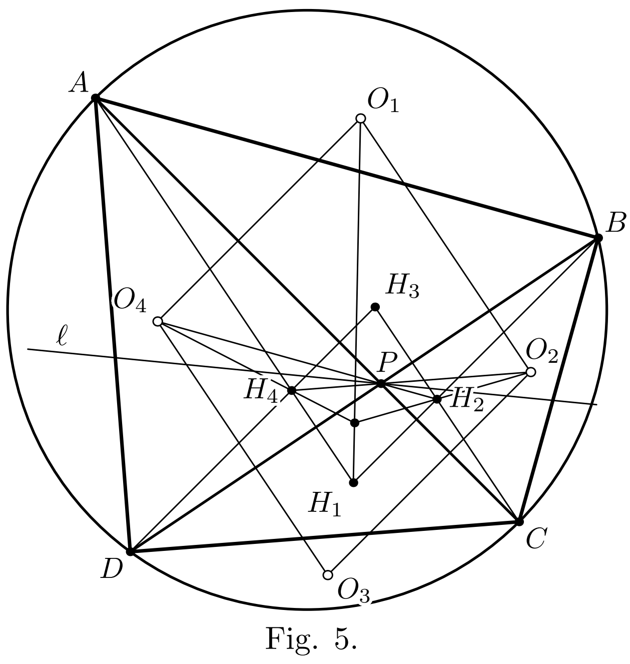

Fig. 5.

Let P' be the intersection of O_2H_4 and O_4H_2.

Then we could see that triangles $P'O_2H_2$ and $P'O_4H_4$ are oppositely similar.

This means that P' is the center of the dilative reflection transforming O_2 and H_2 into O_4 and H_4 respectively.

On the other hand, as mentioned before, the dilative reflection with center P, $H_P^{2\cos\alpha}R_\ell$, transforms O_2 and H_2 into O_4 and H_4 respectively.

This implies that, by Lemmas 3 and 4, P' coincides with P.

So, P lies on the lines O_2H_4 and O_4H_2.

From the result that H_1, H_2, H_3 and H_4 are the images of O_1, O_2, O_3 and O_4 respectively, under the dilative reflection $H_P^{2\cos\alpha}R_\ell$, we learn that the reflection R_ℓ transforms the line PO_2 into PH_2.

As ℓ contains the bisectors of $\angle APD$ and $\angle BPC$, the reflection R_ℓ transforms the line AC into BD.

Thus,

$$\begin{aligned}(AD,\, AC) &\equiv (AD,\, PH_4) + (PO_2,\, AC) \equiv \\ &\equiv \tfrac{\pi}{2} + (BD,\, PH_2) \equiv \tfrac{\pi}{2} + (BD,\, BC) + (BC,\, PH_2) \equiv \\ &\equiv \tfrac{\pi}{2} + (BD,\, BC) + \tfrac{\pi}{2} \equiv (BD,\, BC) \pmod{\pi}.\end{aligned}$$

Hence, $ABCD$ is a concyclic quadrilateral. □

Acknowledgements

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References

[1] K. Vittas. Euler lines in cyclic quadrilateral.
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=48t=107997.

[2] K. Vittas. Hyacinthos messages 12112, Feb 8 2006.
http://tech.groups.yahoo.com/group/Hyacinthos/message/12112.

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EHRMANN'S THIRD LEMOINE CIRCLE

DARIJ GRINBERG

ABSTRACT. The symmedian point of a triangle is known to give rise to two circles, obtained by drawing respectively parallels and antiparallels to the sides of the triangle through the symmedian point. In this note we will explore a third circle with a similar construction — discovered by Jean-Pierre Ehrmann [1]. It is obtained by drawing circles through the symmedian point and two vertices of the triangle, and intersecting these circles with the triangle's sides. We prove the existence of this circle and identify its center and radius.

1. THE FIRST TWO LEMOINE CIRCLES

Let us remind the reader about some classical triangle geometry first. Let L be the symmedian point of a triangle ABC. Then, the following two results are well-known ([3], Chapter 9):

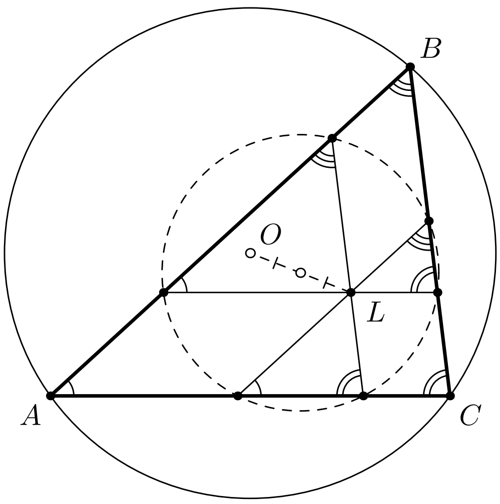

Fig. 1.

Theorem 1. *Let the parallels to the lines BC, BC, CA, CA, AB, AB through L meet the lines CA, AB, AB, BC, BC, CA at six points. These six points lie on one circle, the so-called **first Lemoine circle** of triangle ABC; this circle is a Tucker circle, and its center is the midpoint of the segment OL, where O is the circumcenter of triangle ABC. (See Fig. 1)*

The somewhat uncommon formulation "Let the parallels to the lines BC, BC, CA, CA, AB, AB through L meet the lines CA, AB, AB, BC, BC, CA at six points" means the following: Take the point where the parallel to BC through L

meets CA, the point where the parallel to BC through L meets AB, the point where the parallel to CA through L meets AB, the point where the parallel to CA through L meets BC, the point where the parallel to AB through L meets BC, and the point where the parallel to AB through L meets CA.

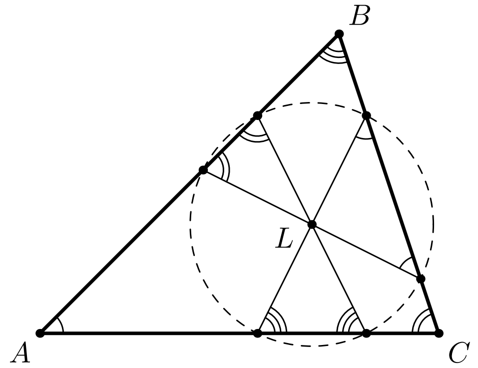

Fig. 2.

Furthermore (see [3] for this as well):

Theorem 2. *Let the antiparallels to the lines BC, BC, CA, CA, AB, AB through L meet the lines CA, AB, AB, BC, BC, CA at six points. These six points lie on one circle, the so-called* second Lemoine circle *(also known as the* cosine circle*) of triangle ABC; this circle is a Tucker circle, and its center is L. (See Fig. 2.)*

We have been using the notion of a *Tucker circle* here. This can be defined as follows:

Theorem 3. *Let ABC be a triangle. Let Q_a and R_a be two points on the line BC. Let R_b and P_b be two points on the line CA. Let P_c and Q_c be two points on the line AB. Assume that the following six conditions hold: The lines Q_bR_c, R_cP_a, P_bQ_a are parallel to the lines BC, CA, AB, respectively; the lines P_bP_c, Q_cQ_a, R_aR_b are antiparallel to the sidelines BC, CA, AB of triangle ABC, respectively. (Actually, requiring five of these conditions is enough, since any five of them imply the sixth one, as one can show.) Then, the points Q_a, R_a, R_b, P_b, P_c and Q_c lie on one circle. Such circles are called* Tucker circles *of triangle ABC. The center of each such circle lies on the line OL, where O is the circumcenter and L the symmedian point of triangle ABC. Notable Tucker circles are the circumcircle of triangle ABC, its first and second Lemoine circles (and the third one we will define below), and its Taylor circle.*

2. The third Lemoine circle

Far less known than these two results is the existence of a third member can be added to this family of Tucker circles related to the symmedian point L. As far as I know, it has been first discovered by Jean-Pierre Ehrmann in 2002 [1]:

Theorem 4. *Let the circumcircle of triangle BLC meet the lines CA and AB at the points A_b and A_c (apart from C and B). Let the circumcircle of triangle*

CLA meet the lines AB and BC at the point B_c and B_a (apart from A and C). Let the circumcircle of triangle ALB meet the lines BC and CA at the points C_a and C_b (apart from B and A). Then, the six points A_b, A_c, B_c, B_a, C_a, C_b lie on one circle. This circle is a Tucker circle, and its midpoint M lies on the line OL and satisfies $LM = -\frac{1}{2} \cdot LO$ (where the segments are directed). The radius of this circle is $\frac{1}{2}\sqrt{9r_1^2 + r^2}$, where r is the circumradius and r_1 is the radius of the second Lemoine circle of triangle ABC.

We propose to denote the circle through the points A_b, A_c, B_c, B_a, C_a, C_b as the *third Lemoine circle* of triangle ABC. (See Fig. 3)

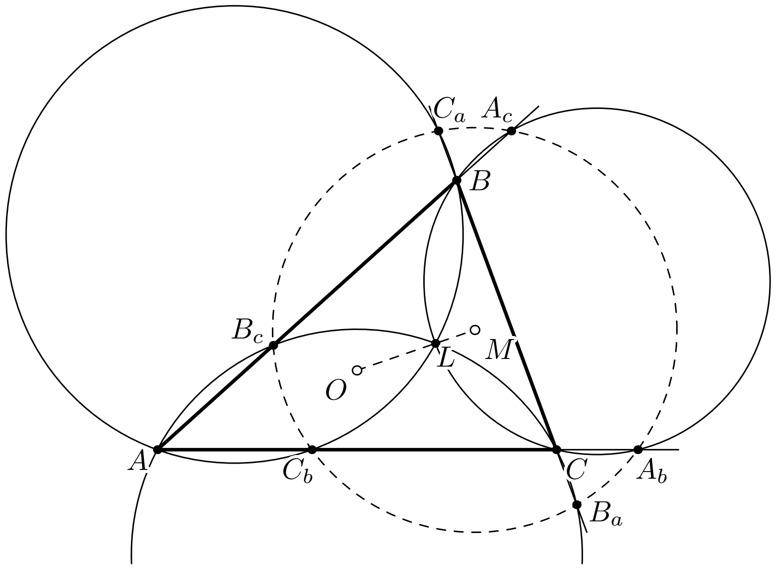

Fig. 3.

The rest of this note will be about proving this theorem. First we will give a complete proof of Theorem 4 in sections 3-5; this proof will use four auxiliary facts (Theorems 5, 6, 7 and 8). Then, in sections 6 and 7, we will give a new argument to show the part of Theorem 4 which claims that the six points A_b, A_c, B_c, B_a, C_a, C_b lie on one circle; this argument will not give us any information about the center of this circle (so that it doesn't extend to a complete second proof of Theorem 4, apparently), but it has the advantage of showing a converse to Theorem 4 (which we formulate as Theorem 10 in the final section 8).

3. A lemma

In triangle geometry, most nontrivial proofs begin by deducing further (and easier) properties of the configuration. These properties are then used as lemmas (and even if they don't turn out directly useful, they are often interesting for themselves). In the case of Theorem 4, the following result plays the role of such a lemma:

Theorem 5. *The point L is the centroid of each of the three triangles AA_bA_c, B_aBB_c, C_aC_bC. (See Fig. 4)*

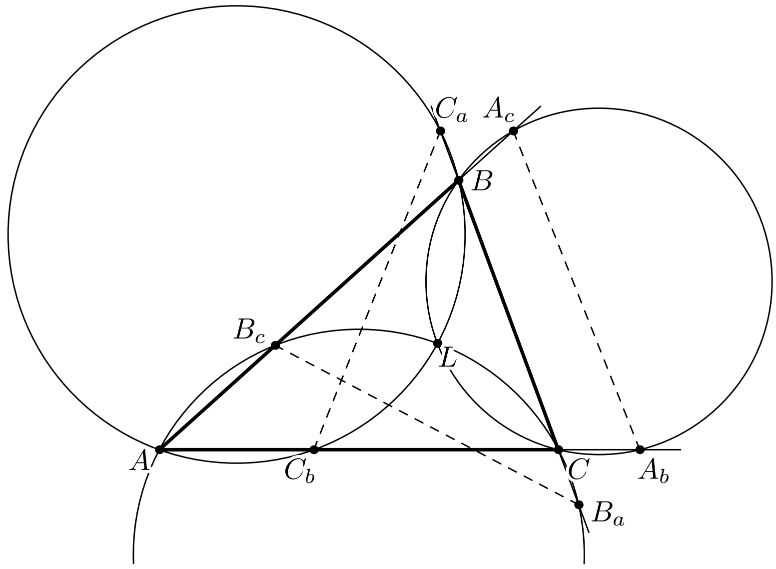

Fig. 4.

Actually this result isn't as much about symmedian points and centroids, as it generalizes to arbitrary isogonal conjugates:

Theorem 6. *Let P and Q be two points isogonally conjugate to each other with respect to triangle ABC. Let the circumcircle of triangle CPA meet the lines AB and BC at the points B_c and B_a (apart from A and C). Then, the triangles B_aBB_c and ABC are oppositely similar, and the points P and Q are corresponding points in the triangles B_aBB_c and ABC. (See Fig. 5)*

Remark 1. Two points P_1 and P_2 are said to be *corresponding points* in two similar triangles Δ_1 and Δ_2 if the similitude transformation that maps triangle Δ_1 to triangle Δ_2 maps the point P_1 to the point P_2.

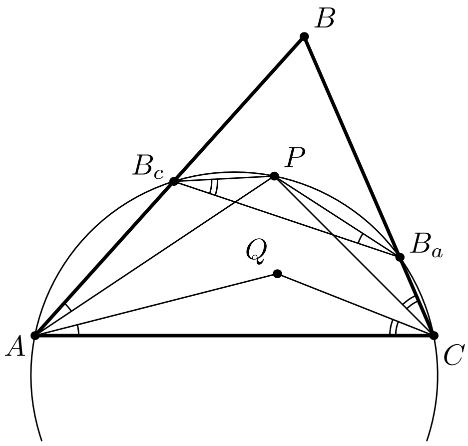

Fig. 5.

Proof of Theorem 6. We will use directed angles modulo 180°. A very readable introduction into this kind of angles can be found in [4]. A list of their important properties has also been given in [2].

The point Q is the isogonal conjugate of the point P with respect to triangle ABC; thus, $\measuredangle PAB = \measuredangle CAQ$.

Since C, A, B_c, B_a are concyclic points, we have $\measuredangle CB_aB_c = \measuredangle CAB_c$, so that $\measuredangle BB_aB_c = -\measuredangle BAC$. Furthermore, $\measuredangle B_cBB_a = -\measuredangle CBA$. Thus, the triangles B_aBB_c and ABC are oppositely similar (having two pairs of oppositely equal angles).

By the chordal angle theorem, $\measuredangle PB_aB_c = \measuredangle PAB_c = \measuredangle PAB = \measuredangle CAQ = -\measuredangle QAC$. Similarly, $\measuredangle PB_cP_a = -\measuredangle QCA$. These two equations show that the triangles B_aPB_c and AQC are oppositely similar. Combining this with the opposite similarity of triangles B_aBB_c and ABC, we obtain that the quadrilaterals B_aBB_cP and $ABCQ$ are oppositely similar. Hence, P and Q are corresponding points in the triangles B_aBB_c and ABC. (See Fig. 6.) Theorem 6 is thus proven. □

Proof of Theorem 5. Now return to the configuration of Theorem 4. To prove Theorem 5, we apply Theorem 6 to the case when P is the symmedian point of triangle ABC; the isogonal conjugate Q of P is, in this case, the centroid of triangle ABC. Now, Theorem 6 says that the points P and Q are corresponding points in the triangles B_aBB_c and ABC. Since Q is the centroid of triangle ABC, this means that P is the centroid of triangle B_aBB_c. But $P = L$; thus, we have shown that L is the centroid of triangle B_aBB_c. Similarly, L is the centroid of triangles AA_bA_c and C_aC_bC, and Theorem 5 follows. □

4. Antiparallels

Theorem 5 was the first piece of our jigsaw. Next we are going to chase some angles.

Since the points B, C, A_b, A_c are concyclic, we have $\measuredangle CA_bA_c = \measuredangle CBA_c$, so that $\measuredangle AA_bA_c = -\measuredangle ABC$. Thus, the line A_bA_c is antiparallel to BC in triangle ABC. Similarly, the lines B_cB_a and C_aC_b are antiparallel to CA and AB. We have thus shown:

Theorem 7. *In the configuration of Theorem 4, the lines A_bA_c, B_cB_a, C_aC_b are antiparallel to BC, CA, AB in triangle ABC.*

Now let X_b, X_c, Y_c, Y_a, Z_a, Z_b be the points where the antiparallels to the lines BC, BC, CA, CA, AB, AB through L meet the lines CA, AB, AB, BC, BC, CA. According to Theorem 2, these points X_b, X_c, Y_c, Y_a, Z_a, Z_b lie on one circle around L; however, to keep this note self-contained, we do not want to depend on Theorem 2 here, but rather prove the necessary facts on our own (Fig. 6):

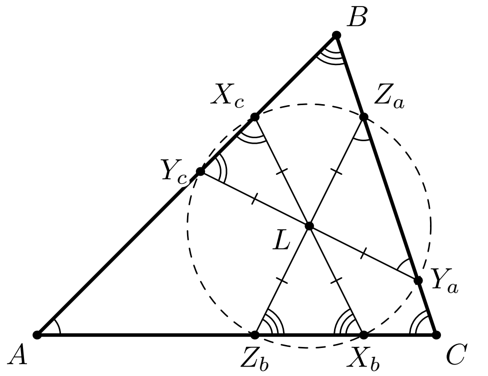

Fig. 6.

Partial proof of Theorem 2. Since symmedians bisect antiparallels, and since the symmedian point L of triangle ABC lies on all three symmedians, the point L must bisect the three antiparallels X_bX_c, Y_cY_a, Z_aZ_b. This means that $LX_b = LX_c$, $LY_c = LY_a$ and $LZ_a = LZ_b$. Furthermore, $\measuredangle AX_cX_b = -\measuredangle ACB$ (since X_bX_c is antiparallel to BC), thus $\measuredangle Y_cX_cL = -\measuredangle ACB$; similarly, $\measuredangle X_cY_cL = -\measuredangle BCA$, thus $\measuredangle LY_cX_c = -\measuredangle X_cY_cL = \measuredangle BCA = -\measuredangle ACB = \measuredangle Y_cX_cL$. Hence, triangle X_cLY_c is isosceles, so that $LX_c = LY_c$. Similarly, $LZ_b = LX_b$ and $LY_a = LZ_a$. Hence,

$$LX_b = LX_c = LY_c = LY_a = LZ_a = LZ_b.$$

This shows that the points X_b, X_c, Y_c, Y_a, Z_a, Z_b lie on one circle around L. This circle is the so-called second Lemoine circle of triangle ABC. Its radius is $r_1 = LX_b = LX_c = LY_c = LY_a = LZ_a = LZ_b$.

We thus have incidentally proven most of Theorem 2 (to complete the proof of Theorem 2, we would only have to show that this circle is a Tucker circle, which is easy); but we have also made headway to the proof of Theorem 4. □

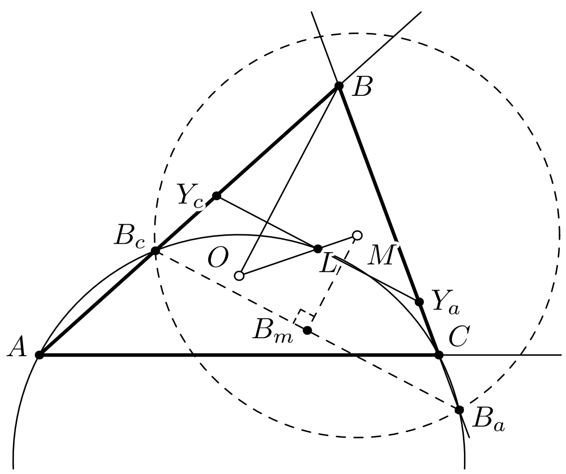

Fig. 7.

Proof of Theorem 9, part 1. Now consider Fig. 8. Let B_m be the midpoint of the segment B_cB_a. According to Theorem 5, the point L is the centroid of triangle

B_aBB_c; thus, it lies on the median BB_m and divides it in the ratio $2:1$. Hence, $BL : LB_m = 2$ (with directed segments).

Let M be the point on the line OL such that $LM = -\frac{1}{2} \cdot LO$ (with directed segments); then, $LO = -2 \cdot LM$, so that $OL = -LO = 2 \cdot LM$, and thus $OL : LM = 2 = BL : LB_m$. By the converse of Thales' theorem, this yields $B_mM \parallel BO$.

The lines Y_cY_a and B_cB_a are both antiparallel to CA, and thus parallel to each other. Hence, Thales' theorem yields $B_mB_c : LY_c = BB_m : BL$. But since $BL : LB_m = 2$, we have $BL = 2 \cdot LB_m$, so that $LB_m = \frac{1}{2} \cdot BL$ and therefore $BB_m = BL + LB_m = BL + \frac{1}{2} \cdot BL = \frac{3}{2} \cdot BL$ and $BB_m : BL = \frac{3}{2}$. Consequently, $B_mB_c : LY_c = \frac{3}{2}$ and $B_mB_c = \frac{3}{2} \cdot LY_c = \frac{3}{2} r_1$.

It is a known fact that every line antiparallel to the side CA of triangle ABC is perpendicular to the line BO (where, as we remind, O is the circumcenter of triangle ABC). Thus, the line B_cB_a (being antiparallel to CA) is perpendicular to the line BO. Since $B_mM \parallel BO$, this yields $B_cB_a \perp B_mM$. Furthermore, $B_mM \parallel BO$ yields $BO : B_mM = BL : LB_m$ (by Thales), so that $BO : B_mM = 2$ and $BO = 2 \cdot B_mM$, and thus $B_mM = \frac{1}{2} \cdot BO = \frac{1}{2} r$, where r is the circumradius of triangle ABC.

Now, Pythagoras' theorem in the right-angled triangle MB_mB_c yields

$$MB_c = \sqrt{B_mB_c{}^2 + B_mM^2} = \sqrt{\left(\frac{3}{2} r_1\right)^2 + \left(\frac{1}{2} r\right)^2} = \sqrt{\frac{9}{4} r_1^2 + \frac{1}{4} r^2} = \frac{1}{2} \sqrt{9r_1^2 + r^2}.$$

Similarly, we obtain the same value $\frac{1}{2}\sqrt{9r_1^2 + r^2}$ for each of the lengths MB_a, MC_a, MC_b, MA_b and MA_c. Hence, the points A_b, A_c, B_c, B_a, C_a, C_b all lie on the circle with center M and radius $\frac{1}{2}\sqrt{9r_1^2 + r^2}$. The point M, in turn, lies on the line OL and satisfies $LM = -\frac{1}{2} \cdot LO$.

This already proves most of Theorem 4. The only part that has yet to be shown is that this circle is a Tucker circle. We will do this next. □

5. Parallels

Since the points A_b, B_a, C_a, C_b are concyclic, we have $\measuredangle C_aB_aA_b = \measuredangle C_aC_bA_b$, thus $\measuredangle CB_aA_b = \measuredangle C_aC_bC$. But since C_aC_b is antiparallel to AB, we have $\measuredangle CC_bC_a = -\measuredangle CBA$, so that $\measuredangle C_aC_bC = -\measuredangle CC_bC_a = \measuredangle CBA$, thus $\measuredangle CB_aA_b = \measuredangle CBA$; consequently, $A_bB_a \parallel AB$. Similarly, $B_cC_b \parallel BC$ and $C_aA_c \parallel CA$. Altogether, we have thus seen:

Theorem 8. *The lines B_cC_b, C_aA_c, A_bB_a are parallel to BC, CA, AB. (See Fig. 8)*

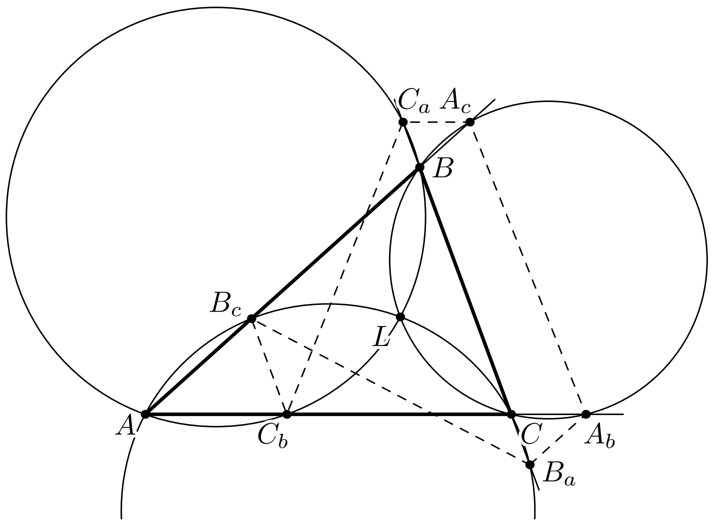

Fig. 8.

Proof of Theorem 4, part 2. If we now combine Theorem 7 and Theorem 8, we conclude that the sides of the hexagon $A_bA_cC_aC_bB_cB_a$ are alternately antiparallel and parallel to the sides of triangle ABC. Thus, $A_bA_cC_aC_bB_cB_a$ is a Tucker hexagon, and the circle passing through its vertices A_b, A_c, B_c, B_a, C_a, C_b is a Tucker circle. This concludes the proof of Theorem 4. □

One remark: It is known that the radius r_1 of the second Lemoine circle ΔABC is $r \tan \omega$, where ω is the Brocard angle of triangle ABC. [1] Thus, the radius of the third Lemoine circle is

$$\frac{1}{2}\sqrt{9r_1^2 + r^2} = \frac{1}{2}\sqrt{9\left(r \tan \omega\right)^2 + r^2} = \frac{1}{2}\sqrt{9r^2 \tan^2 \omega + r^2} = \frac{r}{2}\sqrt{9 \tan^2 \omega + 1}.$$

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[1]For a quick proof of this fact, let X, Y, Z denote the feet of the perpendiculars from the point L to the sides BC, CA, AB of triangle ABC. Then, the radius r_1 of the second Lemoine circle is $r_1 = LY_a$. Working *without* directed angles now, we see that $LY_a = \dfrac{LX}{\sin A}$ (from the right-angled triangle LXY_a), so that $r_1 = LY_a = \dfrac{LX}{\sin A}$. Since $\sin A = \dfrac{a}{2r}$ by the extended law of sines, this rewrites as $r_1 = \dfrac{LX}{\left(\dfrac{a}{2r}\right)} = \dfrac{LX \cdot 2r}{a}$. This rewrites as $\dfrac{a^2}{2r} r_1 = a \cdot LX$. Similarly, $\dfrac{b^2}{2r} r_1 = b \cdot LY$ and $\dfrac{c^2}{2r} r_1 = c \cdot LZ$. Adding these three equations together, we obtain

$$\frac{a^2}{2r} r_1 + \frac{b^2}{2r} r_1 + \frac{c^2}{2r} r_1 = a \cdot LX + b \cdot LY + c \cdot LZ.$$

On the other hand, let S denote the area of triangle ABC. But the area of triangle BLC is $\dfrac{1}{2} a \cdot LX$ (since a is a sidelength of triangle BLC, and LX is the corresponding altitude), and similarly the areas of triangles CLA and ALB are $\dfrac{1}{2} b \cdot LY$ and $\dfrac{1}{2} c \cdot LZ$. Thus,

$$\begin{aligned}\frac{1}{2} a \cdot LX + \frac{1}{2} b \cdot LY + \frac{1}{2} c \cdot LZ &= (\text{area of triangle } BLC) + (\text{area of triangle } CLA) + (\text{area of triangle } ALB) \\ &= (\text{area of triangle } ABC) = S,\end{aligned}$$

6. A different approach: a lemma about four points

At this point, we are done with our job: Theorem 4 is proven. However, our proof depended on the construction of a number of auxiliary points (not only B_m, but also the six points X_b, X_c, Y_c Y_a, Z_a, Z_b). One might wonder whether there isn't also a (possibly more complicated, but) more straightforward approach to proving the concyclicity of the points A_b, A_c, B_c, B_a, C_a, C_b without auxiliary constructions. We will show such an approach now. It will not yield the complete Theorem 4, but on the upside, it helps proving a kind of converse.

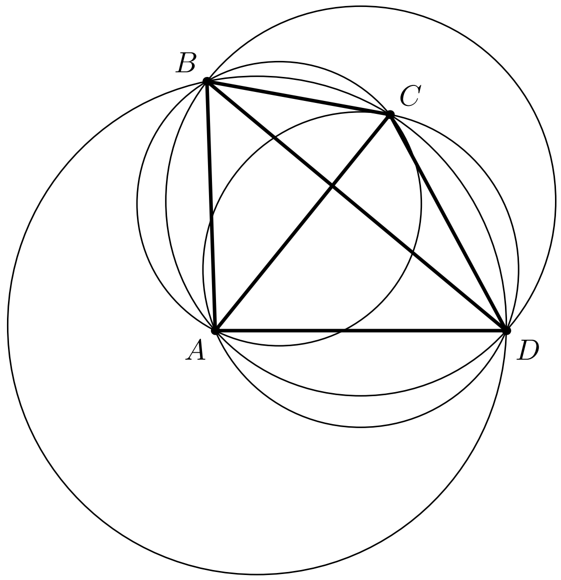

Fig. 9.

Let us use directed areas and powers of points with respect to circles. Our main vehicle is the following fact:

Theorem 9. *Let A, B, C, D be four points. Let p_A, p_B, p_C, p_D denote the powers of the points A, B, C, D with respect to the circumcircles of triangles BCD, CDA, DAB, ABC. Furthermore, we denote by $[P_1P_2P_3]$ the directed area of any triangle $P_1P_2P_3$. Then,*

(1) $$-p_A \cdot [BCD] = p_B \cdot [CDA] = -p_C \cdot [DAB] = p_D \cdot [ABC] .$$

(See Fig. 10.)

---

so that $a \cdot LX + b \cdot LY + c \cdot LZ = 2S$. The equation $\frac{a^2}{2r}r_1 + \frac{b^2}{2r}r_1 + \frac{c^2}{2r}r_1 = a \cdot LX + b \cdot LY + c \cdot LZ$ thus becomes $\frac{a^2}{2r}r_1 + \frac{b^2}{2r}r_1 + \frac{c^2}{2r}r_1 = 2S$, so that

$$r_1 = \frac{2S}{\frac{a^2}{2r}r_1 + \frac{b^2}{2r}r_1 + \frac{c^2}{2r}r_1} = \frac{4rS}{a^2+b^2+c^2} = r \diagup \underbrace{\frac{a^2+b^2+c^2}{4S}}_{=\cot\omega} = r \diagup \cot\omega = r\tan\omega,$$

qed.

Proof of Theorem 9. In the following, all angles are directed angles modulo 180°, and all segments and areas are directed. Let the circumcircle of triangle BCD meet the line AC at a point A' (apart from C). Let the circumcircle of triangle CDA meet the line BD at a point B' (apart from D). Let the lines AC and BD intersect at P. Since the points C, D, A, B' are concyclic, we have $\measuredangle DB'A = \measuredangle DCA$, thus $\measuredangle PB'A = \measuredangle DCA$. But since the points C, D, B, A' are concyclic, we have $\measuredangle DBA' = \measuredangle DCA'$, so that $\measuredangle PBA' = \measuredangle DCA$. Therefore, $\measuredangle PB'A = \measuredangle PBA'$, which leads to $B'A \parallel BA'$. Hence, by the Thales theorem,

$$\frac{A'A}{BB'} = \frac{PA'}{PB}.$$

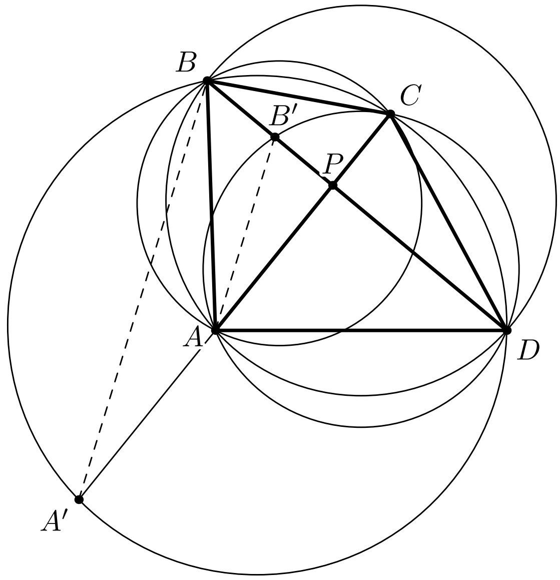

Fig. 10.

The power p_A of the point A with respect to the circumcircle of triangle BCD is $AA' \cdot AC$; similarly, $p_B = BB' \cdot BD$. Thus,

$$\frac{-p_A \cdot [BCD]}{p_B \cdot [CDA]} = \frac{-AA' \cdot AC \cdot [BCD]}{BB' \cdot BD \cdot [CDA]} = \frac{A'A \cdot AC \cdot [BCD]}{BB' \cdot BD \cdot [CDA]} = \frac{A'A}{BB'} \cdot \frac{AC}{BD} \cdot \frac{[BCD]}{[CDA]}$$
$$= \frac{PA'}{PB} \cdot \frac{AC}{BD} \cdot \frac{[BCD]}{[CDA]}.$$

But it is a known fact that whenever $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are three collinear points and $\mathcal{P}$ is a point not collinear with $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, then we have $\frac{\mathcal{AB}}{\mathcal{AC}} = \frac{[\mathcal{APB}]}{[\mathcal{APC}]}$. This fact yields

$$\frac{CP}{CA} = \frac{[CDP]}{[CDA]} \qquad \text{and} \qquad \frac{DP}{DB} = \frac{[DCP]}{[DCB]},$$

so that

$$\frac{CP}{CA} : \frac{DP}{DB} = \frac{[CDP]}{[CDA]} : \frac{[DCP]}{[DCB]} = \frac{[DCB] \cdot [CDP]}{[DCP] \cdot [CDA]} = \frac{-[BCD] \cdot [CDP]}{-[CDP] \cdot [CDA]} = \frac{[BCD]}{[CDA]},$$

and thus

$$\frac{-p_A \cdot [BCD]}{p_B \cdot [CDA]} = \frac{PA'}{PB} \cdot \frac{AC}{BD} \cdot \frac{[BCD]}{[CDA]} = \frac{PA'}{PB} \cdot \frac{AC}{BD} \cdot \left(\frac{CP}{CA} : \frac{DP}{DB}\right)$$
$$= \frac{PA'}{PB} \cdot \frac{AC}{BD} \cdot \left(\frac{PC}{AC} : \frac{PD}{BD}\right) = \frac{PA'}{PB} \cdot \frac{AC}{BD} \cdot \frac{PC}{AC} \cdot \frac{BD}{PD} = \frac{PA' \cdot PC}{PB \cdot PD}.$$

But the intersecting chord theorem yields $PA' \cdot PC = PB \cdot PD$, so that

$$\frac{-p_A \cdot [BCD]}{p_B \cdot [CDA]} = 1;$$

in other words, $-p_A \cdot [BCD] = p_B \cdot [CDA]$. Similarly, $-p_B \cdot [CDA] = p_C \cdot [DAB]$, so that $p_B \cdot [CDA] = -p_C \cdot [DAB]$, and $-p_C \cdot [DAB] = p_D \cdot [ABC]$. Combining these equalities, we get (1). This proves Theorem 9. ☐

7. Application to the triangle

Now the promised alternative proof of the fact that the points A_b, A_c, B_c, B_a, C_a, C_b are concyclic:

Proof. The identity (1) can be rewritten as

$$p_A \cdot [BDC] = p_B \cdot [CDA] = p_C \cdot [ADB] = p_D \cdot [ABC]$$

(since $[BCD] = -[BDC]$ and $[DAB] = -[ADB]$). Applying this equation to the case when D is the symmedian point L of triangle ABC, we get

$$p_A \cdot [BLC] = p_B \cdot [CLA] = p_C \cdot [ALB]$$

(we have dropped the third equality sign since we don't need it), where p_A, p_B, p_C are the powers of the points A, B, C with respect to the circumcircles of triangles BLC, CLA, ALB. But we have $p_A = AC \cdot AA_b$ and $p_B = BC \cdot BB_a$ (Fig. 4); thus, $p_A \cdot [BLC] = p_B \cdot [CLA]$ becomes

$$AC \cdot AA_b \cdot [BLC] = BC \cdot BB_a \cdot [CLA],$$

so that

$$\frac{AA_b}{BB_a} = \frac{BC}{AC} \cdot \frac{[CLA]}{[BLC]}. \tag{2}$$

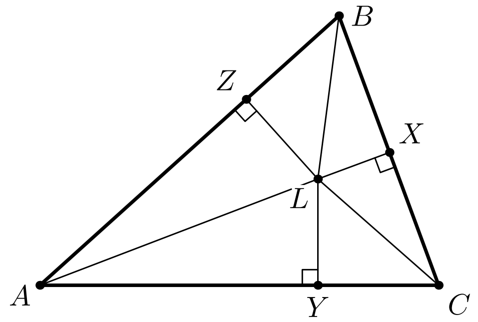

Fig. 11.

Now let X, Y and Z be the feet of the perpendiculars from the symmedian point L onto the lines BC, CA, AB. It is a known fact that the distances from the symmedian point of a triangle to its sides are proportional to these sides; thus, $LX : LY : LZ = BC : CA : AB$. [2] But since the area of a triangle equals $\frac{1}{2}\cdot$ (one of its sidelengths) $\cdot$ (corresponding altitude), we have $[BLC] = \frac{1}{2}\cdot BC\cdot LX$, $[CLA] = \frac{1}{2}\cdot CA\cdot LY$ and $[ALB] = \frac{1}{2}\cdot AB\cdot LZ$. Thus, (2) becomes

$$\frac{AA_b}{BB_a} = \frac{BC}{AC}\cdot\frac{\frac{1}{2}\cdot CA\cdot LY}{\frac{1}{2}\cdot BC\cdot LX} = \frac{BC}{AC}\cdot\frac{CA}{BC}\cdot\frac{LY}{LX} = \frac{BC}{AC}\cdot\frac{CA}{BC}\cdot\frac{CA}{BC}$$
$$= \frac{BC}{AC}\cdot\frac{CA^2}{BC^2} = \frac{BC}{AC}\cdot\frac{AC^2}{BC^2} = \frac{AC}{BC}.$$

By the converse of Thales' theorem, this yields $A_bB_a \parallel AB$. Similarly, $B_cC_b \parallel BC$ and $C_aA_c \parallel CA$; this proves Theorem 8. The proof of Theorem 7 can be done in the same way as in section 4 (it was too trivial to have any reasonable alternative). Combined, this yields that the sides of the hexagon $A_bA_cC_aC_bB_cB_aA_b$ are alternately antiparallel and parallel to the sides of triangle ABC. Consequently, this hexagon is a Tucker hexagon, and since every Tucker hexagon is known to be cyclic, we thus conclude that the points A_b, A_c, B_c, B_a, C_a, C_b lie on one circle. This way we have reproven a part of Theorem 4. □

Here is an alternative way to show that the points A_b, A_c, B_c, B_a, C_a, C_b lie on one circle, without using the theory of Tucker hexagons:

Proof. (See Fig. 9.) Since C_aC_b is antiparallel to AB, we have $\measuredangle CC_bC_a = -\measuredangle CBA$, and thus

$$\measuredangle C_bC_aA_c = \measuredangle(C_aC_b;\ C_aA_c) = \measuredangle(C_aC_b;\ CA) \qquad \text{(since } C_aA_c \parallel CA\text{)}$$
$$= \measuredangle C_aC_bC = -\measuredangle CC_bC_a = \measuredangle CBA.$$

Since A_bA_c is antiparallel to BC, we have $\measuredangle AA_bA_c = -\measuredangle ABC$, so that

$$\measuredangle C_bA_bA_c = \measuredangle AA_bA_c = -\measuredangle ABC = \measuredangle CBA.$$

Therefore, $\measuredangle C_bA_bA_c = \measuredangle C_bC_aA_c$, so that the points C_a, C_b, A_b, A_c lie on one circle. The point B_a also lies on this circle, since

$$\measuredangle C_aB_aA_b = \measuredangle(BC;\ A_bB_a) = \measuredangle(BC;\ AB) \qquad \text{(since } A_bB_a \parallel AB\text{)}$$
$$= \measuredangle CBA = -\measuredangle CC_bC_a \text{ (since } \measuredangle CC_bC_a = -\measuredangle CBA \text{ was shown above)}$$
$$= \measuredangle C_aC_bA_b.$$

Similarly, the point B_c lies on this circle as well. This shows that all six points A_b, A_c, B_c, B_a, C_a, C_b lie on one circle. □

[2] Actually, there is no need to refer to this known fact here, because we have almost completely proven it above. Indeed, while proving that $r_1 = r\tan\omega$, we showed that $r_1 = \frac{LX\cdot 2r}{a}$, so that $LX = \frac{r_1}{2r}a = \frac{r_1}{2r}BC$. Similarly, $LY = \frac{r_1}{2r}CA$ and $LZ = \frac{r_1}{2r}AB$, so that $LX : LY : LZ = BC : CA : AB$.

This argument did never use anything but the facts that the lines A_bA_c, B_cB_a, C_aC_b are antiparallel to BC, CA, AB and that the lines B_cC_b, C_aA_c, A_bB_a are parallel to BC, CA, AB. It can therefore be used as a general argument why Tucker hexagons are cyclic.

8. A converse

The alternative proof in 7 allows us to show a converse of Theorem 4, also found by Ehrmann in [1]:

Theorem 10. *Let P be a point in the plane of a triangle ABC but not on its circumcircle. Let the circumcircle of triangle BPC meet the lines CA and AB at the points A_b and A_c (apart from C and B). Let the circumcircle of triangle CPA meet the lines AB and BC at the point B_c and B_a (apart from A and C). Let the circumcircle of triangle APB meet the lines BC and CA at the points C_a and C_b (apart from B and A). If the six points A_b, A_c, B_c, B_a, C_a, C_b lie on one circle, then P is the symmedian point of triangle ABC.*

We will not give a complete proof of this theorem here, but we only sketch its path: First, it is easy to see that the lines A_bA_c, B_cB_a, C_aC_b are antiparallel to BC, CA, AB. Now, we can reverse the argument from section 7 to show that the lines B_cC_b, C_aA_c, A_bB_a are parallel to BC, CA, AB, and use this to conclude that the distances from P to the sidelines BC, CA, AB of triangle ABC are proportional to the lengths of BC, CA, AB. But this implies that P is the symmedian point of triangle ABC.

References

[1] J. P. Ehrmann. Hyacinthos message 6098, Dec 2 2002.
[2] D. Grinberg. The Neuberg–Mineur circle. *Mathematical Reflections*, 3 2007.
[3] R. Honsberger. *Episodes in nineteenth and twentieth century Euclidean geometry*, volume 37. The Mathematical Association of America, 1995.
[4] R. A. Johnson. Directed angles in elementary geometry. *The American Mathematical Monthly*, 24(3):101–105, 1917.

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AN ELEMENTARY PROOF OF LESTER'S THEOREM

NIKOLAI IVANOV BELUHOV

Abstract. In 1996, J. A. Lester discovered that in every scalene triangle the two Fermat-Torricelli points, the circumcenter, and the center of the nine-point circle are concyclic. We give the first proof of this fact to only employ results from elementary geometry.

In 1996, Professor of Mathematics June A. Lester discovered a remarkable new theorem in triangle geometry:

Lester's theorem. *In every scalene triangle, the two Fermat points, the circumcenter and the nine-point center are concyclic.*

The history of this theorem's discovery is almost as peculiar as the result itself. Here is J. A. Lester's own description:

"I discovered the theorem by searching through a large number of special triangle points for quadruples of points which lie on a circle [...] First, I needed a database of special points and their coordinates; I got it from Clark Kimberling's list of triangle centres [...] I next input everything – points, conversion formulas, cross ratio formulas – into an easy-to-use computer math program, *Theorist* [...] Then I input a single numerical shape [of a triangle] and set a search going for quadruples of special points with a real cross ratio."

The computation took several hours to complete and "had to be repeated multiple times to be sure that those real cross ratios found were not a coincidence." Probably for the first time in Euclidian geometry, a theorem was discovered by applying brute force by a person who specifically set out to do so!

Clearly, this approach does not hint to any actual proof of the theorem. June Lester's own initial proof made extensive use of complex numbers as well as extremely laborious calculations. Later on, simpler analytical proofs were discovered – but a *geometrical* proof was still lacking.

The statement of the theorem was communicated to the author of this note by his mathematics teacher Svetlozar Doychev in 2006, and it has been his dream to work out an elementary proof ever since. Finally, one such proof was discovered: it uses nothing more than the properties of similar figures and cyclic quadrilaterals.

First we establish the following

Lemma. *In a $\triangle ABC$ ($AB \neq AC$), let P be the reflection of B in the line AC and let Q be the reflection of C in the line AB. Let the tangent to the circumcircle* *of $\triangle APQ$ at A meet the line PQ in T. Then, the reflection U of the point T in the point A lies in the line BC (Fig. 1).*

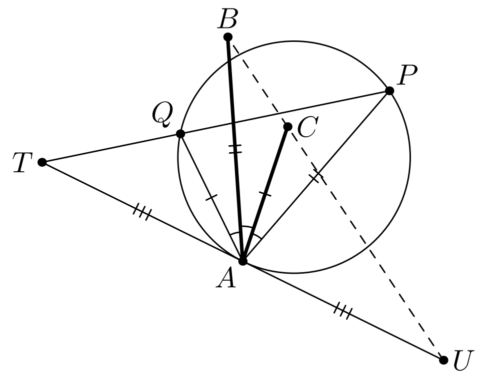

Fig. 1.

Proof. Let K and L be the reflections of the points B and C in the point A. Clearly, it suffices to show that T lies in the line KL (Fig. 2)

We have $AL = AC = AQ$ and $AK = AB = AP$. Besides, $\angle LAQ = 180 - \angle QAC = 180 - 2\angle BAC = 180 - \angle BAP = \angle PAK$. It follows that $\triangle LAQ$ and $\triangle PAK$ are two similar isosceles triangles with base angles equal to $\angle BAC$.

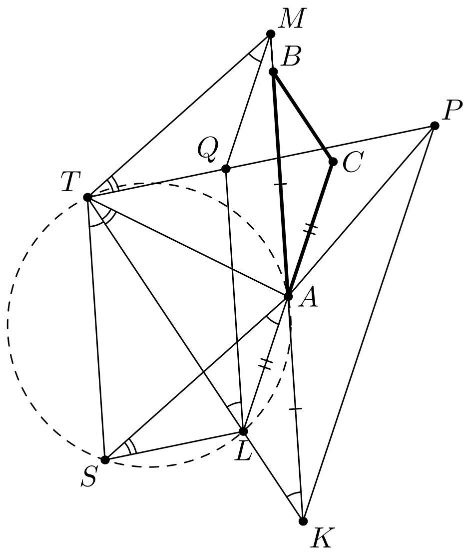

Fig. 2.

By the construction of the point T, the triangles $\triangle TQA$ and $\triangle TAP$ are similar. Therefore, it is possible to construct the point M such that the figures $TQAM$ and $TAPK$ are similar.

Since $\triangle MQA \sim \triangle KAP \sim \triangle QAL \Rightarrow \triangle MQA \simeq \triangle QAL$, the figure $ALQM$ is a parallelogram. Furthermore, $\angle QAM = \angle KPA = \angle BAC = \angle AQB$ shows that the points A, B, K and M are collinear.

Let S be such a point that $\vec{TS} = \vec{QL} = \vec{MA}$ and $\triangle ALS \simeq \triangle MQT$. It follows that $\angle LAS = \angle QMT =$ (as $TQAM \sim TAPK$) $= \angle TKA =$ (as $KA \| LQ$) $= \angle TLQ = \angle LTS$, i.e., that the quadrilateral $LATS$ is cyclic.

Therefore, $\angle ATL = \angle ASL = \angle MTQ =$ (as $TQAM \sim TAPK$) $= \angle ATK \Rightarrow T \in KL$, as needed. This completes the proof of the lemma. □

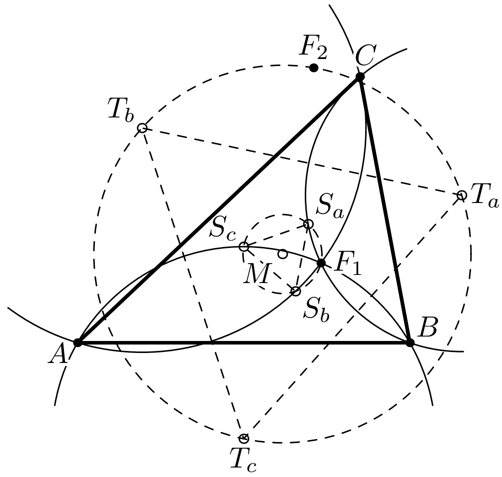

Fig. 3.

Proof of Lester's theorem. Let $\triangle ABC$ be an arbitrary scalene triangle. Construct externally on its sides BC, CA and AB three equilateral triangles δ_a, δ_b and δ_c of centers T_a, T_b and T_c, respectively. Also, construct internally on its sides BC, CA and AB three more equilateral triangles σ_a, σ_b and σ_c of centers S_a, S_b and S_c, respectively.

It is well known that the three circumcircles of the triangles δ_a, δ_b and δ_c meet in the first Fermat point F_1 and that the three circumcircles of the triangles σ_a, σ_b and σ_c meet in the second Fermat point F_2 for the triangle $\triangle ABC$. Also, it is well known that $\triangle T_aT_bT_c$ and $\triangle S_aS_bS_c$ are two equilateral triangles and that the centroid M of $\triangle ABC$ is their common center (Fig. 3).

Notice that $\angle S_bF_1S_c = \angle S_bF_1A + \angle AF_1S_c =$ (as the quadrilaterals AS_bF_1C and BF_1S_cA are both cyclic) $= \angle S_bCA + \angle ABS_c = 30°+30° = 60° = \angle S_bS_aS_c \Rightarrow F_1$ lies in the circumcircle of $\triangle S_aS_bS_c$.

It follows that S_aF_1 is a common chord of two circles of centers M and T_a, respectively, and that F_1 is the reflection of the point S_a in the line MT_a. Analogously, F_2 is the reflection of the point T_a in the line MS_a.

Now we can apply the lemma to $\triangle MS_aT_a$ ($MS_a \neq MT_a$ as $\triangle ABC$ is non-degenerate). Let the tangent to the circumcircle of $\triangle MF_1F_2$ at M meet the line F_1F_2 in Q and let O' be the reflection of Q in the point M. By the lemma, we conclude that the point O' lies in the line S_aT_a.

Analogous application of the lemma to triangles $\triangle MS_bT_b$ and $\triangle MS_cT_c$ yields that the point O' lies in the lines S_bT_b and S_cT_c as well. But these three lines are the perpendicular bisectors of the sides of $\triangle ABC$! It follows that the point O' coincides with the circumcenter O of $\triangle ABC$.

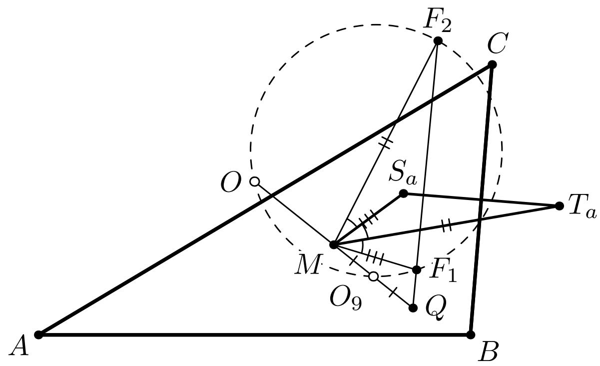

Fig. 4.

Finally, let O_9 be the nine-point center for $\triangle ABC$. It is well-known that the point O_9 lies in the line OM and divides the segment OM externally in ratio $OO_9 : O_9M = 2 : 1$ (Fig. 4).

Therefore, O_9 is the midpoint of the segment MQ. It follows that $QF_1 \cdot QF_2 = QM^2 = (2QM) \cdot (\frac{1}{2}QM) = QO \cdot QO_9$ and the quadrilateral $F_1F_2OO_9$ is cyclic, as needed. This completes the proof of Lester's theorem. □

References

[1] J. A. Lester. The mathematics behind the theorem. *http://thejuniverse.org/PUBLIC/LesterCircle/LCmath.html.*

[2] J. A. Lester. Triangles iii: Complex triangle functions. *Aequationes Mathematicae*, 53(1):4–35, 1997.

[3] P. Yiu. The circles of Lester, Evans, Parry, and their generalizations. In *Forum Geometricorum*, volume 10, pages 175–209, 2010.

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CIRCLES TOUCHING SIDES AND THE CIRCUMCIRCLE FOR INSCRIBED QUADRILATERALS

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ABSTRACT. In an inscribed quadrilateral, four circles touching the circumcircle and two neighboring sides have a radical center.

The main result of the article is the following theorem.

Theorem 1. *Let $ABCD$ be a quadrilateral inscribed to a circle Ω. If Ω_a is the circle touching Ω and segments AB, AD, and circles Ω_b, Ω_c, Ω_d defined similarly (i. e. circles touching Ω and two neighboring sides of $ABCD$), then Ω_a, Ω_b, Ω_c, and Ω_d have a radical center (that is a point having equal powers with respect to Ω_a, Ω_b, Ω_c, and Ω_d).*

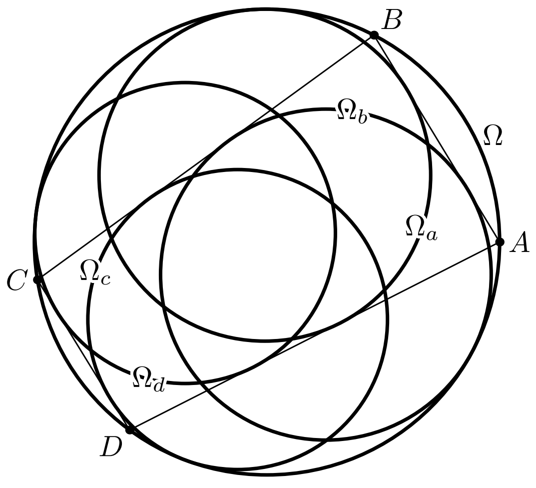

Fig. 1.

We will use the following known Lemmas.

The following lemma is offered on national Russian mathematical olympiad in 2003 year at Number 3 in grade 10 by Berlov.S., Emelyanov.L., Smirnov.A. You can find it in [1].

Lemma 1. *Let $XYZT$ be a quadrilateral with $XZ \perp YT$ inscribed to a circle Ω. Let x, y, z, t be tangents to Ω passing through X, Y, Z, T, respectively. Let $A_1 = t \cap x$, $B_1 = x \cap y$, $C_1 = y \cap z$, $D_1 = z \cap t$. Then the exterior bisectors of quadrilateral $A_1B_1C_1D_1$ form a quadrilateral $X_1Y_1Z_1T_1$ that is homothetic to $XYZT$.*

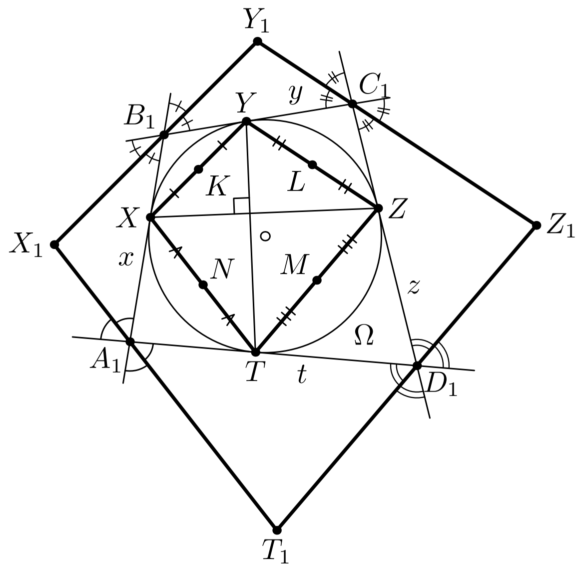

Fig. 2.

Proof. It is obvious that $X_1Y_1 \parallel XY$, $Y_1Z_1 \parallel YZ$, $Z_1T_1 \parallel ZT$, $T_1X_1 \parallel TX$. We will prove that $X_1Z_1 \parallel XZ$. Similarly, $Y_1T_1 \parallel YT$, and the statement of Lemma follows.

Let K, L, M, N be the midpoints of XY, YZ, ZT, TX, respectively. Note that TX is a polar line of A_1 with respect to Ω. Hence T_1X_1 is a polar line of N. Similarly, X_1Y_1 is a polar line of K. Therefore, X_1 is a pole of KN. This means that $OX_1 \perp KN \parallel YT$, where O is the center of Ω. Similarly, $OZ_1 \perp LM \parallel YT$. We get $X_1Z_1 \perp YT$, hence $X_1Z_1 \parallel XZ$. □

Lemma 2. *Let A, B be points on a circle Ω, let X and Y be the midpoints of arcs AB. Suppose that ω is a circle touching the segment AB at P, and touching the arc AYB at Q. Then P, Q, X are collinear, and the power of X with respect to ω equals XA^2.*

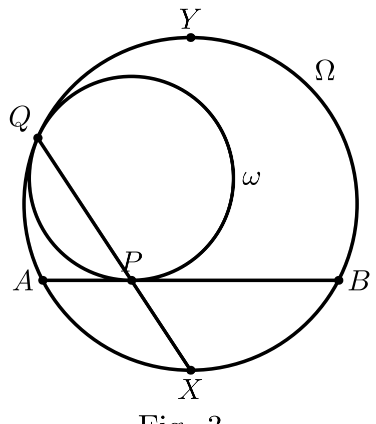

Fig. 3.

Proof. The homothety with center Q taking ω to Ω takes P to X (since the tangent to Ω at X is parallel to AB), hence Q, P, and X are collinear. Note that $\angle AQX = \angle ABX = \angle BAX$, hence triangles XAP and XQA are similar. Therefore, $XP \cdot XQ = XA^2$. □

Lemma 3. *In a projective plane, let $\mathcal{C}$ be a circle (a conic), let ℓ be a line, and let K_1, K_2, K_3, …, K_{2n-1} be points of ℓ. Consider families of $2n$ points X_1, X_2, …, $X_{2n} \in \mathcal{C}$ such that $K_i \in X_iX_{i+1}$, for all $i \in \{1, 2, \dots, 2n-1\}$. Then lines $X_{2n}X_1$ pass through a fixed point $K_{2n} \in \ell$.*

Proof. Since the conditions of Lemma are invariant to projective transformations, it is sufficient to consider the following case: $\mathcal{C}$ is a circle, and ℓ is the line at infinity. In this case given one family X_1, X_2, …, X_{2n} it is easy to obtain the description of all the possible families: for some φ, points X_1, X_3, …, X_{2n-1} could be rotated over the center of $\mathcal{C}$ by φ clockwise, while points X_2, X_4, …, X_{2n} rotated by φ counter clockwise. Now it is obvious that the direction of line $X_{2n}X_1$ is invariant, i. e., $X_{2n}X_1$ passes through a fixed point of the line at infinity. □

The following lemma is equivalent to problem 13 from 2002 IMO shortlist suggested by Bulgaria [2].

Lemma 4. *Let ω_1 and ω_2 be two non-intersecting circles with centers O_1 and O_2, respectively; let m, n be common external tangents, and let k be a common internal tangent of ω_1 and ω_2. Let $A = m \cap k$, $B = n \cap k$, $C = \omega_1 \cap k$. Suppose that the circle σ passes through A and B, and touches ω_1 at D, then D, C, and O_2 are collinear.*

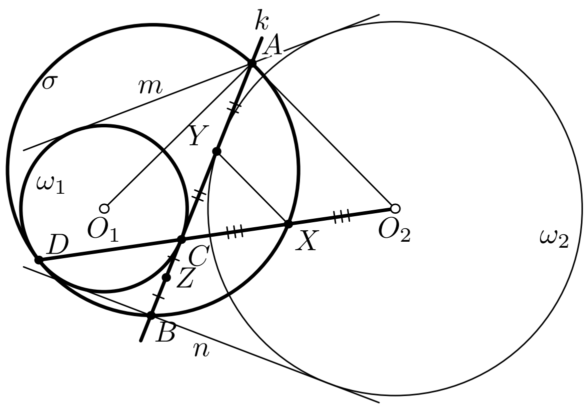

Fig. 4.

Proof. Let X, Y, and Z be the midpoints of CO_2, CA , and CB, respectively. Note that Y has equal powers with respect to ω_1 and A (here A is considered as a circle of radius 0), and $XY \parallel AO_2 \perp AO_1$. Hence XY is the radical axis of ω_1 and A. Similarly, XZ is the radical axis of ω_1 and B. Therefore, X is the radical center of ω_1, A, and B. We obtain that the power of X with respect to ω equals $AX^2 = BX^2$. Using the converse to the statement of Lemma 2 we obtain that X lies on σ (X is the midpoint of the arc AB of σ).

By Lemma 2, D, C, and X are collinear. From that it follows the statement of Lemma. □

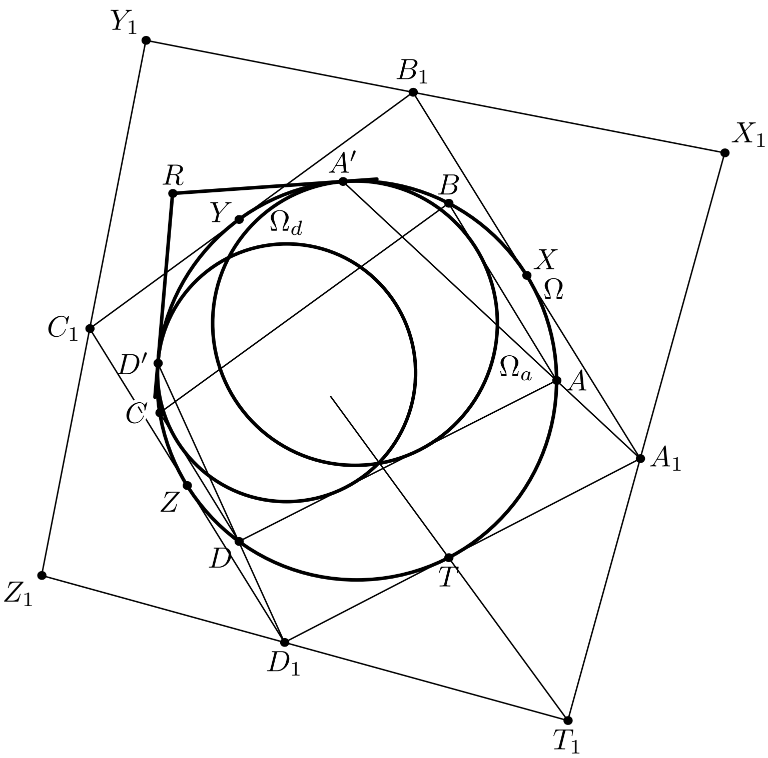

Fig. 5.

Now proceed to the proof of Theorem 1. Let X, Y, Z, T be midpoints of arcs AB, BC, CD, DA containing no other vertices of $ABCD$ (except the endpoints of arcs). Let A_1, B_1, C_1, D_1, X_1, Y_1, Z_1, T_1 be points defined in Lemma 1. Let us fix Ω and quadrilaterals $XYZT$, $X_1Y_1Z_1T_1$, $A_1B_1C_1D_1$; in this construction one can consider a family $\mathcal{F}$ of corresponding quadrilaterals $ABCD$ (starting with any $A \in \Omega$ one can obtain $B \in \Omega$ such that $AB \parallel A_1B_1$, then obtain $C \in \Omega$ such that $BC \parallel B_1C_1$, then obtain $D \in \Omega$ such that $CD \parallel C_1D_1$; hence it is easy to see that $DA \parallel D_1A_1$).

Quadrilateral $XYZT$ has perpendicular diagonals, so by Lemma 1, there exists the center S of homothety that takes $XYZT$ to $X_1Y_1Z_1T_1$. Thus S is a common point of lines XX_1, YY_1, ZZ_1, and TT_1. We will prove that TT_1 is the radical axis of Ω_d and Ω_a (and similarly, XX_1, YY_1, and ZZ_1 are radical axes for pairs Ω_a and Ω_b, Ω_b and Ω_c, Ω_c and Ω_d). From this it follows that S is the radical center of Ω_a, Ω_b, Ω_c, and Ω_d.

By Lemma 2, T has equal powers with respect to Ω_a and Ω_d.

Suppose that Ω_a and Ω_d touch Ω at A' and D', respectively. Tangents to Ω passing through A' and D' intersect at R that is the radical center of Ω, Ω_a, and Ω_d. Note that $A'D'$ is a polar line of R with respect to Ω.

Now it is sufficient to prove that $R \in TT_1$.

Considering homothety with center A' taking Ω_a to Ω we obtain that A_1, A, A' are collinear. Similarly, D_1, D, D' are collinear.

Applying Lemma 3 for $X_1 = A'$, $X_2 = A$, $X_3 = D$, and $X_4 = D'$ ($A'A$ and $D'D$ pass through A_1 and D_1, respectively; AD is parallel to A_1D_1), we obtain that for all $ABCD \in \mathcal{F}$ line $A'D'$ passes through a fixed point of A_1D_1 (or parallel to A_1D_1). Since T is the pole of A_1D_1, this means that R lies on a fixed line passing through T. Thus it is sufficient to prove that $R \in TT_1$ for some particular $ABCD \in \mathcal{F}$.

Now consider $ABCD \in \mathcal{F}$ such that $A' = D' = R$ (thus $\Omega_a = \Omega_d$). Let M be the touch point of Ω_a and AD; let N be the intersection point of external bisectors of angles BAD and CDA. By Lemma 4 (take $\omega_1 = \Omega_a$, $\sigma = \Omega$), we obtain that R, M, and N are collinear. Consider the homothety with center R that takes Ω_a to Ω. This homothety takes M to T, and N to T_1. This means that in the particular case $R \in TT_1$. This completes the proof.

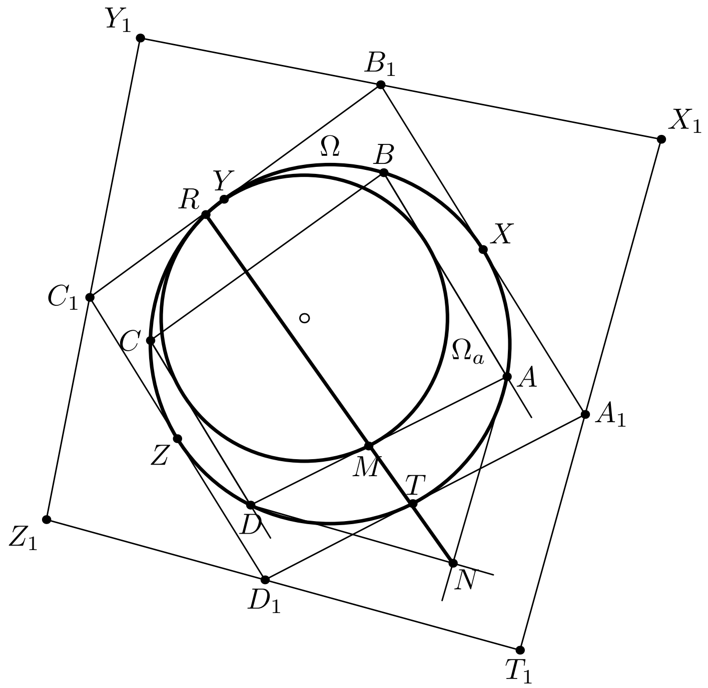

Fig. 6.

Remark. Note that the statement of Theorem 1 is equivalent to the following statement. There exists a circle ω, $\omega \neq \Omega$ such that ω touches each of the circles Ω_a, Ω_b, Ω_c, and Ω_d.

Acknowledgement. The author is grateful to P.A. Kozhevnikov for some remarks that helped to shorten the proof.

References

[1] N. Kh. Agakhanov, I. I. Bogdanov, P. A. Kozhevnikov, O. K. Podlipsky, and D. A. Tereshin. *All-russian mathematical olympiad 1993–2009.* M.:MCCME, 2007.

[2] D. Djukić, V. Janković, I. Matić, and N. Petrović. *The IMO compendium.* Springer, 2006.

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TWO APPLICATIONS OF A LEMMA ON INTERSECTING CIRCLES

VLADIMIR N. DUBROVSKY

Abstract. A useful property of the direct similitude that maps one of two intersecting circles on another and fixes their common point is applied to the configuration consisting of a triangle, its circumcircle, and a circle through its vertex and the feet of its two cevians.

This paper emerged from a discussion in *Hyacinthos* problem solving group at Yahoo started by Luis Lopes [3], which ended up with two theorems about the configuration consisting of a triangle, its circumcircle, and a circle through its vertex and the feet of its two cevians. The proofs of these theorems are given below preceded by a simple, but useful lemma on the direct similitude defined by two intersecting circles.

Lemma. *Let a and b be two intersecting circles and let P and Q be their common points. Then there is a unique direct similitude f with fixed point P that maps a to b, and for any X on a, the line XY, where $Y = f(X)$, passes through Q.*

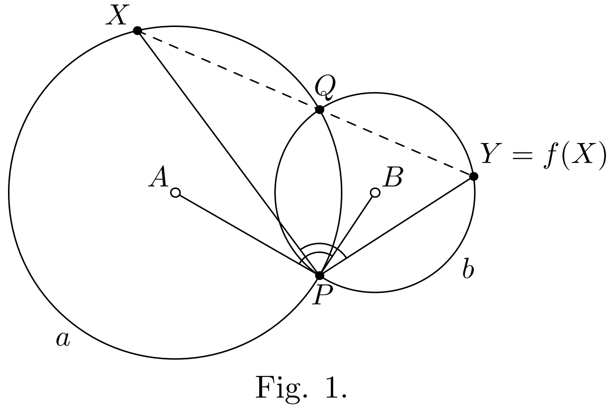

Fig. 1.

Proof. If A and B are the centers of the circles (Fig. 1), then, obviously, f can be represented as scaling by factor PB/PA with respect to P followed by the rotation through $\angle APB$ about P. Then all the triangles PXY are directly similar to triangle PAB, and therefore, the (signed) angle PXY is constant mod π; hence, by the Inscribed Angle Theorem, the second intersection point of XY and a is fixed. But for $X = f^{-1}(Q)$ this point is Q. □

Theorem 1. *Let AA_1, BB_1, and CC_1 be three cevians in a triangle ABC concurrent at P and let ω and α be the circumcircles of triangles ABC and AB_1C_1*

(Fig. 2). Denote by D the second intersection point of α and ω, and extend DA_1 to meet ω again at N. Then AN bisects B_1C_1.

Remark 1. Originally, this fact was reported by A. Zaslavsky [4] in the particular case where P is the triangle's orthocenter.

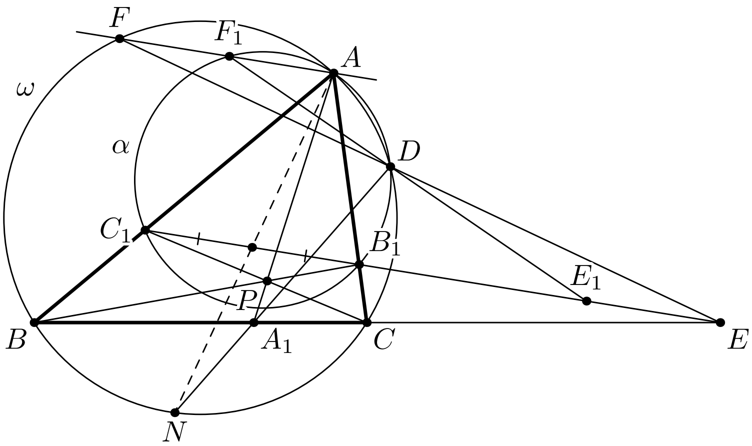

Fig. 2.

Proof. It will suffice to show that the line $n = AN$ is the harmonic conjugate, with respect to $b = AC$ and $c = AB$, of the line through A parallel to B_1C_1. Denote by E the meet of lines BC and B_1C_1 and by F, the second intersection of DE and ω. Since the lines DB, $DN = DA_1$, DC, and DE make a harmonic quadruple (because this is the fact for B, A_1, C, and E), the same is true for the lines AB, AN, AC, and AF. So it only remains to show that AF is parallel to B_1C_1. This can be done by means of the lemma given above.

Indeed, consider the direct similitude d that fixes D and takes ω to α. By the lemma, points B and C are taken by d to C_1 and B_1, respectively, F is taken to the second intersection point F_1 of AF and α, and E is taken to some point E_1. Since E is the meet of BC and FD, point E_1 is the meet of B_1C_1 and F_1D. By the definition of E_1 and F_1, triangles DFF_1 and DEE_1 are directly similar, hence the lines $FF_1(= AF)$ and $EE_1(= B_1C_1)$ are parallel. □

For P satisfying a special condition, we have an additional property of the same configuration.

Theorem 2. *In the setting of Theorem 1, assume that P is concyclic with A, B_1 and C_1. Then the line DP bisects the side BC.*

Proof. Let us draw a third circle γ, the circumcircle of triangle PBC (Fig. 3). We'll consider the composition m of the direct similitude d from the proof of Theorem 1 and another direct similitude q with center Q, the second intersection point of α and γ, which takes α to γ. By the Lemma, we have the following diagrams:

$$B \xrightarrow{d} C_1 \xrightarrow{q} C,\ \ C \xrightarrow{d} B_1 \xrightarrow{q} B,\ \ D \xrightarrow{d} D \xrightarrow{q} D_1,$$

where D_1 is the second intersection point of DP and γ. So m swaps B and C; hence, being a direct similitude, m is the reflection in the midpoint M of BC. It follows that the line $DP = DD_1$ passes through M. □

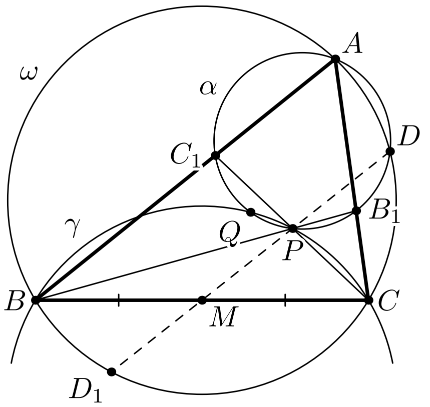

Fig. 3.

Remark 2. Obviously, the circle γ depends only on the triangle, not on point P, and is congruent to ω; it is known well that it passes through the orthocenter H. Thus, the point P in the theorem is, in fact, any point of the circumcircle of triangle BCH. In the original question posed by Lopes, P was just the orthocenter.

References

[1] V. N. Dubrovsky. Hyacinthos messages 20020, 20026, 20030, May 14, 16, 17 2011.
[2] F. Javier. Hyacinthos messages 20016, May 13 2011.
[3] L. Lopes. Hyacinthos messages 20014, May 14 2011.
[4] A. Zaslavsky. Hyacinthos messages 20025, May 16 2011.

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GEOMETRY OF KIEPERT AND GRINBERG–MYAKISHEV HYPERBOLAS

ALEXEY A. ZASLAVSKY

Abstract. A new synthetic proof of the following fact is given: if three points A', B', C' are the apices of isosceles directly-similar triangles BCA', CAB', ABC' erected on the sides BC, CA, AB of a triangle ABC, then the lines AA', BB', CC' concur. Also we prove some interesting properties of the Kiepert hyperbola which is the locus of concurrence points, and of the Grinberg–Myakishev hyperbola which is its generalization.

This paper is devoted to the following well-known construction.

A triangle ABC is given. Let A', B', C' be the apices of three isosceles directly-similar triangles BCA', CAB', ABC'. Then the lines AA', BB', CC' concur. When we change the angles of triangles BCA', CAB', ABC', the common point of these lines moves along an equilateral hyperbola isogonally conjugated to the line OL (where O and L are the circumcenter and the Lemoine point of $\triangle ABC$). It is called the Kiepert hyperbola.

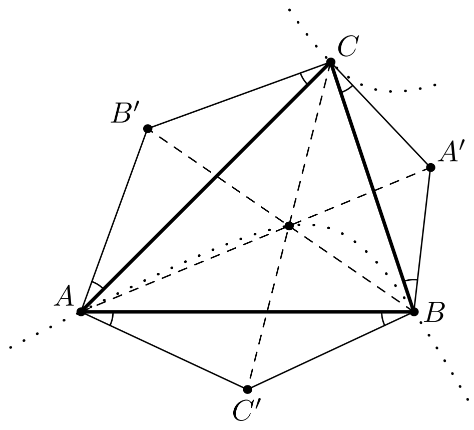

Fig. 1.

Usually this fact is proven using barycentric coordinates. A synthetic proof is based on the following simple but useful lemma.

Lemma 1. *Given two points A and B. Let f be a map sending lines passing through A to lines passing through B, and preserving the cross-ratio of the lines. Then the locus of points $l \cap f(l)$ (where l is a variable line through A) is a conic passing through A and B.*

Indeed, if X, Y, Z are three arbitrary points of this locus, then the lines l and $f(l)$ intersect the conic $ABXYZ$ in the same point (by the known fact that when

U, V, W, T are four fixed points on a conic and P is a variable point on the same conic, the cross-ratio of the lines PU, PV, PW, PT does not depend on P).

Consider now our initial construction. When we change the angles of the three isosceles triangles (keeping them directly similar), the points A' and B' move along the perpendicular bisectors of the segments BC and CA. Since the lines AB' and BA' rotate with equal velocities, the corresponding map between these perpendicular bisectors is projective. Thus, the map between the lines AA' and BB' also is projective. Using the Lemma, we obtain that the common point X of these lines moves along a conic passing through A and B. If the angle on the base of the three isosceles triangles is zero, X is the centroid M of $\triangle ABC$. If this angle is $\pi/2$, the point X becomes the orthocenter H. Also, if the angle on the base is equal to the angle $-C$, then X coincides with C. Therefore, the obtained conic is the hyperbola $ABCMH$. This hyperbola is equilateral because it passes through the vertices and the orthocenter of triangle ABC. The common point of the lines AA' and CC' moves on the same hyperbola (by a similar argument) and thus coincides with X. This proves that the lines AA', BB', CC' concur, and their point of concurrency lies on the equilateral hyperbola $ABCMH$. This hyperbola is clearly the isogonal conjugate of the line OL (since M and H are isogonally conjugate to L and O), and is known as the *Kiepert hyperbola* of triangle ABC.

Now consider some properties of the Kiepert hyperbola.

Denote by $X(\phi)$, $-\pi/2 \le \phi \le \pi/2$ the point on the hyperbola corresponding to isosceles triangles with base angle ϕ (where we say that $\phi > 0$ (respectively, $\phi < 0$) when the isosceles triangles are constructed on the external (respectively, internal) side of $\triangle ABC$). By the above, $X(0) = M$ and $X(\pm\pi/2) = H$. Here come some other examples.

$X(\pm\pi/3) = T_{1,2}$ are the Fermat-Torricelli points. Note that their isogonally conjugated Appollonius points (also known as the isodynamic points) are inverse to each other wrt the circumcircle of $\triangle ABC$. Thus the Torricelli points are two opposite points of the Kiepert hyperbola, i.e. the midpoint of the segment T_1T_2 is the center of the hyperbola and therefore lies on the Euler circle of $\triangle ABC$. (The Euler circle is also known as the nine-point circle.)

$X(\pm\pi/6) = N_{1,2}$. These points are called the *Napoleon points*.

The following properties of Torricelli and Napoleon points are well-known:

– the lines T_1T_2 and N_1N_2 pass through L;

– the lines T_1N_1 and T_2N_2 pass through O;

– the lines T_1N_2 and N_1T_2 pass through the center O_9 of the Euler circle.

Using these properties we obtain the following theorem.

Theorem 2. *For an arbitrary* $\phi \in [-\pi/2, \pi/2]$,

– the line $X(\phi)X(-\phi)$ *passes through* L*;*

– the line $X(\phi)X(\pi/2-\phi)$ *passes through* O*;*

– the line $X(\phi)X(\pi/2+\phi)$ *passes through* O_9*.*

Proof. We will prove only the first assertion of the theorem – the other two can be proven similarly. By projecting from C, we obtain a projective transformation from the Kiepert hyperbola to the perpendicular bisector of AB. This transformation transforms the points $X(\phi)$ and $X(-\phi)$ to two points symmetric with respect to the line AB. Thus the map from the hyperbola to itself which sends every $X(\phi)$ to $X(-\phi)$ is projective. Consider now the map transforming a point X to the second common point of the hyperbola with the line XL. Since these two projective maps coincide in the points T_1, T_2, N_1, and N_2 they coincide in all points of the Kiepert hyperbola. □

Remark. The properties of the Torricelli and Napoleon points we used above can be avoided if so desired. All we needed for the proof of Theorem 2 were three distinct angles ϕ for which the statement of Theorem 2 is known to hold. Using the properties of the Torricelli and Napoleon points, we found four such angles ($\pi/3$, $\pi/6$, $-\pi/3$, and $-\pi/6$), but there are three angles which can be easily seen to satisfy Theorem 2: namely, $-A$, $-B$, $-C$. (These are distinct only if $\triangle ABC$ is scalene – otherwise, a limiting argument will do the trick.) Why do these angles satisfy Theorem 2? Notice first that $X(-A) = A$. To prove that the line $X(-A)X(A)$ passes through L, show that for $\phi = A$ the apex A' is a vertex of the triangle formed by the tangents to the circumcircle of $\triangle ABC$ at A, B and C, and conclude that AX is a symmedian of $\triangle ABC$. To prove that the line $X(-A)X(\pi/2 + A)$ passes through O, notice that for $\phi = \pi/2 + A$ the apex A' is O. To prove that the line $X(-A)X(\pi/2 - A)$ passes through O_9, check that for $\phi = \pi/2 - A$ the apex A' is the reflection of A in O_9.

As a special case of Theorem 2 we obtain that the tangents to the Kiepert hyperbola in the points M and H meet in the point L. The point L thus is the pole of the line HM with respect to the Kiepert hyperbola. Therefore L is the center of the inscribed conic touching the sidelines of the triangle in the feet of its altitudes. (Here we are applying the known fact (see Theorem 4.8 in [1]) that if a conic touches the sidelines of $\triangle ABC$ at the feet of the cevians from a point P, then the center of the conic is the pole of the line PM with respect to the conic.)

Two first assertions of Theorem 2 can be generalized.

Theorem 3. *Consider the pairs of points $X(\phi_1)$, $X(\phi_2)$, where ϕ_1 and ϕ_2 are variable angles satisfying $\phi_1 + \phi_2 = \text{const}$. All such lines $X(\phi_1)X(\phi_2)$ meet the line OL at the same point.*

Proof. Denote the sum $\phi_1 + \phi_2$ by $2\phi_0$. The quadrilateral

$$X(\phi_1)X(\phi_0)X(\phi_2)X(\pi/2 + \phi_0)$$

is harmonic (this can be seen, for instance, by using the projective transformation from the Kiepert hyperbola to the perpendicular bisector of AB – in fact, the quadrilateral $X(\phi_1)X(\phi_0)X(\phi_2)X(\pi/2 + \phi_0)$ is, under this transformation, a preimage of a "quadrilateral" which is more easily seen to be harmonic, and projective transformations preserve harmonicity). Therefore, the line $X(\phi_1)X(\phi_2)$ passes through the pole of the line $X(\phi_0)X(\pi/2 + \phi_0)$ wrt the Kiepert hyperbola.

But this line, for every ϕ_0, passes through O_9 (by Theorem 2). Thus all lines $X(\phi_1)X(\phi_2)$ pass through some fixed point of the polar of O_9 which coincides with OL by Theorem 2. □

Since the Kiepert hyperbola is isogonally conjugated to the line OL, the point obtained in Theorem 3 is the point $X'(\phi_3)$, isogonally conjugated to some point $X(\phi_3)$ of the hyperbola. The relation between ϕ_3 and ϕ_1, ϕ_2 can be obtained using two isogonal pairs theorem.

Let $X(\phi_1)$, $X(\phi_2)$ be two points of the Kiepert hyperbola and $X'(\phi_1)$, $X'(\phi_2)$ be their respective isogonal conjugates on the line OL. By the two isogonal pairs theorem (Corollary from Theorem 3.16 in [1]), the lines $X(\phi_1)X'(\phi_2)$ and $X'(\phi_1)X(\phi_2)$ meet on the hyperbola. By Theorem 3 the corresponding angle is equal to $f(\phi_1)-\phi_2$. On the other hand it is equal to $f(\phi_2)-\phi_1$, where f is some unknown function[1]. Therefore $f(\phi_1)+\phi_1 = f(\phi_2)+\phi_2 = \text{const}$. Taking $\phi_2 = 0$ we obtain $f(\phi) = -\phi$. Thus we can formulate the following Theorem:

Theorem 4. *The three points $X(\alpha)$, $X(\beta)$, $X'(\gamma)$ are collinear iff $\alpha+\beta+\gamma$ is an integer multiple of π.*

An interesting partial case is obtained when $X(\alpha)$ and $X(\beta)$ are the two infinite points of the hyperbola. Then $X(\gamma)$ is isogonally conjugate to the infinite point of the line OL, i.e. it coincides with the fourth common point of the Kiepert hyperbola and the circumcircle. Thus the sum of the angles corresponding to this point and the two infinite points is an integer multiple of π.

Finally note that triangle $A'B'C'$ is not only perspective to ABC, but orthologic to it with orthology center O. The locus of second orthology centers is also the Kiepert hyperbola and this fact can be formulated elementarily.

Theorem 5. *Let $AB'C$ and $CA'B$ be similar isosceles triangles with angles on the bases CA and BC equal to ϕ. Let the perpendicular from C to $A'B'$ meet the perpendicular bisector of AB in point C_1. Then $\angle AC_1B = 2\phi$ and $\angle C_1BA = \angle BAC_1 = \pi/2-\phi$.*

Using this Theorem 5, we immediately see that the point $X(\pi/2-\phi)$ is the orthology center.

Proof. Let A'', B'' be the images of A', B' under the homothety with center C and coefficient 2. Then $\angle CAB'' = \angle CBA'' = \pi/2$ and $A''B'' \parallel A'B'$. Let C_2 be the common point of CC_1 and $A''B''$. Since the quadrilateral CC_2BA'' is inscribed into the circle with diameter CA'', we have $\angle BC_2A'' = \angle BCA'' = \phi$. Similarly $\angle AC_2B'' = \phi$. Thus $A''B''$ is the exterior bisector of angle AC_2B. By orthogonality, C_2C_1 must thus be the bisector of angle AC_2B, and its common point with the perpendicular bisector of AB must lie on the circumcircle of ABC_2

[1]Let $X'(\phi)$ be an arbitrary point of the line OL and $X(\phi_1)$, $X(\phi_2)$ be two common points of the Kiepert hyperbola and some line l passing through $X'(\phi)$. Then by Theorem 3 $\phi_1+\phi_2$ depends only on ϕ and not on l. We denote the corresponding function by $f(\phi)$. By Theorem 2 $f(0)=0$.

(Fig. 2). Therefore $\angle AC_1B = \angle AC_2B = 2\phi$ and $\angle C_1BA = \angle BAC_1 = \pi/2 - \phi$. □

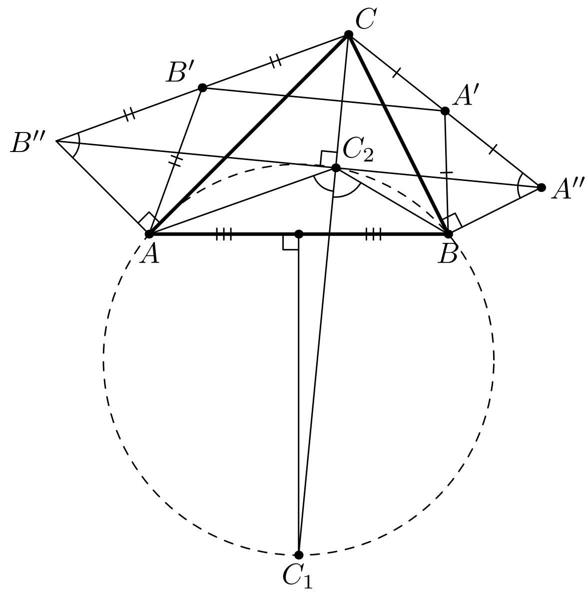

Fig. 2.

D. Grinberg and A. Myakishev [2] found the following generalization of the Kiepert hyperbola.

Theorem 6. *Let $A_1B_1C_1$ be the cevian triangle of a point P with respect to $\triangle ABC$. Let AC_aC_1, C_1C_bB, BA_bA_1, A_1A_cC, CB_cB_1, B_1B_aA be isosceles directly-similar triangles with bases AC_1, C_1B, BA_1, A_1C, CB_1, B_1A. Then, the triangle formed by the lines A_bA_c, B_cB_a, C_aC_b is perspective to $\triangle ABC$.*

If the point P is fixed and the angle ϕ on the bases of isosceles triangles changes, then the perspectivity center moves along some circumconic of $\triangle ABC$.

The authors proposed only an analytic proof of this assertion. The following synthetic proof was found recently.

Proof of Theorem 6. First note that the perspectivity of triangles follows from the Desargues theorem. In fact the ratio of distances from the points C_a and C_b to the line AB is equal to AC_1/BC_1 and does not depend on ϕ. Thus the common point C_2 of the lines AB and C_aC_b also does not depend on ϕ. Defining similarly points A_2 and B_2, we obtain from Ceva and Menelaos theorems that A_2, B_2, C_2 are collinear if the lines AA_1, BB_1, and CC_1 concur (in fact, homothetic triangles yield $\frac{BA_1}{A_1C} = \frac{BA_b}{A_1A_c} = \frac{BA_2}{A_1A_2}$ and similarly $\frac{CA_1}{A_1B} = \frac{CA_2}{A_1A_2}$; dividing these two equalities by each other results in $\left(\frac{BA_1}{A_1C}\right)^2 = \frac{BA_2}{CA_2}$, and similar equalities can be found by cyclic shifting). Therefore it is sufficient to prove that the common point of the lines AA' and BB' moves along some conic, where A', B', C' are the vertices of the triangle formed by A_bA_c, B_cB_a, C_aC_b.

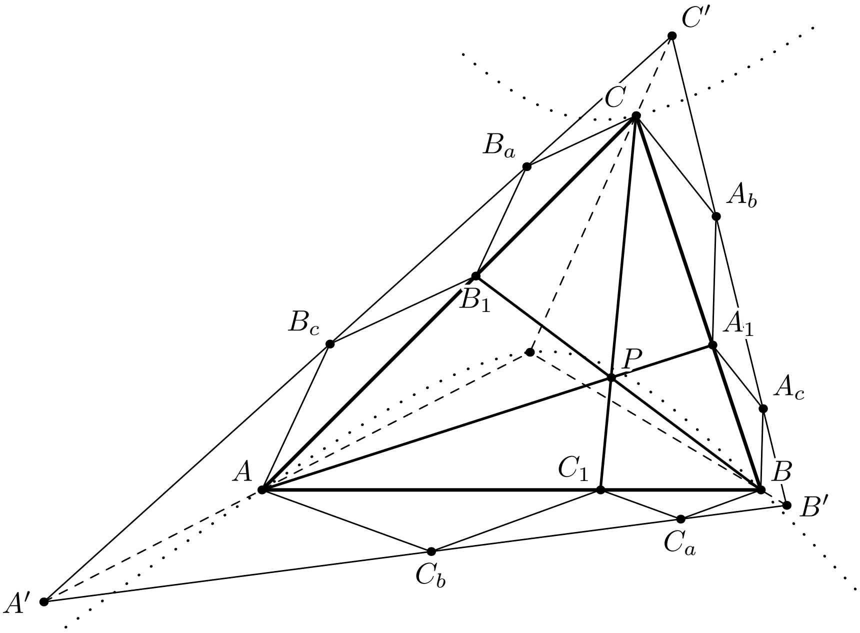

Fig. 3.

Let us find the trajectory of the point A'. This point lies on the lines B_cB_a and C_aC_b passing through the fixed points B_2 and C_2, respectively. These lines also pass through the points B_a and C_a moving on the perpendicular bisectors of the segments AB_1 and AC_1. It is evident that the obtained map between these bisectors is projective, thus by Lemma 1 the point A' moves along some conic α passing through B_2 and C_2. This conic also passes through A, because when $\phi = 0$ the points A and A' coincide. Similarly the point B' moves along a conic β passing through A_2, C_2 and B.

Now, note that the map between A' and B_a is the projection of α from C_2 to the perpendicular bisector of AB_1. Thus this map is projective. Similarly the map between B' and A_b is projective. Therefore the map between A' and B' is a projective map between the conics α and β. Since A lies on α and B lies on β, the map between the lines AA' and BB' is also projective. Using Lemma 1 we obtain that the common point of these lines moves on some conic passing through A and B. Similarly we obtain that this conic passes through C.

$\square$

Grinberg and Myakishev proved by calculations that this conic always is a hyperbola. They also obtained some results concerning the dependence of this hyperbola on P. The main part of their results is very complicated. Therefore we consider only one interesting partial case.

Statement 7. *If P is the orthocenter of $\triangle ABC$, then the Kiepert and Grinberg–Myakishev hyperbolas coincide.*

Proof. It is sufficient to find two common points of these hyperbolas distinct from A, B, C. We are going to prove that $Y\left(\frac{\pi}{4}\right) = X\left(-\frac{\pi}{4}\right)$ and $Y\left(-\frac{\pi}{4}\right) = X\left(\frac{\pi}{4}\right)$, where $Y(\phi)$ is the point of the Grinberg–Myakishev hyperbola corresponding to the angle ϕ.

Let AA_1, BB_1 be the altitudes of $\triangle ABC$, and let AB_aB_1, B_1B_cC, CA_cA_1, A_1A_bB be isosceles right-angled triangles with apices lying on the external side of $\triangle ABC$; let AC_0B be an isosceles right-angled triangle with apex lying on the internal side of $\triangle ABC$. We have to prove that the lines B_cB_a, A_bA_c and CC_0 concur.

Since the points A_1 and C_0 lie on the circle with diameter AB, $\angle CA_1C_0 = \angle BAC_0 = \frac{\pi}{4}$. Thus, the points A_b, A_1, C_0 are collinear and $C_0A_b \parallel CA_c$. Similarly $C_0B_a \parallel CB_c$ (Fig. 4). Also it is clear that

$$\frac{C_0A_b}{C_0B_a} = \frac{\sin\angle C_0BA_b}{\sin\angle C_0AB_a} = \frac{\sin B}{\sin A} = \frac{CA_1}{CB_1} = \frac{CA_c}{CB_c}.$$

Therefore triangles CA_cB_c and $C_0A_bB_a$ are homothetic, which yields $Y\left(\frac{\pi}{4}\right) = X\left(-\frac{\pi}{4}\right)$. Similarly we obtain the second equality. □

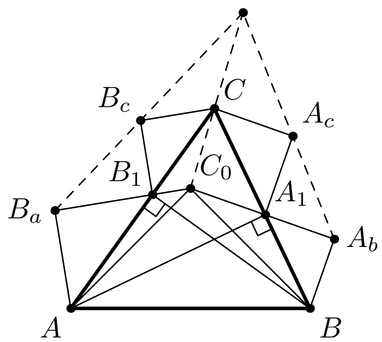

Fig. 4.

Acknowledgements

The author is grateful to Darij Grinberg for editing of this paper and helpful discussions.

References

[1] A. V. Akopyan and A. A. Zaslavsky. *Geometry of conics*, volume 26 of *Mathematical World*. American Mathematical Society, Providence, RI, 2007.

[2] D. Grinberg and A. Myakishev. A generalization of the Kiepert hyperbola. In *Forum Geometricorum*, volume 4, pages 253–260, 2004.

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PROBLEM SECTION

In this section we suggest to solve and discuss problems provided by readers of the journal. The authors of the problems do not have purely geometric proofs. We hope that interesting proofs will be found by readers and will be published. Please send us solutions by email: `editor@jcgeometry.org`, as well as interesting "unsolved" problems for publishing in this Problems Section.

Alexey Zaslavsky, **Brocard's points in quadraleteral.**

Given convex quadrilateral $ABCD$. It is easy to prove that there exists a unique point P such that $\angle PAB = \angle PBC = \angle PCD$. We will call this point *Brocard point* ($Br(ABCD)$) and the respective angle *Brocard angle* ($\phi(ABCD)$) of broken line $ABCD$. Note some properties of Brocard's points and angles:

– $\phi(ABCD) = \phi(DCBA)$ iff $ABCD$ is cyclic;

– if $ABCD$ is harmonic then $\phi(ABCD) = \phi(BCDA)$. Thus there exist two points P, Q such that $\angle PAB = \angle PBC = \angle PCD = \angle PDA = \angle QBA = \angle QCB = \angle QDC = \angle QAD$. These points lye on the circle with diameter OL where O is the circumcenter of $ABCD$, L is the common point of its diagonals and $\angle POL = \angle QOL = \phi(ABCD)$.

Now the problem.

Open Problem. *Let $ABCD$ be a cyclic quadrilateral, $P_1 = Br(ABCD)$, $P_2 = Br(BCDA)$, $P_3 = Br(CDAB)$, $P_4 = Br(DABC)$, $Q_1 = Br(DCBA)$, $Q_2 = Br(ADCB)$, $Q_3 = Br(BADC)$, $Q_4 = Br(CBAD)$. Then $S_{P_1P_2P_3P_4} = S_{Q_1Q_2Q_3Q_4}$.*

This result is obtained by computer and isn't proved.

Lev Emelyanov, **Nagel axis.**

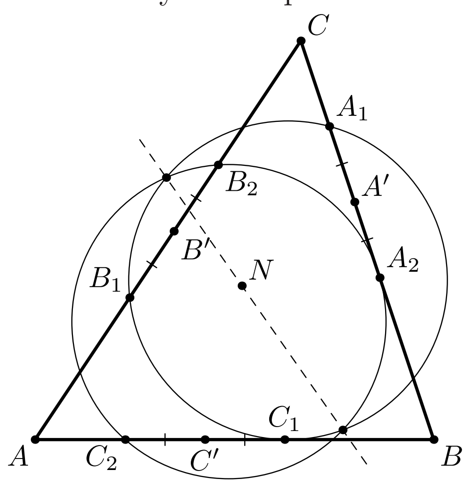
Let N be the Nagel point of a triangle ABC. Let A', B' and C' be the touch points of excircles with the sides BC, CA, AB. Let us consider six points: A_1 and A_2 on the BC, B_1 and B_2 on the CA, C_1 and C_2 on the AB, such that

$$A_1A' = A'A_2 = B_1B' = B'B_2 = C_1C' = C'C_2.$$

Points A_1, B_1, C_1 lie on the rays $A'C$, $B'A$, $C'B$ and A_2, C_2, B_2 lie on the rays $A'B$, $C'A$, $B'C$ respectively.

Then radical axis circumcircles $A_1B_1C_1$ and $A_2B_2C_2$ passes through the Nagel point N, centroid and incenter ABC.

The author does not know the synthetic proof of this fact.



Arseniy Akopyan, **Rotation of isogonal point.**

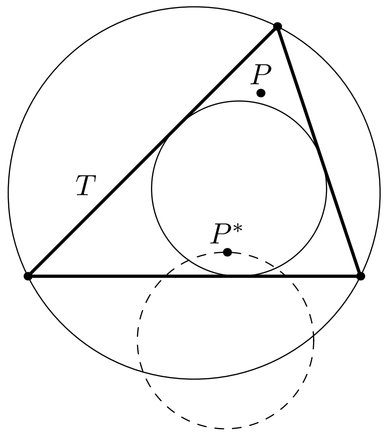
For formulation of the statement of this problem let us recall one corollary of Poncelet's theorem:

Let ω and Ω be inscribed and circumscribed circles of a triangle. Then for any point A on Ω there exists a triangle T with vertex at A inscribed in Ω and circumscribed around ω.

The rotation of the triangle T with the point A we call *Poncelet's rotation.*

The following theorem was found by the author.

Theorem. *Let T be a Poncelet triangle rotated between two circles and P be an any point. Then locus of points P^* isogonal conjugated to P with respect to T is a circle.*



First factful proof was found by François Rideau in [1]. His proof is quite short but not synthetic and use complex number technics. Using his ideas it is not hard to show that the inscribed circle could be substitute by an ellipse.

The synthetic proof of this theorem is not known.

References

[1] F. Rideau. Hyacinthos message 11142, May 29 2005.

GEOMETRICAL OLYMPIAD IN HONOR OF I. F. SHARYGIN

PREPARED BY A. A. ZASLAVSKY

Below is the list of problems for the first (correspondence) round of the VIII Sharygin Geometrical Olympiad and selected problems of the VII Olympiad.

The olympiad is intended for high-school students of 8–11 grades (these are four elder grades in Russian school). In the list below each problem is indicated by the numbers of school grades, for which it is intended. However, the participants are encouraged to solve problems for elder grades as well (solutions for younger grades will not be considered).

Your work containing solutions for the problems, written in Russian or in English, should be sent no later than by April 1, 2012, by e-mail to *geomolymp@mccme.ru* in `pdf`, `doc` or `jpg` files.

Winners of the correspondence round will be invited to take part in the final round to be held in Dubna town (near Moscow, Russia) in Summer 2012 .

More about Sharygin Olympiad see on www.geometry.ru.

Correspondence round of VIII Olympiad

1. *(8)* In triangle ABC a point M is the midpoint of the side AB, and a point D is the foot of altitude CD. Prove that $\angle A = 2\angle B$ if and only if $AC = 2MD$.

2. *(8)* A cyclic n-gon is divided by non-intersecting (inside the n-gon) diagonals to $n-2$ triangles. Each of these triangles is similar to at least one of the remaining ones.

For what n is this possible?

3. *(8)* A circle with center I touches sides AB, BC, CA of a triangle ABC at points C_1, A_1, B_1. Lines AI, CI, B_1I meet A_1C_1 in points X, Y, Z respectively. Prove that $\angle YB_1Z = \angle XB_1Z$

4. *(8)* Given triangle ABC. Point M is the midpoint of the side BC, and point P is the projection of B to the perpendicular bisector of segment AC. Line PM meets AB at a point Q. Prove that the triangle QPB is isosceles.

5. *(8)* Let D be an arbitrary point on the side AC of a triangle ABC. The tangent in D to the circumcircle of triangle BDC meets AB at point C_1; point A_1 is defined similarly. Prove that $A_1C_1||AC$.

6. *(8–9)* Point C_1 of hypotenuse AC of a right triangle ABC is such that $BC = CC_1$. Point C_2 on the cathetus AB is such that $AC_2 = AC_1$; point A_2 is defined similarly. Find angle AMC, where M is the midpoint of A_2C_2.

7. *(8–9)* In a non-isosceles triangle ABC the bisectors of angles A and B are inversely proportional to the respective side lengths. Find angle C.

8. *(8–9)* Let BM be the median of a right triangle ABC ($\angle B = 90^\circ$). The incircle of the triangle ABM touches sides AB, AM in points A_1, A_2; points C_1, C_2 are defined similarly. Prove that the lines A_1A_2 and C_1C_2 meet on the bisector of angle ABC.

9. *(8–9)* In triangle ABC, given lines l_b and l_c containing the bisectors of angles B and C, and the foot L_1 of the bisector of angle A. Reconstruct triangle ABC.

10. In a convex quadrilateral all side lengths and all angles are pairwise different.

a) *(8–9)* Can the largest angle be adjacent to the largest side and at the same time the smallest angle be adjacent to the smallest side?

b) *(9–11)* Can the largest angle be non-adjacent to the smallest side and at the same time the smallest angle be non-adjacent to the largest side?

11. Given triangle ABC and point P. Points A', B', C' are the projections of P to BC, CA, AB respectively. A line passing through P and parallel to AB meets the circumcircle of triangle $PA'B'$ for the second time at point C_1. Points A_1, B_1 are defined similarly. Prove that

a) *(8-10)* lines AA_1, BB_1, CC_1 concur;

b) *(9–11)* triangles ABC and $A_1B_1C_1$ are similar.

12. *(9–10)* Let O be the circumcenter of an acute-angled triangle ABC. A line passing through O and parallel to BC meets AB and AC at points P and Q respectively. The sum of distances from O to AB and to AC is equal to OA. Prove that $PB + QC = PQ$.

13. *(9–10)* Points A, B are given. Find the locus of points C such that C, the midpoints of AC, BC and the centroid of triangle ABC are concyclic.

14. *(9–10)* For a convex quadrilateral $ABCD$, suppose $AC \cap BD = O$ and M is the midpoint of BC. Let $MO \cap AD = E$. Prove that $\frac{AE}{ED} = \frac{S_{\triangle ABO}}{S_{\triangle CDO}}$.

15. *(9–11)* Given triangle ABC. Consider lines l with the following property: the reflections of l in the sidelines of the triangle concur. Prove that all these lines have a common point.

16. *(9–11)* Given right triangle ABC with hypotenuse AB. Let M be the midpoint of AB and O be the center of the circumcircle ω of triangle CMB. Line AC meets ω for the second time in point K. Segment KO meets the circumcircle of triangle ABC in point L. Prove that segments AL and KM meet on the circumcircle of triangle ACM.

17. *(9–11)* A square $ABCD$ is inscribed into a circle. Point M lies on arc BC, AM meets BD at point P, DM meets AC at point Q. Prove that the area of quadrilateral $APQD$ is equal to the half of the area of the square.

18. *(9–11)* A triangle and two points inside it are marked. It is known that one of the triangle's angles is equal to 58°, one of the two remaining angles is equal to 59°, one of the two given points is the incenter of the triangle and the second one is its circumcenter. Using only a non-calibrated ruler determine the locations of the angles and of the centers.

19. *(10–11)* Two circles of radii 1 meet at points X, Y, and the distance between these points is also 1. Point C lies on the first circle, and lines CA, CB are tangents to the second one. These tangents meet the first circle for the second time at points B', A'. Lines AA' and BB' meet at point Z. Find angle XZY.

20. *(10–11)* Point D lies on the side AB of triangle ABC. Let ω_1 and Ω_1, ω_2 and Ω_2 be the incircles and the excircles (touching segment AB) of the triangles ACD and BCD respectively. Prove that the common external tangent to ω_1 and to ω_2, and the common external tangent to Ω_1 and to Ω_2 meet on AB.

21. *(10–11)* Two perpendicular lines pass through the orthocenter of an acute-angled triangle. The sidelines of the triangle cut two segments on each of these lines: one lying inside the triangle and another one lying outside it. Prove that the product of the two internal segments is equal to the product of the two external segments.

22. *(10–11)* A circle ω centered at I is inscribed into a segment of the disk, formed by an arc and a chord AB. Point M is the midpoint of that arc AB, and point N is the midpoint of the complementary arc. The tangents from N touch ω in points C and D. The opposite sidelines AC and BD of quadrilateral $ABCD$ meet at point X, and the diagonals of $ABCD$ meet at point Y. Prove that points X, Y, I and M are collinear.

23. *(10–11)* An arbitrary point is selected on each of twelve diagonals of the faces of a cube. The centroid of these twelve points is determined. Find the locus of all those centroids.

24. *(10–11)* Given are n $(n > 2)$ points on the plane. Assume that there are no three collinear points among them. In how many ways this set of n points can be divided into two non-empty subsets with non-intersecting convex envelopes?

Selected problems of VII Olympiad

1. *(B. Frenkin, correspondence round, 8)* Given triangle ABC. The perpendicular bisector of side AB meets one of the remaining sides at a point C'. Points A' and B' are defined similarly. For which triangles ABC the triangle $A'B'C'$ is regular?

Answer. For regular triangles and for triangles with angles equal to 30, 30 and 120 grades.

Solution. Consider a non-regular triangle ABC. Let AB be its largest side. Then points A', B' lie on segment AB. From the assumption we conclude that $C'C_0$, where C_0 is the midpoint of AB, is the bisector of the segment $A'B'$. Thus, $CA' = A'B = AB' = CB'$, i.e. C' coincides with C, and the triangle ABC is isosceles. Also we have $2\angle A = \angle A + \angle CAB' = \angle CB'B = 60°$, so $\angle A = \angle B = 30°$.

2. *(A. Akopyan, correspondence round, 8)* Two unit circles ω_1 and ω_2 intersect at points A and B. M is an arbitrary point of ω_1, N is an arbitrary point of ω_2. Two unit circles ω_3 and ω_4 pass through both points M and N. Let C be the second common point of ω_1 and ω_3, and D be the second common point of ω_2 and ω_4. Prove that $ACBD$ is a parallelogram.

Solution. Let O_i be the center of circle ω_i. From the assumption it follows that O_1AO_2B, O_1CO_3M, O_3MO_4N, O_4NO_2D are rhombuses with sides equal to 1. Then $\overrightarrow{O_1C} = \overrightarrow{MO_3} = \overrightarrow{O_4N} = \overrightarrow{DO_2}$ and $\overrightarrow{O_1A} = \overrightarrow{BO_2}$. Thus $\overrightarrow{AC} = \overrightarrow{DB}$, q.e.d.

3. *(D. Shvetsov, correspondence round, 8–10)* The excircle of an equilateral triangle ABC ($\angle B = 90°$) touches the side BC at point A_1 and touches the line AC at point A_2. Line A_1A_2 meets the incircle of ABC for the first time at point A'; point C' is defined similarly. Prove that $AC||A'C'$.

Solution. Let I be the incenter and PQ be the diameter of the incircle parallel to AC (Fig. 1). Since $\angle PIC = \angle ACI = \angle BCI$ and $CA_1 = (AB+BC-AC)/2 = r = IP$, the quadrilateral IPA_1C is an isosceles trapezoid. Then the line A_1P is parallel to IC, i.e it coincides with A_1A_2. So P coincides with A', and similarly Q coincides with C'.

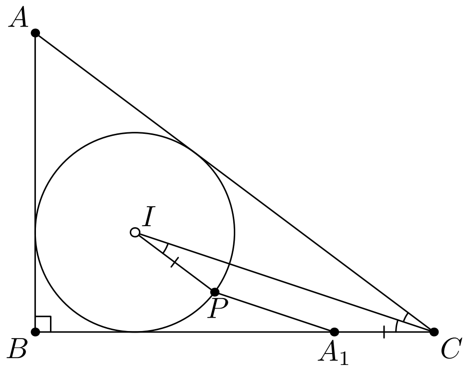


Fig. 1.

4. *(P. Dolgirev, correspondence round, 9–10)* Given are triangle ABC and a line ℓ. The reflections of ℓ with respect to AB and and with respect to AC meet at a point A_1. Points B_1, C_1 are defined similarly. Prove that

a) lines AA_1, BB_1, CC_1 concur;

b) their common point lies on the circumcircle of ABC;

c) two points constructed in this way for two perpendicular lines are opposite.

Solution. At first, note that if ℓ moves parallel with constant velocity, then the reflections of ℓ with respect to AC and BC also move with constant velocities. Therefore, C_1 moves along the line passing through C. This means that the common point of CC_1 with the circumcircle depends only on the direction of l. Now let A', B' be the common points of ℓ with BC and AC respectively (Fig. 2). Then $\angle C_1B'C = \angle CB'A'$, $\angle C'AC = \angle BA'C_1$. Thus, C is the incenter or the excenter of the triangle $A'B'C_1$, i.e. C_1C bisects the angle $A'C_1B'$ or the adjacent angle. However, the angle between lines $A'C_1$ and $B'C_1$ does not depend on l, hence the angle between CC_1 and C_1A' also does not depend on ℓ. So, when ℓ rotates, the lines AA_1, BB_1, CC_1 rotate with the same velocity. This proves all the assertions.

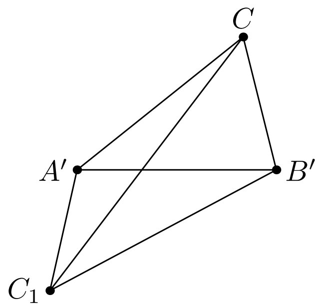


Fig. 2.

5. *(B. Frenkin, correspondence round, 9–11)* a) Does there exist a triangle, whose shortest median is longer that the longest bisector?

b) Does there exist a triangle in which the shortest bisector is longer that the longest altitude?

Solution. a) No, it does not. Let the lengths of sides BC, AC, AB be equal to a, b, c respectively and $a \le b \le c$. Let also CM be the median, AL be the bisector. If the angle C is not acute, then $AL > AC$. Since $BC \le AC$ and $\angle CMA$ is not acute, we see rhat $CM \le AC$ and $CM < AL$.

Now let $\angle C$ be acute. Since AB is the largest side, $\angle C \ge 60^\circ$ and the angles A, B are acute. Then the base H of altitude AH lies on the segment BC. Thus, AH (and AL) is at least $AC \cos\ 60^\circ = b\sqrt{3}/2$. But the square of CM is equal to $\frac{2a^2+2b^2-c^2}{4} \le \frac{2a^2+b^2}{4} \le \frac{3b^2}{4}$. So CM is not greater than $b\sqrt{3}/2$ and can not exceed the bisector of angle A.

b) No, it does not. Let $a \le b \le c$ and l be the bisector of angle C. Then $(al + bl)\sin\frac{C}{2} = 2S_{ABC} = ab\sin C$, i.e. $l = \frac{2ab\cos\frac{C}{2}}{a+b}$. On the other hand, the altitude from A is equal to $h = b\sin C$. Since $a + b \ge 2a$ and $C \ge 60^\circ$, we have $h/l = (a+b)\sin\frac{C}{2}/a \ge 1$.

Remark. It is easy to construct a triangle such that its shortest median is longer than its longest altitude.

6. *(A. Zaslavsky, correspondence round, 9–11)* Given n straight lines in general position on the plane (every two of them are not parallel and every three of them do not concur). These lines divide the plane into several parts. What is

a) the minimal;

b) the maximal

number of these parts that can be angles?

Solution. a) **Answer.** 3. Consider the convex envelope of all common points of those lines. Two lines passing through some vertex of this envelope divide the plane into four angles, and one of them contains all the remaining points. Thus the remaining lines do not intersect the vertical angle and the number of angles can not be less than three. An example with three angles can be constructed by induction: the next line must intersect all previous lines inside the triangle which is the convex envelope of common points.

b) **Answer.** n, in case n is odd; $n-1$ in case $n > 2$ is even. Let us construct a circle containing all common points. Our lines divide it into $2n$ arcs. Let AB, BC

be two adjacent arcs, X, Y be the common points of the line passing through B, with the lines passing through A and C respectively. If X lies on the segment BY, then the part containing the arc BC can not be an angle. Thus, only one of the two parts containing the adjacent arcs can be an angle. Thus, the number of angles is not greater than n, and an equality is possible only when the part containing each second arc is an angle. However, if n is even, this yields that there exist two angles containing the opposite arcs. Since these two angles are formed by the same lines, this is not possible for $n > 2$. If n is odd then n parts formed by the sidelines of a regular n-gon are angles. Obviously, we can add one line without reduction of the number of angles.

Second solution of a). *(A. Goncharuk, Kharkov)* Let a polygon T be the union of all bounded parts. Then all angles are vertical to the angles of T, which are less than $180°$. From the formula for the sum of angles we obtain that there exist three such angles. The polygon with three angles can be constructed in the following way. Take a point D inside the triangle ABC, inscribe a sufficiently small circle in the angle ADB and take $n - 4$ points on the smaller arc formed by the touching points. The tangents in these points and the lines AC, BC, AD, BD form the desired polygon.

7. *(A. Zaslavsky, correspondence round, 9–11)* Does there exist a non-isosceles triangle such that the altitude from one vertex, the bisector from the second one and the median from the third one are equal?

Solution. Yes, it does. Fix some vertices A, B, construct a point D which is the reflection of A wityh respect to B, and consider an arbitrary point C such that $\angle BCD = 150°$. The altitude of triangle ABC from A is equal to the distance DH from D to BC, i.e. $CD/2$. The median BM from B as the medial line in triangle ACD also is equal to $CD/2$ (Fig. 3). Now we move the point C along the arc BD containing angle $150°$. As C tends to B, the bisector from C tends to zero and the median from B tends to $AB/2$. As C tends to D, the median from B tends to zero and the bisector stays not smaller than BC. Thus, there exists a point C for which the bisector is equal to two remaining segments.

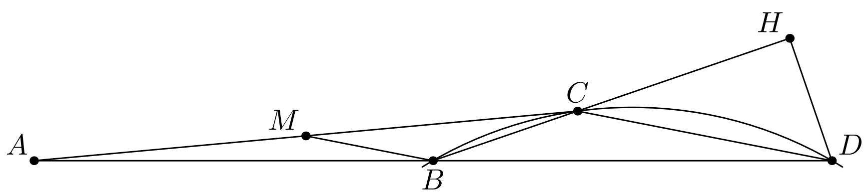


Fig. 3.

Remark. When C moves from B to D, the bisector increases and the altitude decreases. Thus the angles of the desired triangle are determined in a unique way.

8. *(G. Feldman, correspondence round, 10–11)* Let CX and CY be tangents from vertex C of a triangle ABC to the circle passing through the midpoints of its sides. Prove that the lines XY, AB and the tangent to the circumcircle of ABC at point C concur.

Solution. The homothety with center C and factor $1/2$ transforms the line XY to the radical axis of point C and the circle passing through the midpoints A', B', C' of BC, CA, AB respectively. On the other hand, the tangent at C to the circumcircle touches also the circle $A'B'C'$, i.e. it is the radical axis of this circle and the point C. The common point of these radical axes lies on $A'B'$. Using the inverse homothety we obtain the assertion.

9. *(N. Beluhov, correspondence round, 10–11)* Three congruent regular tetrahedrons have a common center. Is it possible that all faces of the polyhedron formed by their intersection are congruent?

Solution. Yes, it is possible. Let the first tetrahedron touch their common inscribed sphere at points A, B, C, D. Rotate these points around the common perpendicular (and bisector) of the segments AB and CD by $120°$ to obtain A', B', C', D' and by $240°$ to obtain A'', B'', C'', D'' (the twelve points form two regular hexagons). The tangential planes to the sphere in these twelve points form the three tetrahedrons needed. Indeed, for any two of these points there exists an isometry that maps this set of twelve points onto itself and maps one of these two points to another one. These isometries enable us to map any facet of the obtained polygon onto any other one.

10. *(T. Golenishcheva-Kutuzova, final round, 8)* Peter made a paper rectangle, put it on an identical rectangle and pasted both rectangles along their perimeters. Then he cut the upper rectangle along one of its diagonals and along the perpendiculars to this diagonal from two remaining vertices. After this he turned back the obtained triangles in such a way that they, along with the lower rectangle, form a new rectangle.

Let this new rectangle be given. Reconstruct the original rectangle using compass and ruler.

Solution. Let $ABCD$ be the obtained rectangle; O be its center; K, M be the midpoints of its shortest sides AB and CD; L, N be the meets of BC and AD respectively with the circle with diameter KM (Fig. 4). Then $KLMN$ is the desired rectangle. In fact, let P be the projection of M to LN. Since $\angle CLM = \angle OML = \angle MLO$, the triangles MCL and MPL are equal. Thus the bend along ML matches these triangles. Similarly the bend along MN matches triangles MDN and MPN. Finally, since the construction is symmetric with respect to the point O, the bend along KL and KN matches triangles BKL and AKN with triangle NKL.

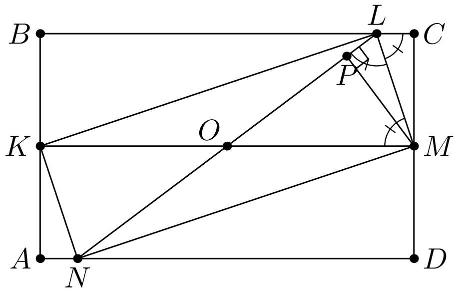


Fig. 4.

11. *(A. Shapovalov, final round, 8)* Given the circle of radius 1 and several its chords with the sum of lengths 1. Prove that one can inscribe a regular hexagon into that circle so that its sides do not intersect those chords.

Solution. Paint the smallest arcs corresponding to given chords. If we rotate the painted arcs in such a way that the corresponding chords form a polygonal line, then the distance between the ends of it is less than 1, and since a chord with length 1 corresponds to an arc equal to $1/6$ of the circle, the total length of painted arcs is less than $1/6$ of the circle.

Now inscribe a regular hexagon into the circle and mark one of its vertices. Rotate the hexagon, and when the marked vertex coincides with a painted point, paint the points corresponding to all remaining vertices. The total length of painted arcs increases at most 6 times, therefore there exists an inscribed regular hexagon with non-painted vertices. Obviously its sides do not intersect the chords.

12. *(A. Zaslavsky, final round, 8)* Using only the ruler, divide the side of a square table into n equal parts. All lines drawn must lie on the surface of the table.

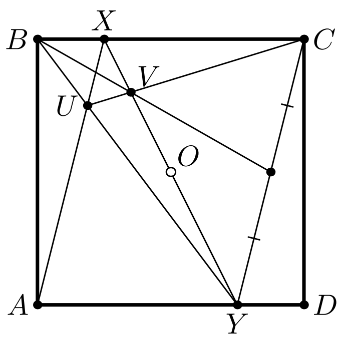


Fig. 5.

Solution. First, we bisect the side. Find center O of square $ABCD$ as a common point of its diagonals. Now let a point X lie on the side BC, Y be a common point of XO and AD, U be a common point of AX and BY, V be a common point of UC and XY (Fig. 5). Then the line BV bisects the bases of the trapezoid $CYUX$. The line passing through O and the midpoint of CY bisects sides AB and CD.

Now suppose that two opposite sides are divided into k equal parts. Let us demonstrate how to divide it into $k+1$ equal parts. Let $AX_1 = X_1X_2 = \cdots = X_{k-1}B$, $DY_1 = Y_1Y_2 = \cdots Y_{k-1}C$. Then by the Thales theorem, the lines $AY_1, X_1Y_2, \ldots, X_{k-1}C$ divide the diagonal BD into $k+1$ equal parts (Fig. 5). Dividing similarly the second diagonal and joining the corresponding points by the lines parallel to BC we divide side AB into $k+1$ equal parts.

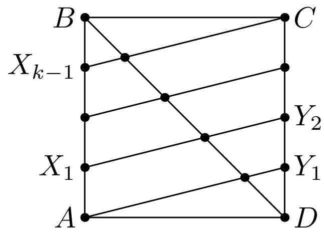


Fig. 6.

13. *(D.Cheian, final round, 9)* In triangle ABC, $\angle B = 2\angle C$. Points P and Q on the perpendicular bisector to CB are such that $\angle CAP = \angle PAQ = \angle QAB = \frac{\angle A}{3}$. Prove that Q is the circumcenter of triangle CPB.

Solution. Let D be the reflection of A in the perpendicular bisector to BC. Then $ABCD$ is the isosceles trapezoid and its diagonal BD is the bisector of angle B. Thus $CD = DA = AB$. Now $\angle DAP = \angle C + \angle A/3 = (\angle A + \angle B + \angle C)/3 = 60^\circ$. Thus triangle ADP is equilateral and $AP = AB$. Since AQ is the bisector of angle PAB, $QP = QB = QC$ (Fig. 7).

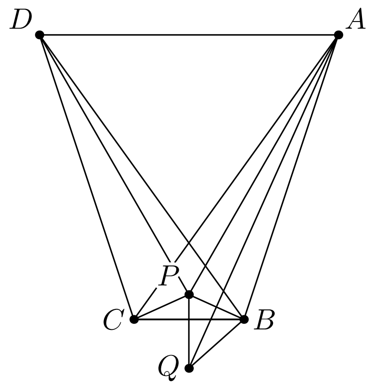


Fig. 7.

14. *(A. Zaslavsky, final round, 9)* A quadrilateral $ABCD$ is inscribed into a circle with center O. The bisectors of its angles form a cyclic quadrilateral with circumcenter I, and its external bisectors form a cyclic quadrilateral with circumcenter J. Prove that O is the midpoint of IJ.

Solution. Let the bisectors of angles A and B, B and C, C and D, D and A meet at points K, L, M, N respectively (Fig. 8). Then line KM bisects the angle formed by lines AD and BC. If this angle is equal to ϕ, then by external angle theorem we obtain that $\angle LKM = \angle B/2 - \phi/2 = (\pi - \angle A)/2 = \angle C/2$ and thus $\angle LIM = \angle C$. On the other hand, the perpendiculars from L and M to BC and CD respectively form the angles with ML equal to $(\pi - \angle C)/2$, i.e. the triangle formed by these perpendiculars and ML is isosceles and the angle at its vertex is equal to C. Thus the vertex of this triangle coincides with I. So the perpendiculars from the vertices of $KLMN$ to the corresponding sidelines of $ABCD$ pass through I. Similarly the perpendiculars from the vertices of triangle formed by external bisectors pass through J.

Now let K' be the common point of external bisectors of angles A and B. Since quadrilateral $AKBK'$ is inscribed into the circle with diameter KK', the projections of K and K' to AB are symmetric with respect to the midpoint of AB. From this and the above assertion, the projections of I and J to each side of $ABCD$ are symmetric with respect to the midpoint of this side, whci this is equivalent to the assertion.

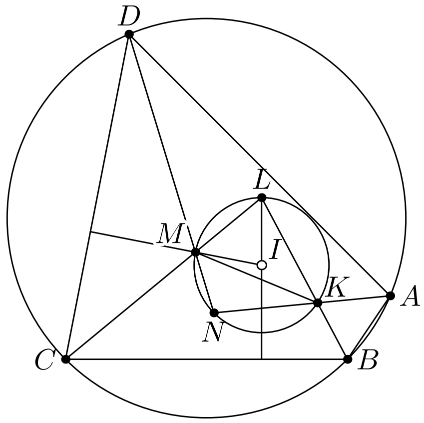


Fig. 8.

Remark. A similar property of a triangle is well-known: the circumcenter is the midpoint of the segment between the incenter and the circumcenter of the triangle formed by its external bisectors.

15. *(I. Bogdanov, final round, 9)* Circles ω and Ω are inscribed into one angle. Line ℓ meets the sides of angles, ω and Ω at points A and F, B and C, D and E respectively (the order of points on the line is A, B, C, D, E, F). It is known that $BC = DE$. Prove that $AB = EF$.

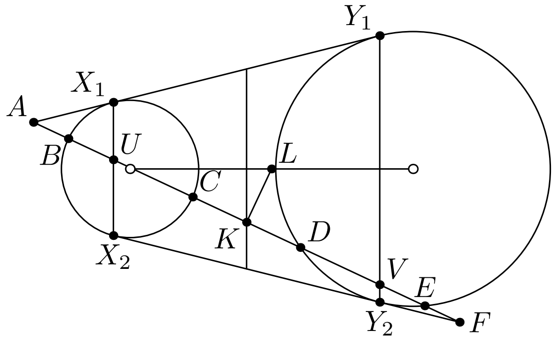


Fig. 9.

First solution. Let one side of the angle touch ω and Ω at points X_1, Y_1, and the second side touch them at points X_2, Y_2; U, V are the common points of X_1X_2 and Y_1Y_2 with AF. The midpoint of CD lies on the radical axis of the circles, i.e. the medial line of trapezoid $X_1Y_1Y_2X_2$, thus $BU = EV$ and $CU = DV$ (Fig. 9). This yields that $X_1U \cdot X_2U = Y_1V \cdot Y_2V$. Hence $FY_2/FX_2 = Y_2V/X_2U = X_1U/Y_1V = AX_1/AY_1$, i.e. $AX_1 = FY_2$. Now from $AB \cdot AC = AX_1^2 = FY_2^2 = FE \cdot FD$ we obtain the assertion of the problem.

Second solution. Prove that there exists exactly one line passing through a fixed point A on a side of the angle which satisfies the condition of the problem. In fact, the distances from the midpoint K of segment CD to the projections of the centers of the circles to the sought line are equal, thus K coincides with the projection of the midpoint L of the segment between the centers. Hence K is the common point of the circle with diameter AL and the radical axis, distinct from the midpoint of segment X_1Y_1. On the other hand, if F is a point such that $AX_1 = Y_2F$ then $AB \cdot AC = FE \cdot FD$ and $AD \cdot AE = FC \cdot FB$, thus AF is the sought line.

16. *(L. Emelyanov, final round, 10)* Quadrilateral $ABCD$ is circumscribed. Its incircle touches sides AB, BC, CD, DA in points K, L, M, N respectively. Points A', B', C', D' are the midpoints of segments LM, MN, NK, KL. Prove that the quadrilateral formed by lines AA', BB', CC', DD' is cyclic.

Solution. Let us begin with the assertion which follows by a calculation of angles.

Lemma. Points A, B, C, D lie on the same circle if and only is the bisectors of angles formed by lines AB and CD are parallel to the bisectors of angles formed by lines AD and BC.

In fact, consider the case when $ABCD$ is a convex quadrilateral, rays BA and DC meet at point E, rays DA and BC meet at point F. Then the angles between the bisectors of angles BED and BFD are equal to half-sums of opposite angles of the quadrilateral. This proves the lemma. Other cases are considered in the same way.

Now let us turn to the solution of the problem. Let I be the incenter of $ABCD$, r be the radius of its incircle. Then $IC' \cdot IA = r^2 = IA' \cdot IC$, i.e. points A, C, A', C' lie on the circle. By the lemma, the bisectors of angles between AA' and CC' are parallel to the bisectors of angles between IA and IC, and hence to the bisectors of the angles between perpendicular lines KN and LM. Similarly the bisectors of the angles between BB' and DD' are parallel to the bisectors of the angles between KL and MN. Using again the lemma we obtain the assertion of the problem.

17. *(V. Mokin, final round, 10)* Point D lies on the side AB of triangle ABC. The circle inscribed in angle ADC touches internally the circumcircle of triangle ACD. Another circle inscribed in angle BDC touches internally the circumcircle of triangle BCD. These two circles touch segment CD in the same point X. Prove that the perpendicular from X to AB passes through the incenter of triangle ABC.

Solution. Let us first prove the following auxiliary fact.

Lemma. Let a circle touch the sides AC, BC of a triangle ABC in points U, V and touch internally its circumcircle in point T. Then the line UV passes through the incenter I of triangle ABC.

Proof of Lemma. Let the lines TU, TV intersect the circumcircle for the second time at points X, Y. Since the circles ABC and TUV are homothetic with center T, points X, Y are the midpoints of arcs AC, BC, i.e. lines AY and BX meet at point I (Fig. 10). Thus the assertion of the lemma follows from Pascal theorem applied to the hexagon $AYTXBC$.

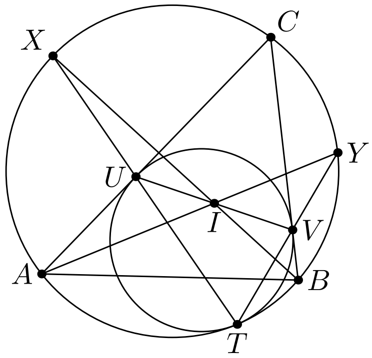


Fig. 10.

From the lemma and from the assumption it follows that DI_1XI_2 is a rectangle, where I_1, I_2 are incenters of triangles ACD and BCD respectively (Fig. 11). Let Y, C_1, C_2 be the projections of points X, I_1, I_2 to AB. Then $BY-AY = BC_2+C_2Y-AC_1-C_1Y = (BC_2-DC_2)-(AC_1-DC_1) = (BC-CD)-(AC-CD) = BC-AC$. Thus, Y is the touching point of AB with the incircle.

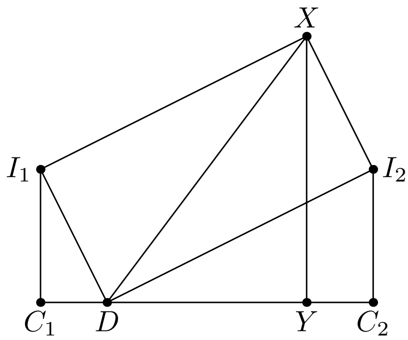


Fig. 11.

18. *(S. Tokarev, final round, 10)* Given a sheet of tin 6×6. It is allowed to bend it and to cut it so that it does not fall to pieces. How to make a cube with edge 2, divided by partitions into unit cubes?

Solution. The desired development is presented on Fig. 12. Bold lines describe the cuts, thin and dotted lines describe the bends up and down. The central 2×2 square corresponds to the horizontal partition of the cube.

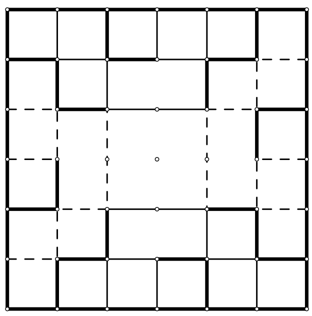


Fig. 12.